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THEORY OF MACHINES

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THEORY OF MACHINES

A TEXTBOOK COVERING THE SYLLABUSES OF
THE B.Sc. (Eng.), Inst.C.E., AND I.Mech.E.
EXAMINATIONS IN THIS SUBJECT

BY
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PREFACE

THE earliest edition of this textbook was written to cover the syllabus of the Universities in the subject of Theory of Machines, and the continued demand for the work is evidence that it has proved useful not only to students in Universities and Technical Colleges, but also to engineers engaged in design.

Developments in mechanical engineering during the twenty-two years which have elapsed since publication of the first edition, including special problems arising during the war period, have necessitated additions to each edition which have extended the scope of the work very appreciably, much of the additional matter being original. The standard of the mathematics required is that which would normally be expected of students pursuing such a course of study as the book provides, along with a parallel course of mathematics.

Additional matter in successive editions includes—

Extension of the theory of lubrication.

Whirling and torsional oscillations of a light loaded shaft.

Equivalent dynamical systems; oscillation of plane mechanisms; torsional oscillations of two shafts connected by gearing.

Free axial vibration of a shaft carrying any number of masses; general case of forced vibrations with damping; the pendulum damper.

Extension of the theory of worm gearing to threads of trapezoidal section.

New examples, dealing with the new matter, have been added.

This edition contains much additional matter, including treatment of inertia governors, the balancing of sleeve valves, the use of the gyroscope for stabilizing vehicles, the gyro compass. The treatment of the vibration of shafts and beams has been extended to include cases in which the load is uniformly distributed, and the vibrations of the gyro compass are examined.

The authors wish to thank the Senate of the London University and the University of London Press for permission to reprint examples from their B.Sc. (Engineering) examination papers; Messrs. W. Clowes & Sons, Ltd., for permission to reprint examples from the examination papers of the Institution of Civil Engineers; and several engineering firms, mentioned in the book, for the loan of blocks.

Thanks are also due to Mr. D. M. Crawford, M.Sc., for much valuable help; to Mr. R. A. Downs, B.Sc., A.M.I.Mech.E., for preparing almost all the illustrations; and to Mr. Luther Barber, A.M.I.Mech.E., and Mr. G. W. C. Coy for assistance with the drawings. Great care has been taken in the preparation of the drawings, as the authors have found by experience that the provision of clear and simple drawings is a necessary preliminary to a clear understanding of the subject.

L. TOFT
A. T. J. KERSEY

NOTE.—The abbreviations used in connection with the examples are (U.L.) or (L.U.), and (I.C.E.). The first and second are given with examples taken from London University papers in the subject. The third abbreviation indicates that the example is from a paper set by the Institution of Civil Engineers.

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THEORY OF MACHINES

CHAPTER I

MOTION—VECTORS—FORCE AND MOTION

1. THE subject of *Mechanics* is the portion of science which deals with the action of forces in producing equilibrium or motion. The name itself would suggest an alternative meaning, namely, the *Science of Machinery*. This textbook deals with the subject in the latter aspect, and will only consider the scientific principles on which the designing of machines is based. Machines are used in order to transform some available form of motion or energy into another form which is necessary for the performance of specified operations. They must transmit both force and motion ; each part must be made of such dimensions that its material is not unduly stressed by the forces transmitted, and the relations between the various parts must be such as to change the available form of motion into the form which it is desired to produce. The theory of machines, therefore, may be divided into two principal sections : (1) consideration of the strength of the various parts of the machine, and (2) consideration of the relative motions of the parts of the machine and of the different points in each part. We are concerned mainly with (2), which may be called the *transmission* side of the subject, and we shall enter, therefore, into a study of (*a*) the relative motions of the machine parts, or the *kinematics of machinery*, and (*b*) the transmission of forces and the effects of force on motion, or the *kinetics of machinery*.

MOTION

2. If a body A occupies different positions, relative to a second body B, at different times, the two bodies are said to be in *relative motion*. In specifying the motion of a body it is always necessary to indicate, either explicitly, or by implication, the second body to which the motion is referred. When we give the speed of a train as thirty miles per hour, we mean that the train is moving relatively to the earth at that speed.

The engineer is concerned with the relative motion of machine parts and in general he gives the motions of the parts with reference to the frame of the machine, or, as is generally the case, with reference to the earth to which the frame is attached. There is no real meaning to the term "Absolute Motion" as we do not know, nor can we ever know, of any body in the universe which is absolutely at rest. When we speak of motion without qualification, we shall mean the motion relative to the earth. The results obtained, however, will apply with equal truth if the machine considered is carried by a body which moves relatively to the earth. Such a body is a ship, a motor-car, a locomotive, etc., and the vehicle carrying the machine is taken as the body of reference.

3. **Speed** is the numerical value of the rate at which a particle is moving. If the particle is moving at a rate which, if continued unchanged, would carry it thirty miles in one hour, the speed of the particle is said to be thirty miles per hour. When it moves through equal intervals of space in equal intervals of time, whatever the magnitude of the intervals of time, it is moving with *uniform speed.* Otherwise, the speed is *variable.* If a particle moves through s feet of space in t seconds of time, its *average speed* is $\frac{s}{t}$ ft. per sec. By definition, the speed at the instant of time t is $\frac{ds}{dt}$.

4. **The Velocity** of a particle is the rate at which it is changing its position. We have defined speed as the numerical part of this rate, the other part being the direction in which the motion is taking place. The velocity may change owing to a change in either or both of these parts. In order to have *uniform velocity* a particle must maintain unchanged both its speed and its direction of motion. The only way in which a particle may have uniform velocity is, therefore, by moving with uniform speed in a straight line. The measure of uniform velocity is then the same as that of uniform speed given above. The velocity at an instant of time t is $\frac{ds}{dt}$, which is the limiting value of $\frac{\delta s}{\delta t}$ where δs is the space described in time δt, and δt is made to approach the value zero. This limiting value

will, of course, have direction as well as magnitude, the direction being that of the tangent to the path at the instantaneous position of the particle.

5. **The Acceleration** of a particle is the rate of increase of its velocity. If there are equal increases of velocity in equal intervals of time, whatever the magnitude of the intervals of time, there is *uniform acceleration*; otherwise there is *variable acceleration.* Acceleration is a quantity like velocity having magnitude and direction. Such quantities are known as vectors and are defined in Art. 12. It is sufficient to note here that if either a velocity or an acceleration is to be uniform the direction, as well as the magnitude, must remain fixed. We have seen that the only possible case of uniform velocity is that of motion with uniform speed along a straight line, and similarly, in all cases of uniform acceleration, the acceleration must always have a fixed direction. With the notation of Arts. 3 and 4 we have, if a = acceleration and v = velocity,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

6. **Units.** As length and time are fundamental quantities we derive from these the units of velocity and acceleration. Let L = unit of length and T = unit of time, then if V = unit of velocity, we have

$$V = \frac{L}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.1)$$

Similarly, if A = unit of acceleration, we have, since the numerical value of acceleration is found by dividing a velocity by an interval of time,

$$A = \frac{V}{T}$$

$$\therefore \quad A = \frac{L}{T^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.2)$$

If the unit of time is the second and the unit of length is the foot, then from the above

$$V = \frac{1 \text{ ft.}}{1 \text{ sec.}} \text{ or } 1 \text{ ft. per sec.}$$

and $$A = \frac{1 \text{ ft.}}{1 \text{ sec.} \times 1 \text{ sec.}} = \frac{1 \text{ ft.}}{1 \text{ sec.}^2}$$

i.e. the unit of acceleration is 1 ft. per sec. per sec.

If other units of length and time are used, the units of velocity and acceleration may be determined in a similar manner.

The equations (I.1) and (I.2) may be used for changing from one system of units to the other.

EXAMPLE.

A particle has a velocity 20 and an acceleration 40, the units of length and time being the foot and the second respectively. Find the velocity V and the acceleration A if the units of length and time are the mile and the hour respectively.

$$V = \frac{20 \text{ ft.}}{1 \text{ sec.}} = \frac{\frac{20}{5280} \text{ mls.}}{\frac{1}{3600} \text{ hrs.}} = 13\cdot 6 \text{ mls. per hr.}$$

$$A = \frac{40 \text{ ft.}}{1 \text{ sec.} \times 1 \text{ sec.}} = \frac{\frac{40}{5280} \text{ mls.}}{\frac{1}{3600} \text{ hrs.} \times \frac{1}{3600} \text{ hrs.}}$$

$$= 98{,}200 \text{ mls. per hr. per hr.}$$

7. Motion of a Particle Along a Straight Line with Constant Acceleration. Let a be the magnitude of the constant acceleration. From the definitions in Arts. 4 and 5

$$\frac{d^2s}{dt^2} = a \text{ and } \frac{ds}{dt} = v$$

where s is the distance measured along the line, v is the velocity and t the time. Integrating the former of these two relations we have

$$\frac{ds}{dt} = at + c \quad . \quad . \quad . \quad . \quad . \quad \text{(I.3)}$$

or

$$v = at + c \quad . \quad . \quad . \quad . \quad . \quad \text{(I.4)}$$

Integrating I.3 $s = \frac{1}{2} at^2 + ct + d$ . . . (I.5)

The constants c and d may be determined if we know the values of v and s for any given value or values of t. Thus if s and t are both zero at the instant when v has the value u, we have, on substituting these values in (I.4) and (I.5), $u = c$ and $o = d$.

Substituting these values of c and d in (I.4) and (I.5)

$$v = u + \alpha t \tag{I.6}$$

$$s = \tfrac{1}{2}\alpha t^2 + ut \tag{I.7}$$

On elimination of t from these two relations we obtain

$$v^2 = u^2 + 2as$$

or

$$v^2 - u^2 = 2as \tag{I.8}$$

By means of the relations (I.6), (I.7) and (I.8), all problems on this kind of motion may be solved. The case of a body moving vertically near the earth's surface is, neglecting air resistance, one of constant acceleration along the line of motion. In this case the value of the constant acceleration is approximately 32·2ft. per sec. per sec., varying slightly at different parts of the earth's surface. Writing g for this number, measuring s vertically upwards, and taking velocities as positive when the motion is upwards, we have—

$$v = u - gt \tag{I.9}$$

$$s = ut - \tfrac{1}{2}gt^2 \tag{I.10}$$

and

$$u^2 - v^2 = 2gs \tag{I.11}$$

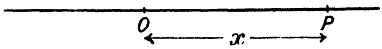


FIG. 1

8. **Motion of a Particle Along a Straight Line with an Acceleration Directed Towards a Fixed Point in the Straight Line, and Varying as the Distance from the Fixed Point.** Let OP be the straight line (Fig. 1), P the particle, and O the fixed point. Let x be the distance OP, x being positive when P is to the right of O. Then the equation relating to the motion is

$$\frac{d^2x}{dt^2} = -k^2x \tag{I.12}$$

where k is a constant and the minus sign occurs because the acceleration is directed in the negative sense along PO. Since $v = \dfrac{dx}{dt}$ this becomes

$$\frac{dv}{dt} = -k^2x$$

$$\therefore \frac{dv}{dx} \cdot \frac{dx}{dt} = -k^2x$$

$$\text{or } v \cdot \frac{dv}{dx} = -k^2x \tag{I.13}$$

Integrating with respect to x

$$\tfrac{1}{2}v^2 = c - \tfrac{1}{2}k^2x^2 \text{ where } c \text{ is a constant.}$$

Hence, $$v^2 = 2c - k^2x^2 \quad . \quad . \quad . \quad . \quad \text{(I.14)}$$

or $$\frac{dx}{dt} = v = \pm\sqrt{2c - k^2x^2}$$

If P is supposed to be moving to the left, the lower sign is taken, and writing k^2a^2 for $2c$,

$$\frac{dx}{dt} = -k\sqrt{a^2 - x^2}$$

$$\therefore\ -k\,dt = \frac{dx}{\sqrt{a^2 - x^2}}$$

Integrating $$-kt = \int\frac{dx}{\sqrt{a^2 - x^2}} + e \text{ where } e \text{ is a constant}$$

$$\therefore\ -kt = -\cos^{-1}\frac{x}{a} + e$$

or $$\cos^{-1}\frac{x}{a} = kt + e$$

$$\therefore\ x = a\cos(kt + e) \quad . \quad . \quad . \quad \text{(I.15)}$$

We shall require this result later when discussing simple harmonic motion.

9. **Angular Velocity** is the rate at which a body is rotating. If the body turns through equal angles in equal intervals of time, whatever the magnitude of the intervals of time, it is moving with *uniform angular velocity.* Otherwise the angular velocity is *variable.* The *average angular velocity* is the angle turned through in a given time divided by the time, i.e. $\frac{\theta}{t}$ where θ is the angle and t the time. The angular velocity at an instant is, from the definition, $\frac{d\theta}{dt}$.

The direction of the axis of rotation is associated with the numerical magnitude of the angular velocity to make up the angular velocity, and the angular velocity may change owing to a change in either or both of these parts. In most applications the axis of rotation is fixed, but in some cases the direction of the axis changes. If the axis is fixed in direction, the consideration of the motion is analogous to that of the rectilinear motion of a particle.

10. **Angular Acceleration** is the rate of increase of the angular velocity. If there are equal increases of the angular velocity in equal intervals of time, whatever the magnitude of those intervals, there is *uniform angular acceleration.* Otherwise there is *variable acceleration.* Angular acceleration has a numerical value as well as a direction. If ω represents the change in angular velocity during time t, the *average angular acceleration* is $\frac{\omega}{t}$, and the angular acceleration at the instant of time t is $\frac{d\omega}{dt}$, or since $\omega = \frac{d\theta}{dt}$ the angular acceleration is $\frac{d^2\theta}{dt^2}$.

The unit angle is the radian, and it is usual to take 1 radian per second as the unit of angular velocity, and 1 radian per second per second as the unit of angular acceleration.

11. **Rotation with Constant Angular Acceleration** about an axis whose direction is fixed.

Let α be the constant angular acceleration,

then, $$\alpha = \frac{d^2\theta}{dt^2} \text{ and } \omega = \frac{d\theta}{dt}.$$

These equations correspond to those of Art. 7, and are solved in the same manner, giving

$$\omega = \alpha t + \omega_o \quad . \quad . \quad . \quad . \quad . \quad (I.16)$$

$$\theta = \tfrac{1}{2}\alpha t^2 + \omega_o t \quad . \quad . \quad . \quad . \quad (I.17)$$

and $$\omega^2 - \omega_o^2 = 2\alpha\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.18)$$

corresponding respectively to (I.6), (I.7) and (I.8) where θ and t are zero together at the instant when $\omega = \omega_o$.

VECTORS—FORCE AND MOTION

12. **Scalar and Vector Quantities.** Quantities which occur in engineering may be divided into two classes, (1) those which are not associated with direction, and (2) those which are associated with direction. Quantities in the first class are known as *scalar quantities,* and calculations relating to the sums, differences, and products of two or more of these quantities of the same kind are carried out by the ordinary arithmetical methods. The sum of a mass of 10 units and one of 20 units is a mass of 30 units. Length, area, volume, speed, and mass are examples of scalar quantities.

Quantities such as displacement, velocity, acceleration, force, couple and momentum are of the second class. Sums, differences, and products of two or more of these quantities are not found by the ordinary arithmetical processes. The sum of two forces each of 10 units, one acting from the north to the south, and the other from east to west, is a force of less than 20 units acting towards the south-west. Quantities with which direction is associated are known as *vector quantities.*

13. **Addition of Vectors.** Consider a particle A, Fig. 2, to be first displaced from A to B along the straight line AB, and then from B to C along the straight line BC. The total change in the position of the particle is the same as if A had been displaced along AC to the position C. In this sense we say that the last displacement is equal to the *sum* of the other two.

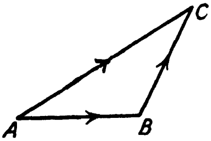


FIG. 2

As we shall require to use the letters A and B to indicate (1) the line AB, (2) the length of the line AB, and (3) the vector AB, in this case the displacement from A to B, we shall use AB to denote the line, $\overline{AB}$ to represent its length, and $\overrightarrow{AB}$ to represent the vector AB. The above statement of equality may be written $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ . . . . . (I.19)

The vector $\overrightarrow{AC}$ (or any other vector) has three characteristics, (1) its length, namely $\overline{AC}$, (2) its direction, which is that of AC, and (3) its sense, which is from A to C along AC and which is represented by the arrow head.

A vector which is the sum of two or more vectors is sometimes called the *resultant vector*, the other vectors being the component vectors. If the two components are at right-angles to each other they are called *resolutes*, and the process of finding two rectangular components is known as *resolving* the vector in the given directions.

If we look upon $\overrightarrow{AB}$ as being the displacement of a steamer, and $\overrightarrow{BC}$ as the displacement of an object across its deck, both displacements taking place simultaneously, we see that $\overrightarrow{AC}$ is the sum or resultant of the two simultaneous displacements $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

14. **The Parallelogram of Vectors.** In Fig. 3, $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and if $ABCD$ is a parallelogram,

$$\overrightarrow{AD} = \overrightarrow{BC}$$

$$\therefore\ \overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{AB}$$

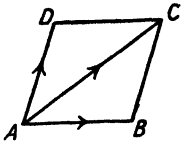


FIG. 3

That is, the diagonal of a parallelogram represents the resultant of the two vectors represented by the two sides which meet on the diagonal. This parallelogram is known as the *parallelogram of vectors*. It is necessary that the arrows on the three vectors should point either all towards their point of intersection or all away from it.

15. **Polygon of Vectors.** To find the resultant of more than two vectors we may proceed by the method of Art. 13 to find the resultant of two vectors, and then to find the resultant of the former resultant with a third vector, and so proceed until we have disposed of all the original vectors, when the last resultant will be the one required.

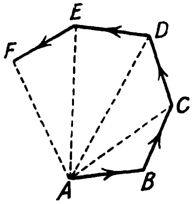


FIG. 4

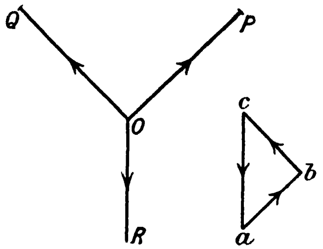


FIG. 5

Thus, in Fig. 4,

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$$

$$\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$$

and

$$\overrightarrow{AF} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF}$$

If $\overrightarrow{AF} = 0$, that is, if the polygon $ABCDEF$ is a closed one, the vectors have no resultant, and are said to balance, or the total displacement is zero.

In the particular case where there are three vectors in equilibrium, we have the *triangle of vectors.*

Let $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$ (Fig. 5) represent three concurrent vectors, and let ab, bc, ca be lines which are respectively parallel and proportional to them. Since abc is a closed triangle $\overrightarrow{OP}$, $\overrightarrow{OQ}$, and $\overrightarrow{OR}$ are in equilibrium.

From the triangle abc,

$$\frac{\overline{ab}}{\sin \widehat{bca}} = \frac{\overline{bc}}{\sin \widehat{cab}} = \frac{\overline{ca}}{\sin \widehat{abc}}$$

But $\widehat{bca}$, $\widehat{cab}$ and $\widehat{abc}$ are respectively the supplements of $\widehat{QOR}$, $\widehat{POR}$ and $\widehat{POQ}$.

Hence $$\frac{\overline{ab}}{\sin \widehat{QOR}} = \frac{\overline{bc}}{\sin \widehat{POR}} = \frac{\overline{ca}}{\sin \widehat{POQ}}$$

If P, Q, R are the respective magnitudes of OP, $\overrightarrow{OQ}$, $\overline{OR}$, we have $$\frac{P}{\overline{ab}} = \frac{Q}{\overline{bc}} = \frac{R}{\overline{ca}}$$

Therefore $$\frac{P}{\sin \widehat{QOR}} = \frac{Q}{\sin \widehat{POR}} = \frac{R}{\sin \widehat{POQ}} \quad . \quad . \quad (I.20)$$

that is, the magnitudes of the vectors are proportional to the sines of the opposite angles. This is *Lami's theorem.*

16. **Extension to Other Vectors.** In the above explanations we have dealt only with displacements. The results obtained, however, apply to all vectors. A velocity is a rectilinear displacement which is produced in unit time, an acceleration is a change of velocity produced in unit time, in fact all vector quantities can be shown to be obtained by multiplying or dividing length with which direction is associated, by numbers not associated with direction such as those representing time, mass, etc. It will appear, therefore, that any results which are shown to be true for displacement vectors, are equally true for vector quantities of any other kind provided that only vectors of the same kind are dealt with in any particular investigation.

17. Difference of Two Vectors. In Art. 13 we found the sum of two vectors, thus in Fig. 2 we saw that

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

We shall define $\overrightarrow{AB}$ as the *difference of the two vectors* $\overrightarrow{AC}$ and $\overrightarrow{BC}$, and $\overrightarrow{BC}$ as the difference of $\overrightarrow{AC}$ and $\overrightarrow{AB}$

$$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} \quad . \quad . \quad . \quad . \quad (I.21)$$

We may then transpose terms from one side to the other of a vector equation if we change the signs. In the case of velocity or acceleration, the difference is the *relative velocity* or the *relative acceleration* of two particles whose motions are represented by the given vectors. We shall represent the velocity of A relative to B by ${}_{B}V_{A}$, using V_{A} to denote the velocity of A with reference to the earth or the frame carrying the machine where it is not necessary to specify the frame of reference. Where the two suffixes are used, the first will indicate the body to which the motion is referred, and the second the body whose motion is under consideration.

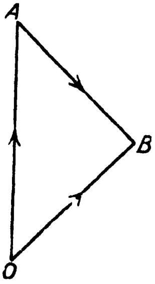


FIG. 6

If $\overrightarrow{AC}$ indicates the velocity of A, and $\overrightarrow{AB}$ the velocity of B, with reference to some frame of reference, then (I.21) becomes

$${}_{B}V_{A} = V_{A} - V_{B} \quad . \quad . \quad . \quad . \quad (I.22)$$

In the case of acceleration this becomes

$${}_{B}A_{A} = A_{A} - A_{B} \quad . \quad . \quad . \quad . \quad (I.23)$$

EXAMPLE.

Two ships, A and B, at sea are moving, A towards the north at 30 miles per hour, and B towards the north-east at 25 miles per hour. Find the speed of the ship B relative to A.

Let ${}_{A}V_{B}$ = velocity of B relative to A, and let $\overrightarrow{OA}$ (Fig. 6) represent V_{A}, and $\overrightarrow{OB}$ represent V_{B}. $\overline{OA} = 30$ units and $\overline{OB} = 25$ units. Then,

$$\begin{aligned} {}_{A}V_{B} &= V_{B} - V_{A} \\ &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \overrightarrow{AB} \end{aligned}$$

From the triangle,

$$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 - 2\,\overline{OA}\;\overline{OB}\cos \widehat{AOB}$$

$$= 30^2 + 25^2 - 2 \times 30 \times 25 \times \cos 45^\circ$$

$$= 464{\cdot}5$$

$$\therefore AB = 21{\cdot}54 \text{ units}$$

also
$$\frac{\sin B}{\overline{AO}} = \frac{\sin 45^\circ}{\overline{AB}}$$

or
$$\sin B = \frac{\overline{AO}\sin 45^\circ}{\overline{AB}}$$

$$= \frac{30 \times \sqrt{2}}{2 \times 21{\cdot}54} = {\cdot}9846$$

and
$$B = 80^\circ \text{ to the nearest degree.}$$

Thus, the magnitude of the velocity of B relative to A is 21·54 miles per hr. and its direction is $80^\circ - 45^\circ = 35^\circ$ south of east.

The values of ${}_AV_B$ and the angle B may be verified by measurement from the figure.

18. **Circular Motion of a Particle.** Let P, Fig. 7, be the position at time t seconds of a particle describing a circle of radius r and centre O. Take rectangular axes of x and y and let the angle $XOP = \theta$.

If PM is perpendicular to OX, then $x = r\cos\theta$ and $y = r\sin\theta$.

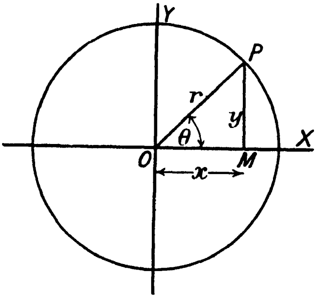


Fig. 7

Also, V_X = component velocity of P along OX

$$= \frac{dx}{dt}$$

$$= \frac{d}{dt}(r\cos\theta)$$

$$= r\frac{d}{dt}(\cos\theta)$$

$$= -r\sin\theta \cdot \frac{d\theta}{dt} \quad \text{(I.24)}$$

Again, A_X = component acceleration of P along OX

$$\begin{aligned} &= \frac{d}{dt}\left(-r\sin\theta\frac{d\theta}{dt}\right) \\ &= -r\frac{d}{dt}\left(\sin\theta\frac{d\theta}{dt}\right) \\ &= -r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2} \end{aligned} \qquad . \quad (\text{I}.25)$$

When OP is horizontal, V_X and A_X are respectively the radial velocity V_R and the radial acceleration A_R. Putting $\theta = 0$ in (I.24) and (I.25) in turn, we have

$$V_R = 0, \text{ as is obvious,}$$

and

$$A_R = -r\left(\frac{d\theta}{dt}\right)^2 \qquad . \quad . \quad . \quad . \quad . \quad (\text{I}.26)$$

Again, when OP is vertical, V_X and A_X are respectively the tangential velocity V_T, and the tangential acceleration A_T. Putting $\theta = \frac{\pi}{2}$ in (I. 24) and (I.25) in turn

$$V_T = -r\frac{d\theta}{dt} \qquad . \quad . \quad . \quad . \quad . \quad (\text{I}.27)$$

and

$$A_T = -r\frac{d^2\theta}{dt^2} \qquad . \quad . \quad . \quad . \quad . \quad (\text{I}.28)$$

The value of V_T given by (I.27) is otherwise obvious from the relation between angular and linear velocities. The acceleration A_R, being negative, is directed along the radius of the circle towards the centre, and is on this account known as *centripetal acceleration.*

Putting ω for $\frac{d\theta}{dt}$, (I.26) becomes

$$A_R = \omega^2 r \qquad . \quad . \quad . \quad . \quad . \quad . \quad (\text{I}.29)$$

The minus sign is dropped here, but it must always be remembered that the acceleration is towards the centre of rotation. It should be noticed that the centripetal acceleration is the same whether ω is constant or variable. In the former case A_T is zero, and the centripetal acceleration is the whole acceleration of the particle; in the latter case there is an additional component acceleration of magnitude $r\frac{dw}{dt} = \frac{dv}{dt}$ (I.28) in the direction of the tangent.

19. **Plane Motion.** In the case of circular motion, r is constant, but if the particle is not constrained to move in a circular path r is variable. In this case, with the notation of the preceding Art., we have,

$$V_X = \text{component velocity of } P \text{ along } OX$$

$$= \frac{dx}{dt} = \frac{d}{dt}(r \cos\theta)$$

$$= \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad (I.30)$$

and $$A_X = \text{component acceleration of } P \text{ along } OX$$

$$= \frac{d}{dt}V_X$$

$$= \frac{d}{dt}\left(\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right)$$

$$= \frac{d^2r}{dt^2}\cos\theta - \sin\theta\frac{dr}{dt}\frac{d\theta}{dt} - r\sin\theta\frac{d^2\theta}{dt^2}$$

$$- \left(r\cos\theta\left(\frac{d\theta}{dt}\right)^2 + \sin\theta \cdot \frac{dr}{dt}\frac{d\theta}{dt}\right)$$

$$= \frac{d^2r}{dt^2}\cos\theta - 2\sin\theta\frac{dr}{dt}\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2$$

$$- r\sin\theta\frac{d^2\theta}{dt^2} \quad . \quad . \quad . \quad . \quad (I.31)$$

Putting $\theta = 0$ and $\theta = \frac{\pi}{2}$ in turn in (I.30) and (I.31), we obtain

$$V_R = \frac{dr}{dt} \quad . \quad . \quad . \quad . \quad . \quad (I.32)$$

$$A_R = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \quad . \quad . \quad . \quad (I.33)$$

$$V_T = -r\frac{d\theta}{dt} \quad . \quad . \quad . \quad . \quad . \quad (I.34)$$

$$A_T = -2\frac{dr}{dt} \cdot \frac{d\theta}{dt} - r\frac{d^2\theta}{dt^2} \quad . \quad . \quad (I.35)$$

V_T and A_T are now the velocity and acceleration respectively in a direction perpendicular to the radius and not along the tangent as in Art. 18.

Putting v for $\frac{dr}{dt}$, and ω for $\frac{d\theta}{dt}$, the last three relations become

$$A_R = \text{radial acceleration} = \frac{dv}{dt} - \omega^2 r \quad . \quad . \quad \text{(I.36)}$$

$$V_T = \text{transverse velocity} = \omega r \quad . \quad . \quad . \quad \text{(I.37)}$$

and $$A_T = \text{transverse acceleration} = 2\omega v + r\frac{d\omega}{dt} \quad . \quad \text{(I.38)}$$

We have dropped the minus signs from the last two relations, as these merely indicate that the components are directed to the left along OX (Fig. 7). If now we put $v = 0$, we obtain results which are the same as those found in Art. 18.

The term $2\omega v$ represents what is called the *Coriolis* acceleration. The effect of the increasing radius is to increase both the radial and the transverse accelerations, the former by $\frac{dv}{dt}$, the latter by the Coriolis component $2\omega v$.

The resultant of these gives the acceleration of a slider moving along a rotating radial rod, relative to an instantaneously coincident point fixed on the rod (see Art. 88).

20. **Force and Motion.** We assume our readers to be familiar with Newton's laws of motion. From his second law we obtain

$$F \propto \frac{d}{dt}(Mv) \quad . \quad . \quad . \quad . \quad . \quad \text{(I.39)}$$

In this formula, F is the force producing the change of motion, M is the mass of the body, v is its velocity, and t the time. The engineers' unit of force is the pound weight, or the pull which the earth exerts on a certain lump of matter at sea level in London. The inertial *mass* of a body is a measure of its inertness or resistance to change of motion, and its value in engineers' units is the ratio of the weight of the body in pounds, at any part of the earth's surface, to the value in feet per second per second of g, the acceleration of a body falling freely under gravity at the same place. The *weight* W lb. of a body is the pull of the earth on the body, and varies slightly with the locality; g also varies with the locality, but the ratio W/g is constant. The unit of mass is then the quantity of matter in a body weighing g lb.

The product Mv is the *momentum* or quantity of motion in the body. From (I.39),

$$F = k\frac{d}{dt}(Mv)$$

where k is a constant. As the mass is constant

$$F = kM\frac{dv}{dt}$$

and since $\frac{dv}{dt}$ is the acceleration, we have writing a for $\frac{dv}{dt}$,

$$F = kMa \qquad . \quad . \quad . \quad . \quad . \quad (I.40)$$

To determine k, we consider the case of a mass of W lb. weight falling freely under gravity; in this case the force acting is W lb., the mass is $\frac{W}{g}$ engineers' units, and the acceleration is g ft. per second per second. Hence, substituting in (I.40)

$$W = k \cdot \frac{W}{g} \cdot g$$

$$\therefore \quad k = 1$$

Substituting this value of k, we have

$$F = Ma. \qquad . \quad . \quad . \quad . \quad . \quad (I.41)$$

or

$$F = M\frac{dv}{dt} \qquad . \quad . \quad . \quad . \quad . \quad (I.42)$$

which are alternative forms of the equation of motion of the body. It should be noticed that F and a $\left(\text{or } \frac{dv}{dt}\right)$ are vectors having the same direction and sense.

21. **Impulse.** If the force F is constant, we have on integrating (I.42),

$$Ft = M(V_2 - V_1) \qquad . \quad . \quad . \quad . \quad (I.43)$$

where t is the time interval and V_1, V_2 are the initial and final velocities respectively. If F is variable, $\int_0^t F dt$ takes the place of Ft, and the relation is

$$\int_0^t F dt = M(V_2 - V_1) \qquad . \quad . \quad . \quad . \quad (I.44)$$

If the interval of time t is indefinitely small, the left-hand side of this relation may still be finite if F is sufficiently great. In such a case, F is called an *impulsive force*, the left-hand side of (I.44) being a measure of the impulse of the force. Though a force which acts during a finite interval of time is not an impulsive force, the time integral of the force is called the impulse of the force in all cases. Relations (I.43) and (I.44) show that the impulse of the force is equal to the change of momentum produced. If the initial velocity is zero and v is the final velocity, (I.43) becomes $Ft = Mv$.

22. **Centripetal Force in Circular Motion.** The relation (I.41) which may be written

$$\text{Force} = \text{mass} \times \text{acceleration}$$

is of general application in all cases of motion of a particle or rigid body. In the case of a rigid body it will be shown that the force is the resultant of all the external forces acting on the body and that the acceleration is that of a point fixed in the body (centre of mass). In the case of a particle describing a circle with uniform speed, we saw that the acceleration of the particle was $\omega^2 r$ directed inwards along the radius of the circle. If m is the mass of the particle, we have from (I.41),

$$\text{Centripetal force} = m\,\omega^2 r$$

Since $m = \dfrac{W}{g}$, and $v = \omega r$, this may be written as

$$F = \text{centripetal force} = \frac{W}{g}\,\omega^2 r \qquad . \quad . \quad (I.45)$$

or

$$F = \text{centripetal force} = \frac{Wv^2}{gr} \qquad . \quad . \quad . \quad (I.46)$$

In other cases where the mass and acceleration are known, the force producing the acceleration is calculated from (I.41), and its direction and sense are the same as those of the acceleration.

23. **The Space-time Graph.** This is a graph (Fig. 8) having the displacement of a particle as ordinate, and the time as abscissa. The space, or displacement, x ft., is the distance measured along the path, which may be curved or straight, between some fixed point in the path and the position occupied by the particle when the time is t seconds. Let the scales be such that 1 in. along Ox represents a space of b feet, and 1 in. along Ot represents a time of a seconds. Then if $\overline{OM}$ and $\overline{MP_x}$

are measured in inches, they represent respectively a time of $a \times \overline{OM}$ seconds, and a space of $b \times \overline{MP_x}$ ft.

24. The Speed-time Graph. The speed v of the particle is given by $v = \dfrac{dx}{dt}$. If a and b are both unity, the slope of the curve $P_xQ_xR_x$ is $\dfrac{dx}{dt}$, and thus the speed of the particle in feet per second is numerically equal to the slope of the graph.

For other values of a and b, the value of v is given by

$$v = \frac{dx}{dt} = \text{slope of graph} \times \frac{b}{a} \quad . \quad . \quad . \quad . \quad \text{(I.47)}$$

By measuring the slope at different points of $P_xQ_xR_x$ and substituting the values in (I.47) we may obtain sufficient values of v and corresponding values of t to plot a speed-time graph (Fig. 9), in which ordinates represent speed, and abscissae represent time. The axis of v is in the same vertical line as the axis of x (Fig. 8), and the time-scale is the same in both figures. A graphical method of obtaining points on the speed-time curve is used, the construction being shown for obtaining the three points P_v, Q_v, and R_v. We shall explain the construction for the point P_v. The axis tO (Fig. 9) is produced to a pole P, where $\overline{OP}$ is any convenient length c inches. A tangent is drawn at P_x (Fig. 8), and Pp is drawn parallel to this, meeting Ov in p. A horizontal line is drawn through p to meet a vertical through P_x in the point P_v. Then we have

$$\text{Ordinate of } P_v = \overline{Op} = \overline{OP} \tan \widehat{OPp}$$

$$= c \times \text{slope of space-time graph at } P_x.$$

Substituting from (I.47) we have

$$\text{Ordinate} = c \times \frac{av}{b}$$

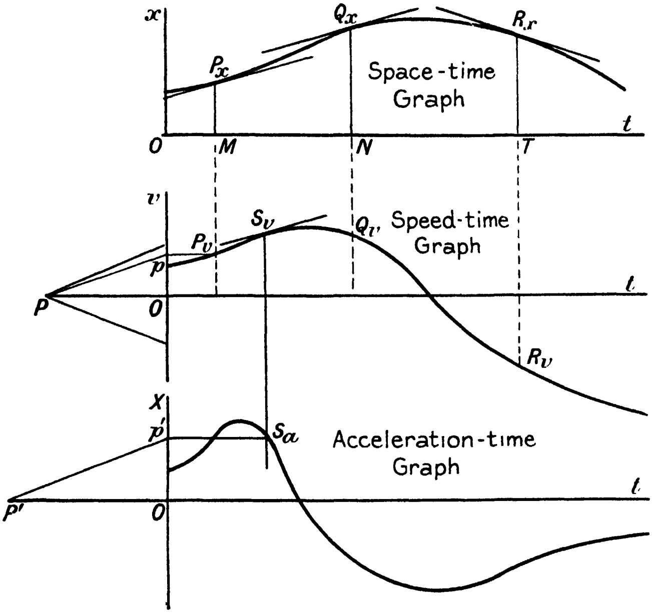
where v is the speed of the particle when the time is $t = \overline{OM}$.

Thus $\quad v = \text{speed in feet per second}$

$$= \frac{b}{ca} \times \text{ordinate} \quad . \quad . \quad . \quad . \quad \text{(I.48)}$$

the ordinate being measured in inches. From this we see that the ordinate represents the speed on a scale on which

1 in. represents a speed of $\frac{b}{ca}$ ft. per sec. This construction is repeated for a sufficient number of points on $P_xQ_xR_x$, and sufficient points are obtained in Fig. 9 to enable us to draw the speed-time graph as shown. In fixing the points P_x, Q_x, R_x, etc., of the space-time graph from which to derive points of the speed-time graph, it is probably more accurate to draw the tangents first, and then to determine their points of contact with the curve. In deriving Fig. 9, tangents were drawn in Fig. 8 by using a tee-square and two set-squares. Using these, tangents were drawn making angles of 30°, 45°, 60°, 0°, 150°, 135° and 120°, respectively, with Ot. The steepest tangent was also drawn. The points of contact of the tangents were then determined, and the above construction carried out for each point. In this way eight points were located on the speed-time graph. The construction was then repeated for intermediate points, until sufficient points were found to enable us to draw the speed-time curve.



FIGS. 8—9—10

25. **Acceleration-time Graph.** In this graph times are plotted as abscissae and component accelerations along the path as ordinates. The graph is obtained from the speed-time graph in the same manner as the latter is obtained from the space-time graph. The construction is shown for one point only, S_a of Fig. 10 being the point derived from S_v of Fig. 9. The polar distance $\overline{OP'}$ may again be any convenient length d in. Let f be the acceleration represented by the ordinate of S_a. Then,

$$\begin{aligned}
\text{Ordinate} &= \overline{Op'} \\
&= \overline{OP'} \tan \widehat{OP'p'} \\
&= d \times \text{slope of speed-time graph at } S_v.
\end{aligned}$$

But,

$$\begin{aligned}
f &= \text{acceleration} \\
&= \frac{dv}{dt} \\
&= \text{slope of speed-time graph} \\
&\quad \times \frac{\text{speed represented by 1 in.}}{\text{time represented by 1 in.}}
\end{aligned}$$

$$\therefore \qquad = \text{slope of speed-time graph} \times \frac{\frac{b}{ca}}{a}$$

$$= \frac{b}{ca^2} \times \text{slope}$$

$$\begin{aligned}
\text{Hence, Ordinate} &= d \times \text{slope} \\
&= \frac{ca^2d}{b} \times f
\end{aligned}$$

and

$$f = \frac{b}{ca^2d} \times \text{ordinate} \quad . \quad . \quad . \quad \text{(I.49)}$$

Thus, the ordinate of Fig. 10 represents the acceleration f on a scale on which 1 in. represents an acceleration of $\frac{b}{ca^2d}$ ft. per second per second. It should be clearly understood that just as Fig. 9 only gives the speed of the particle and not the velocity, so Fig. 10 gives only the numerical value of the component acceleration of the particle along its path. If the path is curved there is a centripetal acceleration which must be compounded with the tangential component given by Fig. 10 (see Art. 18).

EXAMPLE.

A steam engine drop-valve is closed by a spring after the operation of a trip gear. The stiffness of the spring is such that a force of 20 lb. is required per inch of compression. The valve is lifted against the spring, and when fully open the compression is 3 in. When closed the compression is $1\frac{1}{4}$ in. The weight of the valve is 10 lb. and the resistance may be taken as constant and equal to 12 lb. Determine the time taken to close the valve after the operation of the trip. (L.U.A.)

Let x ft. be the displacement of the valve from its highest position at time t seconds. When the valve is closed the value of x is $\frac{1}{12}(3 - 1\frac{1}{4}) = \frac{7}{48}$ ft. In any position the push of the spring is $20\,(3 - 12x)$ lb. If P lb. is the downward force on the valve, we have

$$\begin{aligned} P &= 20\,(3 - 12x) + 10 - 12 \\ &= 58 - 240x \end{aligned}$$

Now Force = mass × acceleration,

$$\therefore \quad 58 - 240x = \frac{10}{g} \times \frac{d^2x}{dt^2}$$

i.e.

$$\frac{d^2x}{dt^2} = \frac{g}{10}\,(58 - 240x)$$

Put X for $x - \frac{58}{240}$

then,

$$\frac{d^2X}{dt^2} = \frac{d^2x}{dt^2} = \frac{g}{10}\,(-\,240X)$$

or,

$$\frac{d^2X}{dt^2} = -\,24gX$$

Comparing this with (I.12), we see from (I.15) that the solution is

$$X = A\cos\left(\sqrt{24g}\,t + \alpha\right)$$

Where a and A are constants to be determined.

Now putting $x = X + \frac{58}{240}$, we have

$$x = \frac{58}{240} + A\cos\left(\sqrt{24g}\,t + \alpha\right)$$

Since $x = 0$ when $t = 0$,

$$0 = \frac{58}{240} + A \cos \alpha$$

$$\therefore \quad \cos \alpha = -\frac{58}{240A}$$

Again, $$\frac{dx}{dt} = -\sqrt{24g}\, A \sin(\sqrt{24g}\, t + \alpha)$$

Since $\frac{dx}{dt} = 0$ when $t = 0$, i.e. the valve starts from rest,

$$0 = -\sqrt{24g}\, A \sin \alpha$$

or, $$\sin \alpha = 0$$

Whence, $$\alpha = 0$$

Hence $\cos \alpha = 1$ and $-\frac{58}{240A} = 1$

Thus, $$A = -\frac{29}{120}$$

The solution is then,

$$x = \frac{29}{120}(1 - \cos \sqrt{24g}\, t)$$

When $x = \frac{7}{48}$

$$\frac{7}{48} = \frac{29}{120}(1 - \cos \sqrt{24g}\, t)$$

$$\therefore \; 1 - \cos \sqrt{24g}\, t = \frac{7}{48} \times \frac{120}{29} = \frac{70}{116}$$

Hence, $\cos \sqrt{24g}\, t = \frac{46}{116} = \cdot 3966$

$$\therefore \; \sqrt{24g}\, t = 1\cdot 163 \text{ and } t = \frac{1\cdot 163}{\sqrt{24 \times 32\cdot 2}}$$

i.e. $t = \cdot 0418$ secs., which is the required time.

EXAMPLES 1.

1. A particle moves with a velocity of 40 ft. per min., and an acceleration of 20 ft. per min. per min. What are the velocity, and acceleration, when the units of time and length are respectively (1) the hour and the mile, (2) the second and the centimetre ? (There are 39·37 in. in a metre.)

2. A cage weighing 5 tons is lifted by a vertical rope connected to a winding motor, with a maximum winding speed of 30 ft. per sec.; this speed is reached under uniform acceleration 6 sec. after the lift starts. Find the tension in the vertical rope whilst the speed is being increased. Neglect the weight of the rope.

3. Two tramway stops are 800 ft. apart; a car starts from one with constant acceleration until it attains a speed of 12 mls. per hr. in 10 secs. It runs at this speed until it is 44 ft. from the second stop, when the brakes are applied, and the car moves with constant retardation, coming to rest at the stop. Find the values of (1) the acceleration, (2) the retardation, (3) the whole time taken.

4. Explain how to find the velocity of a body A relative to a second body B, the velocities of A and B being known relative to a third body C. An aeroplane is flying due east at 54 mls. per hr; if there is a steady wind of 24 mls. per hr. from the north-east, what is the velocity of the aeroplane relative to the air ?

5. A man standing on a train which is moving at 36 mls. per hr. shoots at an object, which is moving away from the railway at right-angles at a speed of 12 miles per hour. If the bullet, which is supposed to move in a straight horizontal line, has a velocity of 880 ft. per sec., and if the line connecting the man and object makes an angle of 45° with the train when he fires, find at what angle to the train he must aim in order to hit the object. (I.C.E.)

6. A wheel weighing W lb., radius of gyration k ft., has its centre of gravity a small distance h ft. out of centre. If it is rotating freely without friction, show that the ratio of the total change of speed during a revolution to the mean speed ω (rad. per sec.) is given approximately by $\frac{2gh}{k^2\omega^2}$.

7. Two ships sail simultaneously from two ports situated on the same meridian, and 50 miles apart. The ship from the more northerly port steers S.W. at 12 knots, and the other steers due W. at 15 knots. How near will the two ships approach each other, and at what time after their departure will they be closest ? (1 knot = 6,080 ft. per hr.) (I.C.E.)

8. A horizontal wire is rotating about one end in a horizontal plane at 20 revs. per min. A small bead is moving along the wire at a constant speed relative to the wire of 2 ft. per sec. Find the velocity and acceleration of the bead at the instant when it is 2 ft. from the fixed end of the wire moving outwards.

9. A flywheel rotating at 20 revs. per min. has an angular retardation of 60 radians per min. per min. Find its angular velocity after 30 sec., and the number of revolutions it makes (1) in 30 sec. and (2) in coming to rest.

10. A locomotive weighing 50 tons is moving on a circular track of radius 200 yds. at a speed of 40 mls. per hr. The steam is suddenly shut off, and the total resistance to motion is 4 tons. Find the radial and tangential accelerations at the instant after the steam is shut off; find also the radial pressure between the wheels and the rails.

11. Using the relation (I.43), show that if two bodies collide the vector representing the change of momentum of one body is equal and opposite to that representing the change of momentum of the other, that is the resultant momentum of the bodies remains constant.

12. The resistance to motion of a motor-cycle combination weighing 950 lb., moving on a level road at V miles per hour, is given by

$$R \text{ (lb.)} = 22{\cdot}6 + {\cdot}038V^2$$

By means of the brakes an additional resistance of 70 lb. is applied when the vehicle is moving at 36 mls. per hr. with the power cut off. Find (*a*) the time taken to stop, (*b*) the distance travelled during that time

13. In a winding gear the cage weighs 25,000 lb. when empty, and raises an additional load of 1,500 lb. The rope weighs 10 lb. per ft. and is 2,600 ft.

long, passing over a sheave whose effective diameter is 20 ft. and weight $2\frac{3}{4}$ tons, the radius of gyration being 9 ft. and the arc of contact 120°. The rope then passes on to a winding drum whose effective diameter is 15 ft. and weight 45 tons, the radius of gyration being 6 ft. Friction may be taken as an additional load of 2,000 lb.

The cage, which is not counterbalanced, is accelerated at a rate of 3 ft./sec.^{2} until a speed of 20 ft./sec.^{2} is attained and decelerated at the end of its travel at 15 ft./sec.^{2}.

(*a*) Calculate the maximum torque required when raising the loaded cage, and (*b*) draw the torque-time and power-time curves for lowering the empty cage. If regenerative braking with an efficiency of 90 per cent is utilized, calculate (*c*) the average power available during lowering. Find also (*d*) the coefficient at the sheave so that slipping of the rope shall not occur. The movement of the cage is from 2,500 ft. to 40 ft. measured vertically from the axis of the head gear. (U.L.A.)

14. A valve is opened by a cam and closed by a helical spring. The weight of the valve is 1 lb.; the initial force on the spring when the valve is closed is 6 lb.; the stiffness of the spring is 10 lb. per in.; the mean diameter of the spring is 1 in. and the spring has 20 convolutions. The constant frictional resistance to the motion of the valve is 0·75 lb.

(*a*) Find the time taken to close the valve from the position of maximum lift, which is 0·25 in.

(*b*) Find the diameter of the steel wire of which the spring is made. (U.L.A.)

15. The maximum drag on the draw-bar of a train is equivalent to $\frac{1}{15}$ of its weight. The maximum braking force is equivalent to $\frac{1}{10}$ of its weight. What is the minimum time to cover a mile from start to stop, and what is the maximum speed attained? (U.L.)

16. Find the average force resisting the motion in number 28 of Examples II. (U.L.)

17. The magnitude of the maximum accelerating force for the valve of an internal-combustion engine is required. The lift in inches of the valve is measured for various angles of rotation of the crankshaft and a curve plotted. Give, in general terms, the remaining data that must be known, and show clearly how the maximum accelerating force may be determined. (I.C.E.)

18. If $x = t^3 + 20t^2 + 100$, plot a graph showing values of x as ordinates and values of t as abscissae from $t = 0$ to $t = 9$, where t is the time in seconds, and x is the distance in feet from the fixed point of a particle moving in a straight line. Obtain from this, by the methods of Arts. 23, 24 and 25,

(1) a speed-time graph,

(2) an acceleration-time graph.

Find from your graphs the speed and acceleration at time $t = 6$ and verify by comparing your results with the values of $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ when $t = 6$.

19. If the particle in the last example is moving on a circular arc of radius 1,000 ft. at the instant when the time is 6 sec., find the resultant acceleration of the particle at that instant.

20. Draw speed-time and acceleration-time graphs for a particle which moves so that $x = t^2$ for the first 10 sec., $x = 10t$ for the next 10 sec., and $x = 400 - 10t$ for the last 10 sec. Find at what times there are sudden changes of speed and, if the weight of the particle is 1 lb., find the impulses necessary to produce the changes of speed. (*Note.* As it is impossible to have infinite forces acting on any body, this state of motion cannot be realized; the changes in speed would take a finite period of time, however small, and there would be no instantaneous changes of slope in the space-time graph.)

21. A train travelling at 30 mls. per hr. has steam shut off and the brakes applied; its speed after t secs. is given on page 25.

t	0	4	12	20	26	35	42	50
v	30	26	21·5	16·7	14·0	10·4	7·7	4·8

(*a*) Plot a curve showing the retardation at any time.

(*b*) Plot a curve showing the distance covered at any time.

(*Note.* The distance covered up to any time t sec., is the area under the speed-time graph from $t = 0$ to $t = t$.)

22. The effective horse-power required to propel a ship of 10,000 tons displacement is 5,000 at a speed of 15 miles per hour. Assuming that the propulsive force is constant and that the resistance is proportional to the square of the speed, find the time taken from rest to attain a speed of 10 miles per hour. Find also the distance travelled during this period.

23. The total resistance to motion of a vehicle weighing W lb., moving at v ft. per sec. on a level track, is given by

$$R\ (\text{lb.}) = k + m \, . \, v^2$$

where k and m are constants.

If the propelling force is withdrawn when the speed is V ft. per sec., show that the distance travelled before coming to rest is given by

$$S\ (\text{ft.}) = \frac{W}{2g \, . \, m} \, . \log_e \left(1 + \frac{m}{k} \, . \, V^2\right)$$

24. A pile of mass w lb. is driven into the ground by n blows of a hammer of mass W lb. falling from rest through a vertical distance of h ft. for each blow. If R lb. is the constant resistance of the ground, find how far the pile is driven into the ground. Find an expression for the loss of energy during each impact. If $w = 1000$, $W = 3000$, $h = 20$, $n = 30$, and $R = 100{,}000$, find the total distance moved, and the energy lost at each impact.

25. The resistance to motion of a body weighing 200 lb., falling vertically, is 4 oz. when the speed is 10 ft. per sec., and varies as the square of the speed. If the body is dropped from a height of 10,000 ft., find (*a*) the maximum speed reached, (*b*) the time taken to reach the ground.

26. In the production of pipes by the method of centrifugal casting the liquid metal is fed into a mould which is rotated at high speed about a horizontal axis. Show that the least possible speed of rotation of the mould is $\frac{266}{\sqrt{d}}$ r.p.m., where d is the internal diameter of the casting in inches. (In practice much higher speeds of revolution are used.)

27. A body weighing 322 lb. is lifted by a varying force F. Values of x, the distance lifted, and F are given below.

x (ft.)	0	1	2	3	4	5·5	7	9	11 and above
F (lb.)	540	540	540	530	500	460	310	220	190

If the velocity is 5 ft. per sec., when $x = 0$, tabulate values of the velocity at the end of each foot of lift and the time taken to reach each position. At what height will the body cease to rise?

28. A pile weighing $\frac{1}{2}$ ton is driven into the ground by blows of a hammer weighing 2 tons dropping through a height of 5 ft. If the ground exerts a constant resistance of 40 tons, determine the distance through which the pile is driven per blow and the loss of energy at each blow. (U.L.)

CHAPTER II

CENTRE OF GRAVITY—MOMENTS—COUPLE—MOMENT OF INERTIA—DYNAMICS OF SIMPLE ENGINE MECHANISM

26. **Centre of Gravity.** The weight of a body is the vector sum or resultant of the weights of the particles of which it is composed. The weights of the latter are forces acting along vertical lines whose directions are practically parallel. Thus, the weight of the body is the resultant of a system of parallel forces of fixed magnitudes acting through a set of rigidly connected points. It is proved in textbooks of mechanics that, whatever the direction of the parallel forces, there is in such a system a fixed point through which the resultant force always acts. This point is known as the centre of the system of parallel forces, and the point in a rigid body through which the weight always acts is called the *centre of gravity* or the *centre of mass* of the body.

If x and y are the co-ordinates, and w the weight of a small element of a rigid plane lamina, the co-ordinates $\bar{x}$ and $\bar{y}$ of the centre of gravity of the lamina are given by

$$\bar{x} = \frac{\Sigma wx}{\Sigma w}$$

$$\bar{y} = \frac{\Sigma wy}{\Sigma w}$$

If the particles of a rigid body move in parallel straight lines the body may be treated as a particle of the same mass collected at the centre of gravity, and its equation of motion is (I.41). For the general case of plane motion of a rigid body it is necessary to investigate both the translational and the rotational effects of forces. The plane motion of the rigid body may be resolved into two parts, (1) the motion of its centre of gravity, which is the same as that of a particle of the same mass as the body when acted on by a force equal to the resultant of the forces acting on the body, and (2) the rotational motion of the body with the centre of gravity as centre of rotation, due to the turning effects of the forces.

27. **Moment of a Force.** By experiments on levers and other rigid bodies, it is proved that the effect of a force in

rotating, or tending to rotate, a rigid body about an axis which is perpendicular to the force, is measured by the product of the force into the perpendicular distance between the axis and the line of action of the force. This product is called the *torque*, or the *moment of the force about the axis*. If the body is a plane lamina, the axis becomes a point, and we then speak of the moment of the force *about a point*. The turning tendency may be in either of two senses, i.e. the clockwise, or positive, sense, or the counter-clockwise, or negative, sense.

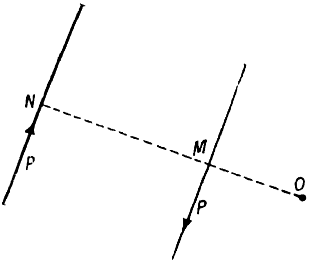


FIG. 11. COUPLE

28. Couple. Moment of a Couple. Two forces of equal magnitude and of opposite senses acting one along each of two parallel straight lines form a *couple*. Fig. 11 shows two such forces, each of P lb. weight, acting along parallel lines at a distance a apart. O is any point in the plane of the forces, and OMN is a straight line drawn perpendicular to the direction of the forces and cutting their lines of action in M and N. The algebraic sum of the moments of the two forces is given by

$$\begin{aligned} C &= \text{moment of forces about } O \\ &= P\,.\,\overline{ON} - P\,.\,\overline{OM} \\ &= P\,(\overline{ON} - \overline{OM}) \\ &= P\,.\,\overline{NM} \\ &= P\,a, \text{ which is independent of the position of } O \end{aligned}$$

Thus, a couple has the same moment about all points in its plane. On this account we speak of the moment of a couple without reference to any particular point and the *moment of a couple* is the product of the magnitude of one of the two forces and the distance between their lines of action. A couple is a vector, and the direction of the vector is that of a perpendicular to the plane of the couple. The sense of the vector is determined by the right-hand screw rule, which is explained in Art. 204. As the pound is the engineers' unit of force, and the foot that of distance, the unit of turning moment is the pound-foot (lb.-ft.).

29. A Single Force is Equivalent to an Equal Parallel Force

together with a Couple. *If in Fig. 11 we introduce at M a force equal and opposite to the force already acting there, the system of three forces will reduce, on cancelling the equal and* opposite forces, to the single force P at N. But the system of three forces consists of a couple of moment Pa, and a single force P at M of the same senses as the force P at N. Thus, the force P at N is equivalent to a single parallel force in the same sense at M together with a couple of moment Pa. By means of this equivalence we may replace a single force at one point by an equal force at any other point and a couple.

In this way a plane system of non-concurrent forces may be replaced by a concurrent system of forces, at any given point, and a couple. The resultant of the concurrent system of forces has the same magnitude, direction and sense as has the resultant of the non-concurrent system. In dealing with the motion of a rigid body, we sometimes reduce the acting forces to a single resultant force, acting through the centre of gravity, and a couple. We are thus able to find the motion of the centre of gravity by treating the body as a particle at the centre of gravity; the motion of the body relative to the centre of gravity being one of rotation, depends upon the value of the couple and upon the mass of the body, and its disposition relative to the centre of gravity.

As an example of the use of the methods of this section we will take the case of a nut being tightened by means of a spanner. Let a force P be applied at the outer end of the spanner in a direction perpendicular to the length of the spanner, and let the distance between the line of the force and the axis of the nut be h. The force P may be replaced by an equal force P through the axis of the nut, and a couple of moment Ph tending to rotate the nut.

30. **Moment of Inertia. Angular Momentum.** From the formula, force = mass × acceleration, we see that, if the force is constant, the acceleration produced is inversely proportional to the mass of the body. The mass is therefore a measure of the *inertia* of the body, or of its " resistance " to change of velocity. In the case of a rotating body, the couple or moment produces angular acceleration, and the measure of the " resistance " to change of angular velocity is a quantity called the *moment of inertia* of the body. Consider a particle of mass $m = \frac{w}{g}$ engineers' units, moving in a circle of radius r ft., the

angular velocity at any instant being ω radians per second. In Art. 18 we saw that the force acting on the particle has at any instant a tangential component of $\frac{w}{g}\frac{dv}{dt}$ lb., and a radial component of $\frac{w}{g}\,\omega^2 r$ lb., directed inwards. The latter component has no moment about the centre of the circle, and the former component has a moment about the centre of magnitude $\frac{wr}{g}\frac{d}{dt}(r\omega) = \frac{wr^2}{g}\,\alpha$ lb.-ft. units, where $\alpha = \frac{d\omega}{dt}$ is the angular acceleration. Thus the moment needed to produce the angular acceleration is given by,

C = turning moment or couple

i.e. $$C = \frac{wr^2}{g}\,\alpha \text{ lb.-ft. units} \qquad \text{(II.1)}$$

For a system of rigidly connected particles, or for a rigid body which may be treated as an assemblage of particles, the total turning moment is given by

$$C = \Sigma \frac{wr^2}{g}\,\alpha$$

$$\begin{aligned} C &= \alpha\,\Sigma \frac{w}{g} r^2 \\ &= \alpha\,\Sigma\, mr^2 \end{aligned}$$

Or, writing I for Σmr^2, we have

$$C = I\alpha \qquad \text{(II.2)}$$

I is known as the *moment of inertia*, and if m is in engineers' units, and r in feet, I is measured in mass (feet)^2 units.

The relation (II.2) may be written

Couple = moment of inertia × angular acceleration

which should be compared with the relation

Force = mass × acceleration

Putting $\alpha = \frac{d\omega}{dt}$ in (II.2), and integrating, on the assumption that C is constant, we obtain

$$Ct = I\omega \qquad \text{(II.3)}$$

which should be compared with

$$Ft = Mv \text{ (Art. 21)} \qquad \text{(II.4)}$$

ω is the increase in the angular velocity.

If the body has initial angular velocity ω_1, we get instead of

(II.3) $Ct = I(\omega_2 - \omega_1)$, where ω_2 is the angular velocity after time t.

Ct is the *impulse of the moment or couple*, and $I\omega$ is the *angular momentum*, or *moment of momentum* of the body about the axis of rotation. The term moment of momentum is used because $I\omega$ is the sum of the products of the momenta of the particles and their respective distances from the axis.

Compare these quantities with the impulse of a force, and the momentum of a particle respectively (Art. 21). $I\alpha = \frac{d}{dt}(I\omega)$ is the rate of change of the angular momentum, and (II.2) may be written

Couple = rate of change of angular momentum

This relation is true for rotation about any axis whatever, the moment of inertia referring, of course, to the axis of rotation.

Radius of Gyration. It is sometimes convenient to be able to treat the matter in a rotating body as if it were all collected at a fixed distance from the axis of rotation. This we may do so far as the application of (II.2) is concerned, if we do not alter the value of I.

Let k be the distance from the axis of rotation at which the whole mass of a body weight W lb. may be considered as concentrated to have the same moment of inertia as the body itself about that axis. k is known as the *radius of gyration* of the body.

Then, $\frac{W}{g} \cdot k^2 = I$

$$\therefore k^2 = \frac{g \cdot I}{W} \qquad \text{(II.5)}$$

Since g is usually in feet and second units, k must be measured in feet, and I in engineers' units.

31. Related Moments of Inertia. If x is the distance of w from a plane, then $I = \Sigma \frac{w}{g} x^2$ is the moment of inertia of the body about the plane.

Consider an element of a body, weight w lb., at P, the distances of which from the planes YZ, ZX, and XY are x, y. and z ft. respectively (Fig. 12).

Distance of P from $OX = \sqrt{y^2 + z^2}$
,, ,, ,, $OY = \sqrt{x^2 + z^2}$
,, ,, ,, $OZ = \sqrt{x^2 + y^2}$
,, ,, ,, $O = \sqrt{x^2 + y^2 + z^2}$

Let I_{OX}, I_{OY}, I_{OZ} be the moments of inertia of the body about the lines *OX*, *OY*, *OZ* respectively, and let I_{YZ}, I_{ZX}, I_{XY} be the moments of inertia of the body about the planes *YZ*, *ZX*, *XY* respectively.

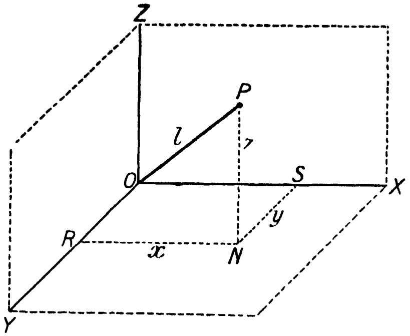


FIG. 12. MOMENTS OF INERTIA

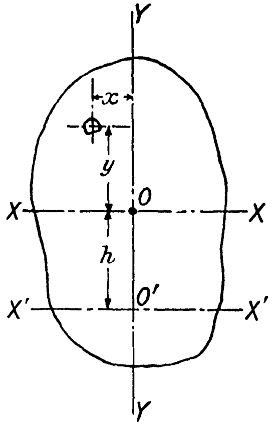


FIG. 13

Since, $$\frac{w}{g}(y^2 + z^2) = \frac{w}{g} \cdot y^2 + \frac{w}{g} \cdot z^2$$

$$\therefore \Sigma\frac{w}{g} \cdot (y^2 + z^2) = \Sigma\frac{w}{g} \cdot y^2 + \Sigma\frac{w}{g} \cdot z^2$$

$$\therefore I_{OX} = I_{XZ} + I_{XY} \quad . \quad . \quad . \quad . \qquad (II.6)$$

Similarly, $$I_{OY} = I_{YZ} + I_{XY} \quad . \quad . \quad . \quad . \qquad (II.7)$$

and $$I_{OZ} = I_{YZ} + I_{ZX} \quad . \quad . \quad . \quad . \qquad (II.8)$$

Let *XX* and *YY* (Fig. 13) represent the traces of two planes at right-angles to each other, and perpendicular to the plane of the paper. Then, from the foregoing, the moment of inertia of the body about an axis *OZ* through *O* perpendicular to the plane of the paper = moment of inertia about the plane *YY* + moment of inertia about the plane *XX*.

In certain cases, where the lateral dimensions are relatively small, the moment of inertia about the plane *YY* may be neglected.

If *OZ* passes through the centre of mass, and the moment of inertia about a new axis $O'Z'$, parallel to OZ and distant h from it, be required, then

$$I_{O'Z} = \Sigma\frac{w}{g}(y+h)^2 + \Sigma\frac{w}{g} \,.\, x^2$$

$$= \Sigma\frac{w}{g} \,.\, y^2 + 2h \,.\, \Sigma\frac{w}{g} \,.\, y + \Sigma\frac{w}{g} \,.\, h^2 + \Sigma\frac{w}{g} \,.\, x^2$$

Since XX passes through the centre of mass $\Sigma\frac{w}{g} \,.\, y = o$

$$\therefore I_{O'Z'} = I_{ZX} + \frac{W}{g} \,.\, h^2 + I_{YZ}$$

$$= I_{OZ} + \frac{W}{g} \,.\, h^2 \qquad . \quad . \quad . \quad . \quad . \qquad (II.9)$$

If k_1 is the radius of gyration about $O'Z'$, and k is the radius of gyration about OZ, then

$$k_1{}^2 = k^2 + h^2 \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (II.10)$$

32. **Work.** If a force F (Fig. 14) acts on a particle as it moves from A to B along the straight line AB the force is said to do work on the particle. If F is constant, and the angle $\theta = GAB$ is also constant, work done is measured by the product of the component of F along AB into the length $\overline{AB}$, or $W = F \cos\theta \times \overline{AB}$, where W represents the work done. Re-writing this in the form

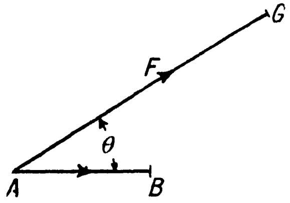


FIG. 14. WORK DONE BY A FORCE

$$W = F\,\overline{AB}\cos\theta \qquad . \quad . \quad . \qquad (II.11)$$

we see that the work done is also the product of the force into the component displacement in the direction of the force. If $\theta = 0$ the work done is $F\,\overline{AB}$, and if $\theta = 90°$ the work done is zero. If θ is an obtuse angle $\cos\theta$ is negative, and the work done is negative. In such a case we say that work is done against the force. The unit of force being the pound, and that of distance the foot, the unit of work is the foot-pound (ft.-lb.). If either F or θ is not constant, the work done in the displacement is given by

$$W = \int_{0}^{AB} F\cos\theta\, ds \qquad . \quad . \quad . \qquad (II.12)$$

If AB is curved, the value of θ in (II.11) is the angle between the line of action of F and the tangent to the path.

The *work done by a couple* is the algebraic sum of the amounts of work done by its component forces. Referring to Fig. 11 (page 27), if the body on which the couple acts turns through an angle $\delta\theta$ radians in the sense of the couple, the work done by the force P at N is $P\,\overline{ON}\,\delta\theta$, ($\overline{ON}\,\delta\theta$ being the displacement of N in the direction of the force), and the work done by the force P at M is $-P\,\overline{OM}\,\delta\theta$. The work done by the couple is, therefore,

$$\begin{aligned} W &= P\,\overline{ON}\,\delta\theta - P\,\overline{OM}\,\delta\theta \\ &= P(\overline{ON} - \overline{OM})\,\delta\theta \\ &= P\,\overline{NM}\,\delta\theta \\ &= C\,\delta\theta \text{ (Art. 28)} \quad . \quad . \quad . \quad \text{(II.13)} \end{aligned}$$

The work done in turning through an angle θ is, therefore,

$$W = \int_0^\theta C\,d\theta \quad . \quad . \quad . \quad . \quad \text{(II.14)}$$

If C is constant $W = C\theta$ . . . . . . (II.15)

33. **Power** is the rate of doing work. The British unit is the horse-power, which is 33,000 ft.-lb. per min. If a constant force F lb. acts on a particle moving with a constant speed of v ft. per sec. in the direction of the force, the work done per minute is $60Fv$ ft.-lb. and the horse-power (h.p.) is given by

$$\begin{aligned} \text{h.p.} &= \frac{60Fv}{33{,}000} \\ &= \frac{Fv}{550} \quad . \quad . \quad . \quad . \quad . \quad \text{(II.16)} \end{aligned}$$

If a constant couple or torque of moment C lb.-ft. acts on a rigid body rotating at N revs. per min., the work done per minute is $C \times 2\pi N$ ft.-lb. and

$$\text{h.p.} = \frac{2\pi NC}{33{,}000} \quad . \quad . \quad . \quad . \quad \text{(II.17)}$$

34. **Load Lifting Machines.** Appliances for raising loads or overcoming resistances, such as the lever, screw jack, pulley blocks, cranes, etc., are called load lifting machines. A machine such as a punching machine, while not strictly a *lifting* machine, may be classed with these.

The *velocity ratio* of a load lifting machine is the ratio of the speed of the point of application of the effort to the speed of the point at which useful work is delivered. This ratio is independent of the forces transmitted and depends only on the dimensions of the machine.

It is assumed that the dimensions of the machine are not altered appreciably by the action of the forces transmitted.

The *force ratio* (or mechanical advantage) of a machine is the ratio of the *useful* resistance overcome to the effort applied.

Principle of Work (or *Law of Work*).

If in any machine whatever an effort E, moving with linear velocity x, overcomes a total resistance R, moving with velocity y, then

$$E \times x = R \times y$$

$$\therefore R = E \times \frac{x}{y}$$

$\therefore$ Total resistance = effort × velocity ratio.

If the useful resistance is W, then $\frac{W}{R}$ is known as the *mechanical efficiency* (η).

$$\therefore \eta = \frac{W}{R} = \frac{W}{E} \div \frac{x}{y} = \frac{\text{force ratio}}{\text{velocity ratio}}$$

This is the same as $\eta = \dfrac{\text{useful work done}}{\text{total work done}}$

35. **Overhauling of Machines.** A machine is said to be able to "overhaul" if, when the effort is removed, the useful resisting force remaining the same, the motion is reversed, i.e. an effort applied at the driven end instead of the driving end will reverse the motion of the machine.

Let W = useful resistance

F = load at driven end which is equivalent to the friction of the machine.

Then, $$\eta = \frac{W}{W + F}$$ from which,

$$(W + F)\eta = W$$

or, $$F = W\left(\frac{1}{\eta} - 1\right)$$

If the effort is removed F is reversed, and if the machine is to overhaul, we must have

$$W > F$$

$$\text{i.e.} \qquad W > W\left(\frac{1}{\eta} - 1\right)$$

$$\eta > 0{\cdot}5$$

The machine will, therefore, not overhaul if its efficiency is less than 50 per cent.

The above assumes that the removal of the effort does not alter appreciably the value of F. This assumption is permissible (1) where the velocity ratio is large, (2) with a belt drive, in which the total pull of the belt is independent of the power transmitted, (3) where the effort is applied as a couple, as in an electric motor.

36. **Energy.** If a body is capable of doing work, either by changing its position in space, or by altering its length, shape, or volume, or by losing some of its speed (angular or linear) it is said to possess energy.

The amount of energy stored in a body is the amount of work it must do before it loses all its energy.

The energy a body possesses due to its velocity is known as kinetic energy; any other energy is known as potential energy.

To Find the Kinetic Energy of a Mass of W lb. weight, moving with a Linear Velocity of v ft. per sec. Imagine the body brought to rest by a constant resistance R lb. in moving through a distance x ft. The equation of motion is

$$\text{Force} = \text{mass} \times \text{acceleration}$$

$$\therefore \frac{W}{g}\frac{d^2x}{dt^2} = -R$$

The case of motion of this kind is investigated in Art. 7, and (I.8) becomes

$$v^2 - u^2 = -2 \cdot \frac{g \cdot R}{W} \cdot x$$

where v is the final velocity in this case zero), and u is the initial velocity (in this case v).

$$\text{Then,} \qquad v^2 = \frac{2g \cdot R}{W} \cdot x$$

$$\therefore R \cdot x = \frac{W \cdot v^2}{2g} \qquad . \quad . \quad . \quad . \quad \text{(II.18)}$$

$R \,.\, x$ is the work done by the body in coming to rest, i.e. in losing all its kinetic energy, so that, writing K.E. for kinetic energy,

$$\text{K.E.} = \frac{W \,.\, v^2}{2g} \quad . \quad . \quad . \quad . \quad (\text{II}.19)$$

In changing its velocity from u to v $(v > u)$ a body increases its energy by an amount given by

$$\text{Increase of K.E.} = \frac{W}{2g}(v^2 - u^2). \quad . \quad . \quad . \quad (\text{II}.20)$$

To Find the K.E. of a Rotating Body. Let I_0 engineers' units be the moment of inertia about its axis of rotation O, and let ω be the angular velocity in radians per second. Consider a small element of the body of weight w lb. at a distance of x ft. from O. We have

$$\begin{aligned}
\text{Linear velocity of element} &= \omega x \text{ ft. sec.} \\
\text{K.E. of element} &= \frac{w}{2g}\omega^2 x^2 \text{ ft.-lb.} \\
\text{Total K.E.} &= \Sigma \frac{w}{2g}\omega^2 x^2 \text{ ft.-lb.} \\
&= \tfrac{1}{2}\omega^2 \, \Sigma \frac{w}{g} x^2 \\
&= \tfrac{1}{2} I_o \omega^2 \text{ ft.-lb.} \quad . \quad (\text{II}.21)
\end{aligned}$$

Let I_G be the moment of inertia about a parallel axis through G, the centre of gravity of the body. Then from (II.9),

$$I_o = I_G + \frac{W}{g}\overline{OG}^2$$

and the

$$\text{K.E.} = \tfrac{1}{2} I_G \omega^2 + \tfrac{1}{2}\frac{W}{g} \,.\, \overline{OG}^2 \omega^2$$

But if v is the velocity of G, $v = \omega \,.\, OG$, hence

$$\text{K.E.} = \tfrac{1}{2} I_G \omega^2 + \tfrac{1}{2}\frac{W}{g} v^2 \quad . \quad (\text{II}.22)$$

(II.21) is used when the position of the axis of rotation is known, and (II.22) when the angular velocity and the speed of the centre of gravity are known. The latter relation tells us that the kinetic energy may be considered as made up of two parts, (1) that due to the body rotating about the centre of gravity, and (2) that due to a particle of mass equal to the total mass moving with the centre of gravity.

37. **"M" of a Rotating Body.** The "M" of a rotating body is the kinetic energy when making 1 rev. per min. From (II.21)

$$\text{K.E.} = \tfrac{1}{2}I\omega^2$$

and at 1 rev. per minute

$$\omega = \frac{2\pi}{60}\text{ radians per sec.}$$

hence,

$$M = \frac{\pi^2}{1800} I \text{ ft.-lb.} \quad . \quad \text{(II.23)}$$

At any other speed of N revs. per min., the K.E. is then given by

$$\text{K.E} = MN^2 \text{ ft.-lb.} \quad . \quad \text{(II.24)}$$

From (II.23)

$$M = \frac{I}{180}\text{ approximately.}$$

(II.24) is a convenient form of expression for kinetic energy.

Potential Energy of a Weight *W* lb. at a Height *h* ft. above the Ground. If the weight is allowed to fall to the ground with uniform speed, it will exert a force of W lb. through a distance of h ft. It has, therefore, in virtue of its height, the capacity to do Wh ft.-lb. of work. Its potential energy is therefore given by

$$\text{Potential energy} = Wh \text{ ft.-lb.} \quad . \quad . \quad \text{(II.25)}$$

A coiled spring or deflected beam possesses potential energy, the energy being equal to the work done on the elastic body, i.e. the work which it would do in returning to its unstrained position.

38. **The Principle of Conservation of Energy.** Energy can be changed in form, but cannot be created or destroyed. If one body loses energy, some other body or bodies gain an equivalent amount of energy. This energy may not be of the same form, e.g. a body sliding on a rough horizontal surface will be brought to rest by friction and its kinetic energy will be turned into heat. Heat is thus a form of energy (kinetic energy of the molecules of the bodies) and the equivalent of 1 British Thermal Unit (B.Th.U.) of heat is 778 ft.-lb. of energy. This principle of the constancy of the total amount of energy is known as *the principle of conservation of energy.* In cases where there is no friction, impact, or other energy-dissipating agency, the principle may be put in the following form—

Total energy of system - Work done on system = Constant.

Writing K.E. for kinetic energy, P.E. for potential energy, and W for the work done on the system, we have

$$\text{K.E.} + \text{P.E.} - W = \text{Constant} \qquad \text{(II.26)}$$

If the work done on the system is negative, i.e. if the bodies of the system do work against external resistances, W is negative. If W is zero the energy of the system is constant :

$$\text{K.E.} + \text{P.E.} = \text{Constant} \qquad \text{(II.27)}$$

The value of W in (II.26) is the total work done by all the external forces, with the exception of the weights of the bodies. The following example shows the application of the above principle.

EXAMPLE.

A cylinder, weight W lb., external radius r ft., radius of gyration k ft., rolls without slipping down an inclined plane, length l ft., height h ft. Find the velocity of translation at the bottom of the plane, assuming that the resistance due to friction is negligible.

Let v = velocity at bottom of plane (ft. per sec.), then $\frac{v}{r}$ = angular velocity (radians per sec.).

From (II.27) the gain of K.E. = Loss of P.E.

$$\therefore W \cdot \frac{v^2}{2g} + \frac{W}{2g} \cdot k^2 \cdot \frac{v^2}{r^2} = W \cdot h$$

$$v^2\left(1 + \frac{k^2}{r^2}\right) = 2gh$$

$$\therefore \quad v = \sqrt{\frac{2gh}{1 + \frac{k^2}{r^2}}}$$

If there is a constant resistance to motion = R lb., and v_1 is the final velocity, then, by (II.26),

$$\frac{W \cdot v_1^2}{2g} + \frac{W}{2g} \cdot k^2 \cdot \frac{v_1^2}{r^2} + R \cdot l = W \cdot h$$

$$\therefore \frac{R}{W} = \frac{h - \frac{v_1^2}{2g}\left(1 + \frac{k^2}{r^2}\right)}{l}$$

Since v_1 is twice the average speed down the plane, the value of $\frac{R}{W}$ can be calculated by an observation of the time taken to move a measured distance down the plane, provided k is known. Two such observations, varying h only, will enable the value of k to be determined accurately by experiment, from the equation

$$\frac{k^2}{r^2} = \frac{2g\,(h_1 - h_2)}{v_1{}^2 - v_2{}^2} - 1$$

The final velocity v_1 will be the same as if friction were absent, and the height of the plane reduced by $\frac{R}{W} \,.\, l$.

It should be noticed that the principle of work (Art. 34) is merely a special application of the principle of conservation of energy.

TABLE OF ANALOGIES BETWEEN QUANTITIES, AND FORMULAE, RELATING TO THE LINEAR MOTION OF A PARTICLE AND THE ANGULAR MOTION OF A RIGID BODY

LINEAR MOTION	ANGULAR MOTION
x = distance	θ = angle in radians
t = time	t = time
M = mass = $\frac{W}{g}$	I = moment of inertia = $\sum \frac{w}{g} x^2$
v = velocity = $\frac{dx}{dt}$	ω = angular velocity = $\frac{d\theta}{dt}$
a = acceleration = $\frac{d^2x}{dt^2}$	α = angular acceleration = $\frac{d^2\theta}{dt^2}$
F = force = Ma	C = couple = $I \,.\, \alpha$
Momentum = Mv	Angular momentum = $I \,.\, \omega$
Impulse = Ft	Impulse of couple = Ct
K.E. = kinetic energy = $\frac{1}{2}Mv^2$	K.E. = $\frac{1}{2}I\omega^2$
Work = $F \,.\, s$ or $\int Fds$	Work = $C \,.\, \theta$ or $\int Cd\theta$
Motion with uniform acceleration :—	Motion with uniform angular acceleration :—
$\left\{\begin{array}{l}\text{Initial velocity} = u_0 \\ v = u_0 + at \\ s = u_0 t + \frac{1}{2}at^2 \\ v^2 - u_0{}^2 = 2ax\end{array}\right.$	$\left\{\begin{array}{l}\text{Initial angular velocity} = \omega_0 \\ \omega = \omega_0 + \alpha t \\ \theta = \omega_0 t + \frac{1}{2}\alpha t^2 \\ \omega^2 - \omega_0{}^2 = 2\alpha\theta\end{array}\right.$

39. Centripetal Force on a Rigid Body. If a particle of weight w lb. rotates in a circle of radius r ft. with angular velocity ω radians per second, the centripetal force is $w/g\ \omega^2 r$ lb. directed inwards along the radius. If w/g is the mass of a small portion of a rigid body assumed collected at the point P (Fig. 15), and the body is rotating about the axis whose end view is O, the centripetal force on the small mass is $\frac{w}{g}\ \omega^2\ \overrightarrow{PO}$. Let G be an axis through the centre of gravity; join OG, and draw PM perpendicular to OG produced. The resultant centripetal force on the whole body is given by

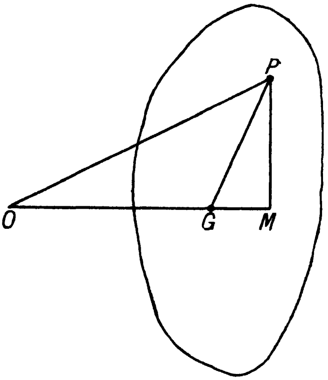


FIG. 15

$$F = \Sigma \frac{w}{g} \omega^2 \overrightarrow{PO}$$

$$\frac{w}{g} \omega^2 (\overrightarrow{PM} + \overrightarrow{MG} + \overrightarrow{GO})$$

and putting $x = \overline{GM}$, $y = \overline{MP}$ and omitting the senses in the first two terms on the right

$$F = \Sigma \frac{w}{g} \omega^2 y + \Sigma \frac{w}{g} \omega^2 x + \Sigma \frac{w}{g} \omega^2 \,.\, \overrightarrow{GO}$$

$$= \frac{\omega^2}{g} \Sigma wy + \frac{\omega^2}{g} \Sigma\, wx + \frac{\omega^2}{g} \overrightarrow{GO}\, \Sigma w$$

But since the centre of gravity G is the origin of the co-ordinates x and y, $\bar{x} = \bar{y} = 0$, and therefore $\Sigma wx = \Sigma wy = 0$ (Art. 26). Also $\Sigma w = W$, the total weight of the body,

$$\therefore F = \frac{W\omega^2}{g} \overrightarrow{GO} \quad . \quad . \quad . \quad . \quad . \quad (II.28)$$

The centripetal force is the same as if the whole mass were collected in a single particle situated at the centre of gravity of the body.

This proof assumes the body to be a plane lamina perpendicular to the axis of rotation. If the body has finite thickness the centripetal forces may also produce a couple in the plane of the axis of rotation.

621.8

40. Acceleration, and Accelerating Forces, in the Simple Engine Mechanism. In Fig. 16 let

r = crank radius (ft.)
$n \cdot r$ = length of connecting rod (ft.)
θ = angle in radians made by crank BC with line of stroke, reckoning from inner dead centre
ω = angular velocity of crank (rads. per sec.)
x = displacement of A from its inner dead centre position (ft.)
P = effective force at crosshead A (lb.)
R = upward thrust of the guide-bar on the crosshead (lb.)
T = turning moment or torque (lb.-ft.) on crank BC due to P

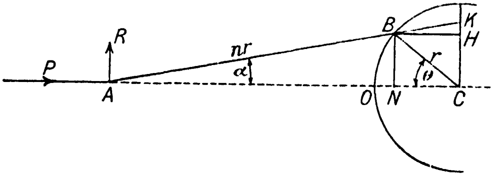


Fig. 16. Simple Engine Mechanism

From the triangle of forces for the point A, it will be seen that $\frac{R}{P} = \tan \alpha$

also $\quad x = nr + r - \overline{AC}$

and $\quad \overline{AC} = \overline{CN} + \overline{AN} = r \cdot \cos \theta + n \cdot r \cos \alpha$

where $\sin \alpha = \frac{\sin \theta}{n}$ and therefore $\cos \alpha = \frac{1}{n}\sqrt{n^2 - \sin^2 \theta}$

$$\therefore x = r\left\{n + 1 - \cos \theta - \sqrt{n^2 - \sin^2 \theta}\right\}$$

The velocity of A is given by

$$v = \frac{dx}{dt} = r\left\{\sin \theta \cdot \frac{d\theta}{dt} + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2 \theta}} \cdot \frac{d\theta}{dt}\right\}$$

$$\therefore v = r \cdot \omega \left\{\sin \theta + \frac{\sin 2\theta}{2n}\right\} \text{ approx.} \qquad \text{(II.29)}$$

The acceleration of A is given by

$$a = \frac{dv}{dt} = r \cdot \omega^2 \left\{\cos \theta + \frac{\cos 2\theta}{n}\right\} \text{ approx.} \qquad \text{(II.30)}$$

(*Note.* The expressions for v and a are approximate only, and ignore some very small terms which may, however, become important with very high speeds of revolution. See Appendix B.)

Let $d\theta$ (radians) be the angle moved through by BC when P moves forward a distance dx (ft.).

Work done by $T = T \, . \, d\theta$ Work done by $P = P \, . \, dx$

$\therefore$ By the principle of work,

$$T \, . \, d\theta = P \, . \, dx$$

$$\therefore T = P \, . \, \frac{dx}{d\theta} = P \frac{dx}{dt} \frac{dt}{d\theta} = \frac{P}{\omega} \frac{dx}{dt}$$

$\therefore$ From (II.29)

$$T = P \, . \, r \left\{ \sin \theta + \frac{\sin 2\theta}{2n} \right\} \text{ approx.} \qquad \text{(II.31)}$$

To find an expression for the torque T (neglecting friction).

The thrust in the connecting rod is $\dfrac{P}{\cos \alpha} = P \sec . \alpha$

The distance from C to the line of the connecting rod is $r \sin \widehat{CBK} = r \sin (\theta + \alpha)$. The turning moment T is the product of these, hence

$$\begin{aligned} T &= P \sec \alpha \, . \, r \sin (\theta + \alpha) \\ &= Pr \frac{\sin \theta \cos \alpha + \cos \theta \sin \alpha}{\cos \alpha} \\ &= P \, (r \sin \theta + r \cos \theta \tan \alpha) \end{aligned}$$

Let AB be produced to meet the vertical through C in K, and draw BH parallel to AC. Then

$$r \sin \theta = \overline{BN} = \overline{CH}$$

and

$$\begin{aligned} r \cos \theta \tan \alpha &= \overline{CN} \tan \alpha \\ &= \overline{BH} \tan \alpha \\ &= \overline{HK} \end{aligned}$$

and from the above,

$$T = P \, (\overline{CH} + \overline{HK})$$

i.e.

$$T = P \, . \, \overline{CK} \qquad \text{(II.32)}$$

Let W = weight of reciprocating parts at A (lb.).

Then by (II.30), since force = mass × acceleration, the inertia force F at A is given approximately by

$$F = \frac{W}{g} \cdot r \cdot \omega^2 \left\{ \cos\theta + \frac{\cos 2\theta}{n} \right\} \quad . \quad \text{(II.33)}$$

F must be *deducted* from the net steam or gas load on the piston in order to obtain the *effective* force P at the crosshead A, since

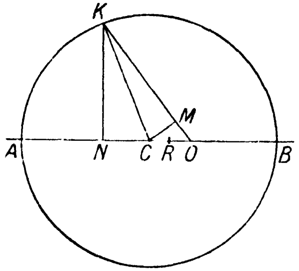


FIG. 17

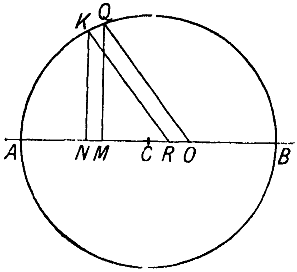


FIG. 18

HARRISON'S CONSTRUCTION

$$\text{Net load on piston} - P = \text{accelerating force } F$$

$$\therefore \text{Net load on piston} - F = P$$

The following approximate constructions for the velocity and acceleration of a crosshead are due to Professor Harrison—

Let C be the centre of a circle of radius r (Fig. 17). Choose a point O on the diameter AB such that $\overline{CO} = c$, where c is small, and draw OK making an angle θ with AO. Draw CM perpendicular to OK. Since the angle CKM is small $\overline{KM} = \overline{KC} = r$ (approx.).

$$\therefore \overline{KO} = r + c \cdot \cos\theta \text{ (approx.)}$$

$$\therefore \overline{KN} = \overline{KO} \cdot \sin\theta = r \cdot \sin\theta + \frac{c}{2} \cdot \sin 2\theta \quad . \quad \text{(II.34)}$$

$$\text{Also } \overline{ON} = \overline{KO} \cdot \cos\theta = r \cdot \cos\theta + c \cdot \cos^2\theta$$

$$= r \cdot \cos\theta + \frac{c}{2} + \frac{c}{2} \cdot \cos 2\theta$$

$$\therefore \overline{ON} - \frac{c}{2} = r \cdot \cos\theta + \frac{c}{2} \cdot \cos 2\theta \quad . \quad \text{(II.35)}$$

If R is the mid-point of OC, then $\overline{RN} = \overline{ON} - \frac{c}{2}$

$$\therefore RN = r \cdot \cos\theta + \frac{c}{2} \cdot \cos 2\theta \qquad . \quad \text{(II.36)}$$

$$\omega \cdot \overline{KN} \text{ corresponds to } v, \text{ if } c = \frac{r}{n} \quad . \quad . \quad . \quad \text{(II.37)}$$

$$\omega^2 \cdot \overline{RN} \text{ corresponds to } a, \text{ if } c = \frac{2r}{n} \quad . \quad . \quad . \quad \text{(II.38)}$$

(*Note.* The angle CKM is least when the acceleration is greatest.)

The construction will then be as follows (Fig. 18)—

Describe a circle with centre C and radius CA representing the crank radius to scale. Make $\overline{CR} = \overline{RO} = \frac{r}{n}$. Draw KR, making an angle equal to the crank angle θ with AB, and draw OQ parallel to KR. Draw the perpendiculars KN, QM.

Then (1) velocity of crosshead $= \omega \, . \overline{KN}$
(2) acceleration of crosshead $= \omega^2 \, . \overline{RM}$
(3) torque due to steam or gas pressure $= P \times \overline{KN}$
(4) inertia torque $= \frac{W}{g} \cdot \omega^2 \cdot \overline{RM} \cdot \overline{KN}$ (see Art. 41)
(5) net torque = (3) – (4)

If, in Fig. 17, c is made equal to $\frac{r}{2n}$, then

$$AN = r(1 - \cos\theta) + \frac{r}{4n}(1 - \cos 2\theta)$$

= displacement of piston from its inner dead centre position.

41. **Inertia Torque.** Inertia torque may be defined as the torque required to preserve uniform angular velocity of a crankshaft to which reciprocating parts are connected, assuming no external resistance to rotation. This torque is regarded as positive when acting in the direction of rotation, and must be subtracted from the torque due to steam or gas pressure in order to obtain the effective torque at any instant.

Obviously the average inertia torque during a complete revolution of the crank is zero, so that in calculation of horsepower (which depends upon average torque only) it is not necessary to take this into account. Its effect at high speeds of revolution is to modify (*a*) the shape of the diagram of net torque, (*b*) the maximum net torque, (*c*) the ratio of maximum to mean torque, (*d*) the tendency to torsional oscillations of the crankshaft.

If W = weight of reciprocating parts (lb.)
r = crank radius (ft.)
ω = angular velocity of crank (rads. per sec.)
T = inertia torque (lb.-ft.)

then, from (II.31) and (II.33),

$$\begin{aligned}T &= \frac{W}{g} \,.\, r^2 \,.\, \omega^2 \left(\sin\theta + \frac{\sin 2\theta}{2n}\right)\left(\cos\theta + \frac{\cos 2\theta}{n}\right)\\ &= \frac{W}{g} \,.\, r^2 \,.\, \omega^2 \Bigg(\sin\theta \,.\, \cos\theta + \\ &\qquad \frac{\sin 2\theta \,.\, \cos\theta + 2\cos 2\theta .\, \sin\theta}{2n} + \frac{\sin 2\theta \,.\, \cos 2\theta}{2n^2}\Bigg)\\ &= \frac{W}{4g} \,.\, r^2 \,.\, \omega^2 \Bigg(2\sin 2\theta + \frac{3\sin 3\theta - \sin\theta}{n}\\ &\qquad + \frac{\sin 4\theta}{n^2}\Bigg) \text{ approx} \quad . \quad . \quad . \quad . \quad . \qquad \text{(II.39)}\end{aligned}$$

This expression is of use when it is desired to obtain the net variation of inertia torque due to a given arrangement of cranks. The following are expressions for the net inertia torque (derived from the above expression) for three typical arrangements of cranks, the proofs of these being left to the reader.

(1) Two cranks at 90°,

$$T = \frac{W\omega^2 r}{4g}\left[\frac{3(\sin 3\theta - \cos 3\theta)}{n} - \frac{\sin\theta + \cos\theta}{n} + \frac{2\sin 4\theta}{n^2}\right]$$

(2) Two cranks at 180°,

$$T = \frac{W}{4g} \,.\, r^2 \,.\, \omega^2 \left(4\sin 2\theta + \frac{2\sin 4\theta}{n^2}\right)$$

(3) Three cranks at 120°,

$$T = \frac{W}{4g} \,.\, r^2 \,.\, \omega^2 \times \frac{9}{n}\sin 3\theta$$

In considering the balancing of the connecting rod (see Art. 193), the usual method is to replace the connecting rod by two masses m_1 and m_2 at A and B (Fig. 19) respectively, such that

$m_1 + m_2 = M$, i.e. the total mass is the same

$m_1 q = m_2 (1-q)$, i.e. the masses have the same centre of gravity as the rod, where M is the mass of

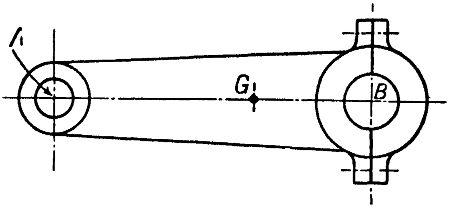


FIG. 19. DYNAMICALLY EQUIVALENT MASSES

the rod and $q = \frac{\overline{AG}}{\overline{AB}}$, G being the centre of mass of the rod. For complete dynamical equivalence the moments of inertia about G should be the same with the particles as with the rod, i.e.

$$m_1 . (\overline{AG})^2 + m_2 (\overline{BG})^2 = M . k^2 \text{ (same I)}$$

These three conditions can only be satisfied simultaneously if $\frac{k^2}{(\overline{AB})^2} = q(1-q)$, which is not possible unless the rod is extended beyond B. When the three conditions are satisfied the two masses are known as an *equivalent dynamical system.* (See Art. 226.)

42. Angular Velocity and Acceleration of Connecting Rod.

Angular velocity of AB (Fig. 16) $= \frac{d\alpha}{dt}$

By trigonometry $\sin\alpha = \frac{\sin\theta}{n}$; hence $\cos\alpha = \frac{1}{n}\sqrt{n^2 - \sin^2\theta}$

$$\therefore \cos\alpha . \frac{d\alpha}{dt} = \frac{\cos\theta}{n} . \frac{d\theta}{dt}$$

$$\text{i.e. } \frac{d\alpha}{dt} = \frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}} . \frac{d\theta}{dt} = \omega . \frac{\cos\theta}{n\left(1 - \frac{\sin^2\theta}{2n^2}\right)} \text{ approx.}$$

$$= \omega . \frac{\cos\theta}{n} . \left(1 + \frac{\sin^2\theta}{2n^2}\right) \text{approx.}$$

$$= \omega . \frac{\cos\theta}{n} . \left(1 + \frac{1 - \cos 2\theta}{4n^2}\right) \text{approx.} \quad \text{(II.40)}$$

Taking the exact value of $\frac{d\alpha}{dt}$, and differentiating, then angular acceleration of AB is given by

$$\frac{d^2\alpha}{dt^2} = \frac{-(n^2 - \sin^2\theta) \,.\, \sin\theta + \cos^2\theta \,.\, \sin\theta}{(n^2 - \sin^2\theta)^{\frac{3}{2}}} \,.\, \omega^2$$

$$= \frac{-n^2 \,.\, \sin\theta + \sin\theta\,(\sin^2\theta + \cos^2\theta)}{(n^2 - \sin^2\theta)^{\frac{3}{2}}} \,.\, \omega^2$$

$$= -\frac{(n^2 - 1)\sin\theta}{n^3\left(1 - \frac{\sin^2\theta}{n^2}\right)^{\frac{3}{2}}} \,.\, \omega^2$$

$$= -\omega^2 \,.\, \frac{n^2 - 1}{n^3} \,.\, \sin\theta \,.\, \left(1 + \frac{3}{2} \,.\, \frac{\sin^2\theta}{n^2}\right) \text{ approx.}$$

$$= -\omega^2 \,.\, \frac{n^2 - 1}{n^3} \,.\, \sin\theta \left\{1 + \frac{3}{4n^2}(1 - \cos 2\theta)\right\} \text{ approx.} \quad . \quad \text{(II.41)}$$

For values of n of 4 or over, close approximations for angular velocity and acceleration of AB are respectively

$$\frac{d\alpha}{dt} = \frac{\cos\theta}{n} \,.\, \omega. \quad . \quad . \quad . \quad . \quad . \quad \text{(II.42)}$$

$$\frac{d^2\alpha}{dt^2} = -\frac{n^2 - 1}{n^3} \,.\, \sin\theta \,.\, \omega^2 \quad . \quad . \quad . \quad \text{(II.43)}$$

Except in the case of high-speed engines, the error involved in the approximations is usually less than the error involved in the assumption that ω is constant throughout the revolution of the crank.

43. **Construction of Crank Effort Diagram and Torque Diagram.** The crank effort E, for a given position of the crank, is the effective force at right-angles to the crank due to the thrust of the connecting rod. The torque due to this is given by

$$T = E \,.\, r$$

If the effective force P at the crosshead is known, and the lengths of crank and connecting rod are given, the value of T may be found from the formula (II.31), or more conveniently from Fig. 18.

To obtain P we require to have an indicator diagram, such as is shown in Fig. 20, and to know the area of the piston.

Draw 06 equal and parallel to the zero pressure line AB.

On 06 describe a semicircle, centre c. Set out $cO' = \frac{r}{2n}$ (see Art. 40), and draw 0′1, 2′0, etc., at convenient angular distances apart (30° in the figure). From 1 project a line 1$\overline{nab}$, cutting AB at n. Then $\frac{\overline{An}}{\overline{AB}}$ represents the fraction of the stroke completed by the piston when the crank is in the position 0′1. At this point, the forward pressure on the piston (lb. sq. in.) is given by $\overline{nb}$, the point b being on the left-hand diagram. The back pressure is given by $\overline{na}$, the point a being on the right-hand diagram. The net forward pressure in lb. per sq. in. is thus given by $\overline{ab}$. For position 0′5 of the the net forward pressure is given by $\overline{cd}$.

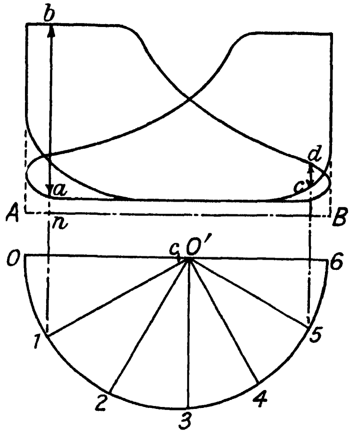


FIG. 20

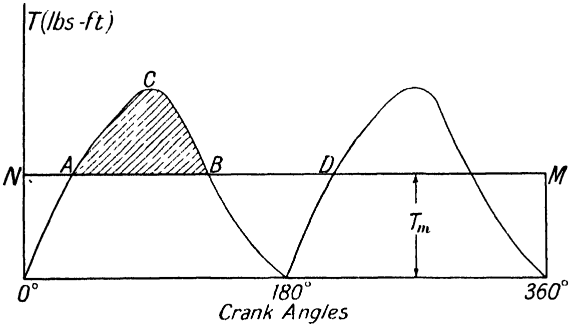


FIG. 21. TURNING MOMENT DIAGRAM

Quantities may now be tabulated, as follows—

Crank Angle	Net pressure (p) (lb. sq. in.)	P $= p \times$ area of piston in sq. in.	$\overline{KN}$ (ft.) (Fig. 18)	Torque $= P \times \overline{KN}$
30°				
60°				
⋮				
360°				

A diagram (Fig. 21) may now be constructed, showing the variation of torque during one complete revolution of the crank, the mean torque T_m being represented by NM. If the mean effective pressure (p_e) is obtained from the indicator diagram, d is the diameter of piston in inches, and S is the stroke in feet, T_m may be checked from the equation

$$2\pi \, . \, T_m = p_e \, . \, \frac{\pi}{4} d^2 \, . \, S \times 2$$

$$\therefore T_m = \frac{p_e \, . \, d^2 \, . \, S}{4} \quad . \quad . \quad . \quad . \quad \text{(II.44)}$$

If more accurate values of the torque are required, the quantity $\overline{KN}$ in the last column should be replaced by $\overline{CK}$ (Fig. 16). (This entails drawing the mechanism of Fig. 16 in a series of positions and measuring $\overline{CK}$ from each.) If inertia is taken into account, values of T given by (II.39) should be deducted from the above values of the torque.

44. **Flywheels.** Referring to Fig. 21, if the resisting torque (assumed to be constant) is T_m, the average speed (revs. per min.) of the engine will be constant, but the speed will fluctuate during the revolution according to the equation

Moment of inertia × angular acceleration

= couple acting (Art. 30)

or $$I \, . \, \frac{d^2\theta}{dt^2} = T - T_m$$

where T is the instantaneous torque, and I the moment of inertia of the rotating parts. The angular velocity will increase from A to B and will decrease from B to D.

Let ω_1 = minimum angular velocity (at A)

ω_2 = maximum angular velocity (at B)

ω_o = mean angular velocity (from revs. per min.)

If $\frac{\omega_2 - \omega_1}{\omega_o}$ is small, then ω_o is given very closely by

$$\omega_o = \frac{\omega_2 + \omega_1}{2}$$

Energy stored by rotating parts from A to B

$$= \text{shaded area } ACB \text{ (Fig. 21)}$$
$$= E \text{ ft.-lb.}$$
$$= \tfrac{1}{2}I\,(\omega_2{}^2 - \omega_1{}^2) \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(II.45)}$$

If 1 in. on vertical scale of diagram $= x$ lb.-ft., and 1 in. on horizontal scale $= y$ radians, then

$$E \text{ (ft.-lb.)} = \text{area } ACB \text{ (sq. in.)} \times x\,.\,y \quad . \quad . \qquad \text{(II.46)}$$

From (II.45) $$(\omega_2 - \omega_1\,(\omega_2 + \omega_1) = \frac{2E}{I}$$

$$\therefore (\omega_2 - \omega_1) \times 2\omega_o = \frac{2E}{I}$$

$$\therefore \omega_2 - \omega_1 = \frac{E}{I\,.\,\omega_o} \quad . \qquad \text{(II.47)}$$

$\therefore$ Fractional change of speed above and below mean

$$n = \pm\frac{\omega_2 - \omega_1}{2\omega_o} = \pm\frac{E}{2I\,.\,\omega_o{}^2} \quad . \quad . \quad . \qquad \text{(II.48)}$$

Thus, for given values of E and I, the fractional change of speed is inversely proportional to the square of the mean speed of revolution.

If N is the mean speed of the engine in revs. per min., then

$$\omega_o{}^2 = \frac{4\pi^2N^2}{3600} = \frac{N^2}{91{\cdot}4}$$

$$\therefore n = \pm\frac{45{\cdot}7E}{I\,.\,N^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(II.49)}$$

If $P =$ horse-power of engine, then work done per revolution $= \dfrac{33{,}000\,P}{N}$

If $$f = \frac{E}{\text{work done per rev.}}, \text{ then}$$

$$n = \pm\frac{45{\cdot}7f}{I\,.\,N^2}\,.\,\frac{33{,}000\,P}{N}$$

$$= \pm 1{\cdot}5 \times 10^6 \times \frac{f\,.\,P}{I\,.\,N^3} \quad . \quad . \quad . \qquad \text{(II.50)}$$

f is usually known as the *coefficient of fluctuation of energy*, and c $(= 2n)$ is the coefficient of fluctuation of speed. The value of c is usually specified according to the nature of the drive, and the value of f depends upon the shape of the indicator diagram and the number and sequence of cranks.

It will be obvious that in the case of slow-speed engines, where ω may vary 1 or 2 per cent from the mean value, it would be illogical to use very close approximations in the expressions for piston velocity and acceleration, etc. In the case of high-speed engines, however, variations of angular velocity are extremely small.

EXAMPLE.

If an engine, rotating at N revs. per min., is driving a vehicle, weighing W tons, at V miles per hour, then if I is the moment of inertia of the rotating parts of the engine, and assuming that the energy of rotation of the wheels is negligible, the kinetic energy stored is

$$\begin{aligned} K &= \tfrac{1}{2} I \cdot \frac{N^2}{91{\cdot}4} + \frac{2240\,W}{2g} \cdot \left(\frac{88V}{60}\right)^2 \\ &= \frac{I \cdot N^2}{183} + 75W \cdot V^2 \\ &= \frac{N^2}{183}\left\{I + 13{,}740\,W \cdot \left(\frac{V}{N}\right)^2\right\} \quad . \quad . \quad \text{(II.51)} \end{aligned}$$

If $G = \dfrac{\text{r.p.m. of engine}}{\text{r.p.m. of wheels}}$

D = diameter of driving wheels in inches

then, $$V = \frac{N}{G} \cdot \frac{3{\cdot}14D}{12} \cdot \frac{60}{5280} = \frac{N \cdot D}{336\,G} \quad . \quad . \quad \text{(II.52)}$$

Substituting in (II.51) we obtain,

$$K = \frac{N^2}{183}\left\{I + \frac{W}{8{\cdot}24} \cdot \left(\frac{D}{G}\right)^2\right\} \quad . \quad . \quad . \quad \text{(II.53)}$$

The effect of variations of engine torque will thus be the same as if the vehicle had no mass and the I of the rotating parts of the engine were increased by $\dfrac{W}{8{\cdot}24} \cdot \left(\dfrac{D}{G}\right)^2$.

EXAMPLE.

Weight of vehicle, $1\tfrac{1}{4}$ tons ; $D = 32$ in. ; $G = 4$; weight of rotating parts, 300 lb. ; radius of gyration 8 in.

$$I = \frac{300}{32{\cdot}2} \times \left(\frac{2}{3}\right)^2 = 4{\cdot}14$$

$$\frac{W}{8{\cdot}24}\left(\frac{D}{G}\right)^2 = \frac{1{\cdot}25 \times 64}{8{\cdot}24} = 9{\cdot}7$$

$\therefore$ Equivalent $I = 13{\cdot}8$ engineers' units

EXAMPLE.

The flywheel of a steam engine has a mass of 50 tons, and its radius of gyration is 12 ft. The other moving parts of the engine are comparatively light.

When the engine is running unloaded at 52 revs. per min., the steam is shut off suddenly, and the engine comes to rest in 4 min. 10 sec.

Assume that the frictional resistances are constant during this period, and find the horse-power required to drive the engine when running unloaded at 52 revs. per min. (U.L.)

E = energy stored in flywheel running at 52 r.p.m.

$$= \tfrac{1}{2}I\omega^2$$

$$= \tfrac{1}{2}\,\frac{50 \times 2240}{32{\cdot}2} \times 12^2 \times \left(\frac{52}{60} \times 2\pi\right)^2$$

$$= 7{,}420{,}000 \text{ ft.-lb.}$$

N = number of revs. made in coming to rest

= mean speed × time

$$= \frac{52}{2} \times \frac{250}{60} = \frac{325}{3}$$

Energy absorbed by friction per rev.

$$= \frac{E}{N} = \frac{7{,}420{,}000 \times 3}{325}$$

$$= 68{,}600 \text{ ft.-lb.}$$

Energy absorbed per min. at 52 r.p.m.

$$= 68{,}600 \times 52$$

$$= 3{,}570{,}000 \text{ ft.-lb.}$$

and, finally,

P = horse-power required to drive the engine when running unloaded at 52 r.p.m.

$$= \frac{3{,}570{,}000}{33{,}000} = 108{\cdot}2$$

Horse-power required = 108·2.

EXAMPLE.

The crank effort diagram for an engine is given by the equation T lb·ft. = 7850 sin θ + 1500 sin 2θ, where θ is the crank displacement. The crank

has a mean speed of 100 revs. per min., and the resisting torque is constant, Find—

(1) The power of the engine.

(2) The weight of a flywheel rim having a mean radius of 4 ft. so that the total fluctuation of speed shall not be more than 2 per cent.

(3) The angular acceleration of the flywheel when the crank is on the dead centre, and also when the crank has turned through 30° from the dead centre. (U.L.A.)

(1) The work done per stroke is $\int_0^\pi T d\theta$

$$= \int_0^\pi (7850 \sin \theta + 1500 \sin 2\theta)\, d\theta$$

$$= \Big[- 7850 \cos \theta - 750 \cos 2\theta \Big]_0^\pi$$

$= 15{,}700$ ft.-lb. Hence, if the engine is double-acting

$$\text{Horse-power} = \frac{15{,}700 \times 200}{33{,}000} = 95{\cdot}15.$$

$$\text{The resisting torque is } \frac{15{,}700}{\pi} = 4996 \text{ ft.-lb.}$$

(2) The turning moment is equal to the resisting torque when

$$7850 \sin \theta + 1500 \sin 2\theta = 4996.$$

To solve this equation it is necessary to draw the turning moment diagram, Fig. 22. This is drawn by tabulating values of θ, $\sin \theta$, and $\sin 2\theta$, as below, and then plotting the results.

(1)	θ = angle in degrees	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
(2)	$\sin \theta$ . .	0	·2588	5000	·7071	·8660	·9659	1	·9659	·8660	·7071	·5000	·2588	0
(3)	$\sin 2\theta$. .	0	·5000	·8660	1·0000	·8660	·5000	0	–·5000	–·8660	–1	–·8660	–·5000	0
(4)	$7850 \sin \theta$.	0	2032	3925	5551	6798	7582	7850	7582	6798	5551	3925	2032	0
(5)	$1500 \sin 2\theta$	0	750	1299	1500	1299	750	0	–750	–1299	–1500	–1299	–750	0
(6)	T = (4) + (5)	0	2782	5224	7051	8097	8332	7850	6832	5499	4051	2626	1202	0

The above calculations could have been made easier by dividing both sides of the equation by 7,850. If this were done, however, the scale readings in the resulting graph would not be actual turning moments as in Fig. 22.

From the graph, we see that $T = T_m$ when $\theta = {\cdot}4974$ radians, and also when $\theta = 2{\cdot}182$ radians. Thus the excess energy to be stored up by the flywheel is given by

$$
\begin{aligned}
E &= \int_{\cdot 4974}^{2\cdot 182} (7850 \sin\theta + 1500 \sin 2\theta - 4996)\, d\theta \\
&= \Big[-7850 \cos\theta - 750 \cos 2\theta - 4966\theta \Big]_{\cdot 4974}^{2\cdot 182} \\
&= 7850 \ \cos .\, 4974 + 750 \ \cos .\, 9948 - 7850 \ \cos \\
&\qquad 2{\cdot}182 - 750 \cos 4{\cdot}364 - 4966\, (2{\cdot}182 - {\cdot}4974) \\
&= 7850 \times {\cdot}8787 + 750 \times {\cdot}5446 + 7850 \times {\cdot}5736 \\
&\qquad + 750 \times {\cdot}3420 - 4966 \times 1{\cdot}685 \\
&= 3697 \text{ ft.-lb.}
\end{aligned}
$$

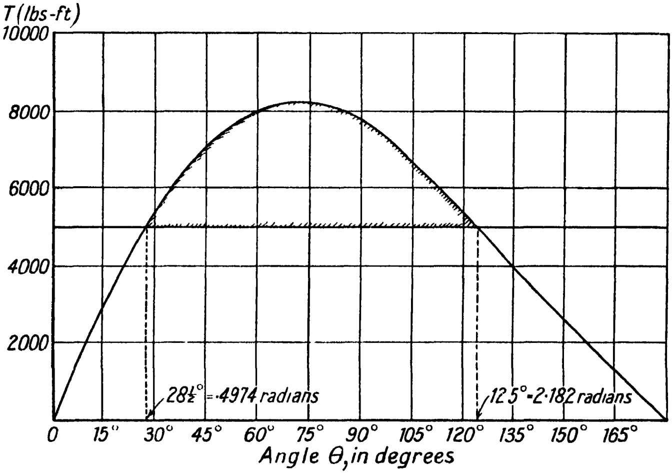


FIG. 22. TURNING MOMENT DIAGRAM

The percentage fluctuation of energy of the engine is

$$\frac{3697}{15{,}700 \times 2} \times 100.$$

If ω_1 and ω_2 are the highest and lowest speeds respectively. and I the moment of inertia of the flywheel, then

$$\tfrac{1}{2}I\,(\omega_1{}^2 - \omega_2{}^2) = 3697$$

or

$$\tfrac{1}{2}I\,(\omega_1 + \omega_2)\,(\omega_1 - \omega_2) = 3697$$

$$\therefore\ I\,\frac{\omega_1 + \omega_2}{2}\,(\omega_1 - \omega_2) = 3697$$

But $\frac{\omega_1 + \omega_2}{2}$ = mean angular velocity

$$= \frac{100 \times 2\pi}{60} = \frac{10\pi}{3} \text{ radians per sec.}$$

And $\omega_1 - \omega_2 = \frac{2}{100} \times$ mean angular velocity

$$= \frac{\pi}{15} \text{ radians per sec.}$$

$$\therefore \frac{10\pi}{3} \times \frac{\pi}{15} I = 3697$$

or
$$I = \frac{3697 \times 9}{2\pi^2}$$

$$= 1681 \text{ engineers' units.}$$

But
$$I = \frac{W}{g} k^2$$

$$\therefore 1681 = \frac{16W}{g}$$

or
$$W = \frac{1681g}{16}$$

= 3382 lb., which is the weight of the flywheel rim.

(3) Couple = I × angular acceleration

$$\therefore T - T_m = I \times \frac{d^2\theta}{dt^2}$$

i.e.
$$\frac{d^2\theta}{dt^2} = \frac{1}{I}(T - T_m)$$

$$= \frac{T - T_m}{1681}$$

When $\theta = 0$, $T - T_m = -4996$ lb.-ft., and the angular acceleration is $\frac{d^2\theta}{dt^2} = -\frac{4996}{1681} = -2{\cdot}97$ radians per sec. per sec.

When $\theta = 30°$, $T - T_m = 7850 \sin 30° + 1500 \sin 60° - 4996$.

$\therefore T - T_m = 5224 - 4996$ from the table $= 228$ lb.-ft., and the angular acceleration is $\frac{228}{1681} = {\cdot}136$ radians per sec. per sec.

The answers are—

(1) Horse-power = 95·2.

(2) Weight of rim = 3382 lb.

(3) (*a*) Acceleration at dead centre = - 2·97 radians per sec. per sec.

(*b*) Acceleration at 30° crank angle = ·136 radians per sec. per sec.

The area under the turning moment diagram from 0 to π, and that enclosed by the graph of T and the line $T = T_m$, could have been found by any of the ordinary methods for finding areas under curves. Generally such a method is necessary, as the equation to the turning moment curve is not known and direct integration is impossible.

EXAMPLE.

The maximum load raised and lowered by a winding drum 5 ft. diameter is 5000 lb. Attached to the drum is a brake drum, the combined weights of the drums being 2000 lb., and their radius of gyration 4 ft. The maximum speed of descent is 20 ft. per sec., and when descending at this speed the brake must be capable of stopping the load in 20 ft. Find

(*a*) The tension of the rope during stopping at the above rate.

(*b*) The friction torque necessary at the brake, neglecting the inertia of the rope.

In a descent of 100 ft. the load starts from rest and falls freely until its speed is 20 ft. per sec. The brake is then applied, and the speed kept constant at 20 ft. per sec. until the load is 30 ft. from the bottom. The brake is then tightened so as to give uniform retardation, and the load is brought to rest at the bottom. Find

(*c*) The total time of descent.

(*a*) Average speed of load during stopping = 10 ft. per sec. The time of stopping $= \frac{20}{10} = 2$ sec. Also, retardation $= \frac{20}{2}$ = 10 ft. per sec. per sec.

Let T lb. be the tension in the rope. Then

$$\text{Force} = \text{mass} \times \text{acceleration}$$

$$T - 5000 = \frac{5000}{g} \times 10$$

$$\therefore T = 5000\left(1 + \frac{10}{32\cdot2}\right)$$

$$= 6550$$

that is, the tension is 6550 lb.

(*b*) Moment of inertia of the drums

$$= \frac{2000}{32 \cdot 2} \times 16 = 994 \text{ engineers' units}$$

Angular retardation of drum

$$= \frac{10}{2 \cdot 5} = 4 \text{ radians per sec. per sec.}$$

Torque to retard drum

$$= 994 \times 4 = 3980 \text{ lb.-ft.}$$

Torque due to T $= 2 \cdot 5 \times 6550 = 16{,}380$ ft.-lb.

Total friction torque necessary

$$= 20{,}360 \text{ lb.-ft.}$$

(*c*) Let ω = angular speed of drum, corresponding to a linear speed of the load of 20 ft. per sec. Then $2 \cdot 5\omega = 20$, from which $\omega = 8$ radians per sec. Let h be the distance in feet fallen by the load up to the instant of maximum speed. Then by the principle of the conservation of energy

$$5000h = \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}\frac{W}{g}v^2$$

i.e.

$$5000h = \tfrac{1}{2} \times 994 \times 64 + \frac{5000 \times 400}{64 \cdot 4}$$
$$= 31{,}800 + 31{,}070$$
$$= 62{,}870$$

and

$$h = 12 \cdot 57 \text{ ft.}$$

The average speed during this fall is $\frac{20}{2} = 10$ ft. per sec., and the time to fall h ft. is $\frac{12 \cdot 57}{10} = 1 \cdot 257$ sec.

Distance covered at uniform speed $= 100 - 12 \cdot 57 - 30$
$= 57 \cdot 43$ ft.

Time moving at uniform speed $= \frac{57 \cdot 43}{20} = 2 \cdot 871$ sec.

also average speed in slowing down $= 10$ ft. per sec.

$\therefore$ Time taken in slowing down $= \frac{30}{10} = 3$ sec.

And the total time of descent is $1 \cdot 257 + 2 \cdot 871 + 3 = 7 \cdot 13$ sec.

EXAMPLE

In a double acting engine having two cylinders in the same plane and cranks at right angles, the line of mean resistance cuts off from the turning effort diagram eight areas of fluctuation of energy above and below the mean. In a particular engine it is found that these areas, taken in order, represent: 20·5; 35·9; −9·1; 21·1; −19·7; 8·4; −33·1; 17·1 ft.-tons. Make a rough sketch of the turning-effort diagram and mark on it the positions of the several maximum and minimum crank velocities, indicating which is the greatest and which the least. Also find the weight of a flywheel (radius of gyration 3 ft.) which will restrict the variation in speed to within 1 per cent of a mean speed of 300 r.p.m. (L.U.)

The effort diagram is sketched in Fig. 22A, the horizontal line represents the mean turning-effort and the wavy line represents the actual turning effort on the shaft. *A*, *B*, *C*, *D*, etc., are the points of intersection of the two diagrams. Let *E* be the energy in the flywheel in ft.-tons at the instant corresponding to *A*. At *B* the energy will be *E* − 20·5, that at *C* will be *E* − 20·5 + 35·9, etc. The table gives the energy at each of the points *A*, *B*, *C*, etc.—

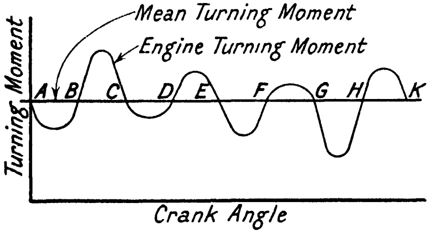


FIG. 22A

Point .	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*K*
Energy stored in fly-wheel	*E*	*E* − 20·5	*E* + 15·4	*E* + 6·3	*E* + 27·4	*E* + 7·7	*E* + 16·1	*E* − 17·0	*E* + ·1

Minimum speeds occur at the points *B*, *D*, *F* and *H*, and maximum speeds occur at the other points. By inspection of the table we see that the smallest minimum occurs at *B* and the largest maximum at *E*. The greatest fluctuation of energy is therefore $(E + 27\cdot4) - (E - 20\cdot5) = 47\cdot9$ ft.-tons.

Let I be the moment of inertia in engineers' units and let ω_1 and ω_2 be the greatest and least speeds in radians per sec. Then

$$\omega_1 + \omega_2 = 2 \times \text{mean speed in radians per sec.}$$

$$= 2 \times \frac{300}{60} \times 2\pi = 20\pi$$

and since there is a total change of speed of 2 per cent

$$\omega_1 - \omega_2 = \frac{2}{100} \times \frac{\omega_1 + \omega_2}{2} = \frac{\pi}{5}$$

Note. The reader should notice that the last paragraph of the question (which is taken from an examination paper) is ambiguous and might be taken to mean a total variation of only 1 per cent of the mean speed. Any solution should state clearly which interpretation has been accepted.

Now change of energy of flywheel

$$= \tfrac{1}{2}I(\omega_1^2 - \omega_2^2)$$

$$= \tfrac{1}{2}I(\omega_1 + \omega_2)(\omega_1 - \omega_2) \text{ ft.-lb.}$$

and substituting

$$47{\cdot}9 \times 2240 = \tfrac{1}{2}I \times 20\pi \times \frac{\pi}{5}$$

or,

$$I = \frac{47{\cdot}9 \times 2240}{2\pi^2} \text{ engineers' units}$$

but

$$I = \frac{W}{g}k^2$$

$$\therefore \quad W = \frac{32{\cdot}2}{9} \times \frac{47{\cdot}9 \times 2240}{2\pi^2} \text{ lb.}$$

$$= \frac{32{\cdot}2 \times 47{\cdot}9}{18\pi^2}$$

$$= 8{\cdot}68 \text{ tons} = \text{weight of flywheel}$$

EXAMPLES 2.

1. A thin uniform rod 7 ft. long, and weighing 21 lb., rotates at 60 revs. per min. in a horizontal plane about one end. Find the centripetal force acting on any cross section of the rod, and find the tension in the rod caused by the rotation at points distant 2 ft., 4 ft., and 6 ft. respectively from the centre of rotation. (I.C.E.)

2. Find in direction and magnitude the force required to compel a body weighing 10 lb. to move in a curved path, the radius of curvature at the point considered being 20 ft., the velocity of the body 40 ft. per sec., and the acceleration in its path 48 ft. per sec. per sec.

3. A tramcar weighing 8 tons is moving round a curve of 50 ft. radius at 8 miles per hour, the plane of the rails being horizontal and the gauge 3 ft. 6 in. If the surface of the car is taken as equivalent to a rectangle 15 ft. high and 20 ft. long, and its centre of gravity is 5 ft. above the rails, find the least normal wind pressure which would overturn the car. (I.C E.)

4. A bullet of 1 oz. weight impinges directly upon a board 4 in. thick with a velocity of 1,200 ft. per sec. It emerges from the board with a velocity of 100 ft. per sec. Assuming constant retardation, find (1) the average

force exerted by the bullet, (2) its horse-power during penetration, (3) kinetic energy on entering, (4) kinetic energy on leaving, (5) work done during penetration.

5. Find the force acting on the piston-rod at each end of the stroke in a double-acting horizontal steam engine when running under the following conditions—

Weight of piston and rod	275 lb.
Radius of crank	15 in.
Length of connecting rod	60 in.
Revs. per min.	150
Steam pressure—constant throughout stroke	100 lb. sq. in.
Back ,, ,, ,, ,,	3 lb. sq. in.
Diameter of cylinder	16 in.

(I.C.E.)

6. Find the horse-power developed by the engine in example (5). Find the average turning moment on the crank-shaft, and the actual turning moment on the crank-shaft when the crank and connecting-rod are at right-angles. (Neglect all frictional resistances.)

7. Show that when a body is moving about a fixed axis with angular acceleration a under the influence of a turning moment T, then $T = I\alpha$ where I is the moment of inertia of the body with reference to the axis of rotation.

The rotor of a hydraulic turbine weighs 25 tons, and has a radius of gyration of 5 ft. When running at 200 revs. per min. the rotor is suddenly relieved of part of its load, so that its speed rises to 205 revs. per min. in one second. Find the unbalanced turning moment exerted, assuming this uniform throughout the change of speed. (U.L.)

8. Define "fluctuation of energy" and the "coefficient of fluctuation of energy" of an engine. An engine of 50 h.p. runs at 100 r.p.m. The energy to be stored in the flywheel between maximum and minimum speeds is 10 per cent of the work done per revolution of the engine. Find the moment of inertia of the flywheel required if the coefficient of fluctuation of speed is 0·02.

[*Note.* In each case the coefficient is the ratio $\frac{\text{variation of quantity}}{\text{quantity}}$]

9. In a simple engine mechanism the length of the crank is 6 in., and the connecting-rod is 2 ft. long. If the crank is moving uniformly at 60 revs. per min., find the angular velocity and angular acceleration of the connecting-rod when the crank has moved through the following angles from its inner dead-centre position, (1) 0°, (2) 60°, (3) 90°, and (4) 180°.

10. A machine punching 1½ in. holes in a 1¼ in. plate does 15 in.-tons of work per sq. in. of sheared area. The punch has a stroke of 4 in. and punches one hole every 10 sec. The minimum speed of the flywheel at the radius of gyration is 90 ft. per sec. Find the weight of the wheel if the speed at this radius is not to fall below 80 ft. per sec. during each punch. (U.L.)

11. The fluctuation of energy of a double-acting engine of 100 h.p. when making 100 revs. per min. is $\pm$ 25 per cent of the energy developed per stroke. Find the weight of flywheel of 8 ft. mean diameter necessary to keep the fluctuation of speed within $\pm$ 2 per cent.

12. The cage of a goods hoist weighs 9 cwt. and carries a maximum load of 15 cwt. It is raised by a rope passing over a 4 ft. diameter drum of weight 800 lb. and radius of gyration 18 in. The other end of the rope is connected to a balance weight; the cage being overbalanced when empty to the extent of 40 per cent of the full load. If the drive, when raising the maximum load, is to be capable of a performance equivalent to an acceleration of 4 ft. per sec. per sec. at a speed of 10 ft. per sec., calculate the drum torque and power necessary for the masses given. (U.L.)

13. A four-stroke cycle internal combustion engine indicates 16 h.p. at 160 r.p.m. with 79 explosions or power strokes per min. The work done on the gases during the compression stroke is 0·30 of the effective work as represented by the area of the indicator diagram. If the total fluctuation of speed is limited to $2\frac{1}{2}$ per cent of the normal speed, find the mass in tons of each of the two flywheels, each wheel having a radius of gyration of 3 ft. (U.L.)

14. Find the expression for the force required to overcome the inertia of the reciprocating parts of a direct-acting engine at each end of the stroke, in terms of the mass in lb., the revs. per min. N, the length of the crank R ft., and the length of the connecting-rod L ft. Apply the expression to find these forces in an engine in which $m = 360$ lb., $N = 126$ revs. per min., $R = 1{\cdot}5$ ft., $L = 6$ ft. By reference to outline sketches, explain how these forces are obtained in a gas engine having one end of the cylinder open to the atmosphere. (U.L.)

15. If the effective pressure on the piston of the engine of example 14 is 60 lb. per sq. in. when the crank has turned through 30° from the inner dead centre position, and the diameter of the piston is 12 in., find the normal thrust between the crosshead and the guide bar. Obtain a general expression for this as a function of the steam pressure, engine dimensions, and crank angle.

16. In a main rope haulage gear an electric motor operates a drum 7 ft. diameter through gearing of 14 to 1 reduction ratio. The rope from the drum is attached to a load of 8 tons, which is to be hauled 1500 yds. on a gradient of 1 in 35. The axle friction is 30 lb. per ton. Neglecting rope friction, determine the motor power necessary for a haulage speed of 7 miles per hour if 20 per cent of the motor output is spent on friction within the gear. Assuming the rotating parts of this gear to be equivalent to a mass of 10 tons rotating at the drum speed and at a radius of 3 ft., determine the average torque to be exerted by the motor during the acceleration period if full speed is to be attained in 1 minute from rest. Assume the frictional torque of the gear to be constant. (U.L.)

17. The crank effort exerted by an engine is given in lb.-ft., by $T = 2000 \sin\theta + 500 \sin 2\theta$, where θ is the crank cycle. Find (*a*) the horse-power of the engine at 360 r.p.m., and (*b*) the fluctuation of energy in ft.-lb. in one revolution. (U.L.)

18. The turning moment T (lb.-ft.) on a simple engine is given by $T = Pr \sin\theta$. (This neglects the variation in the steam pressure and the obliquity of the connecting-rod.) The resisting torque T_m is uniform; find it. Find for what values of θ between 0 and π radians the turning moment is equal to the resisting torque. Find the value $\int_A^B T - T_m) d\theta$ (Art. 44). What does this represent? The mean speed of the engine is N revs. per min., and the fluctuation of speed is $\pm$ 1 per cent. Find the moment of inertia of the flywheel. Find an expression for the angular acceleration of the flywheel.

19. The crank effort diagram for an engine is given by T (lb.-ft.) $= 8{,}000 \sin\theta + 1.200 \sin 2\theta + 100 \sin 3\theta + 5{,}000$, where θ is the crank angle. The resisting torque is uniform. If the engine makes 120 revs. per min. determine

(1) The horse-power of the engine.

(2) The weight of a flywheel rim of mean radius 4 ft., so that the fluctuation in speed shall not exceed $\pm$ 1 per cent.

(3) The angular acceleration of the flywheel when $\theta = 0°$, and when $\theta = 45°$. (U.L.A.)

20. Find the particulars asked for in example 19, assuming that the crank effort diagram from $\theta = 0°$ to $\theta = 360°$ consists of two isosceles triangles having their bases along the axes of θ from $\theta = 0$ to $\theta = 180°$, and from $\theta = 180°$ to $\theta = 360°$ respectively and their vertices at a height above the axis representing a turning moment of 14,000 lb.-ft. Would it make any difference

to the fluctuation of speed if the triangles, though having the same bases and heights as those above, were not isosceles ?

21. In a series of experiments on a small crab the following results were obtained for the effort P lb. required to raise steadily a load of W lb.—

$$P = 3{\cdot}5\,\text{lb}, \; W = 22{\cdot}7\,\text{lb}. \qquad P = 16\,\text{lb}., \; W = 120\,\text{lb}$$

The graph of P and W is a straight line. The velocity ratio of the machine $= 8{\cdot}78$. Find an expression for the efficiency of the crab in terms of W and numerical constants. The maximum safe load for this crab is 250 lb. Find the efficiency and the effort for this load. When the load is on the machine, what will happen if the effort is removed ? Give reasons for your answer. (U.L.)

22. A locomotive whose cranks make n revs. per min. is moving at V mls. per hr. If I is the moment of inertia of the rotating masses (all supposed rotating with the crank-shaft) and M is the mass in pounds of the locomotive, find the total kinetic energy. If steam is shut off, how far will the locomotive move in coming to rest, there being a constant resistance to motion of R lb. ?

23. A straight bar of uniform section is 10 ft. long and weighs 9 lb. It oscillates about one end in a vertical plane through an angle of 10° on each side of the vertical. Find the potential energy, kinetic energy, and angular velocity of the bar when it makes angles of 0°, 5°, and 10° with the vertical.

24. A cone of height h and of small radius of base r is made of uniform material, and its mass is W lb. If k is its radius of gyration about the centre of gravity of the cone, find by the method of Art. 41 the masses and positions of two particles which would have the same linear and angular inertia. (*a*) if one of the particles is at the apex of the cone, (*b*) if one of the particles is at the centre of the base of the cone.

25. *ABCDEF* is a regular hexagon, whose side is of length a. Forces act along the sides thus : 5 lb. from A to B, 7 lb. from C to B, 8 lb. from D to C, 4 lb. from D to E, and 6 lb. from E to F. Reduce all these to a single force at A and a couple. Find a single force to replace the above system of forces.

26. Show that the velocity of rubbing of the crankpin of an engine is

$$v = \omega r\left(1 + \frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}}\right) \text{ ft. per second.}$$

r = radius of crankpin in feet, n = ratio of connecting rod to crank.

27. A simple engine mechanism has a crank 9 in. long, and a connecting-rod 3 ft. long. Draw a rectangular diagram giving values of the crank-angle along the horizontal line, and piston displacement along the vertical, (1) by measuring the displacements from drawings of the mechanism, and (2) using Harrison's method (Art. 40). Draw both diagrams in the same figure and find the greatest difference between the displacements obtained.

28. A moving staircase has a speed of 2 ft. per sec. The slope is 45°, and the vertical rise 50 ft. Two hundred people of an average weight 9 stones use the staircase per minute. Just before stepping on at the bottom they are moving forward at 4 ft. per sec. During the rise they walk up at a rate of 2 ft. per sec. relative to the staircase. At the top they step off with a relative backward velocity of 1 ft. per sec. Neglecting friction, find the horse-power required to maintain the motion. (U.L.A.)

29. The external radius of a hollow cylinder is 8 in., and starting from rest it rolls 5 ft. down a plane inclined at 5° to the horizontal, the final velocity being 3·2 ft. per sec. When the inclination of the plane is altered to 3°, the final velocity is 1·9 ft. per sec. Find (*a*) the radius of gyration of the cylinder, (*b*) the resistance to rolling per pound of weight (Art. 38).

30. The following data are given for a motor vehicle : Weight of vehicle, 1½ tons total ; diameter of wheels, 32 in. ; gear ratio, 4·32 ; weight of rotating parts of engine, 370 lb. ; radius of gyration of rotating parts, 8½ in. Neglecting the energy of rotation of the wheels, and assuming that the frictional

resistance remains constant, find the increase of engine torque required to accelerate the speed of the vehicle from 20 m.p.h. to 30 m.p.h. in 7 secs.

31. Find the horizontal force at the axle required to pull a wheel D in. diameter out of a rut x in. wide, assuming that only the edges of the rut are in contact with the wheel. The load on the wheel is W lb. Find also the pressure on the edge of the rut when the wheel is just lifting.

32. A symmetrical body on wheels (e.g. a motor vehicle), weight W, is tilted sideways through an angle α by raising the wheels on one side. If b is the breadth of the wheel base, and $n \,.\, W$ is the reduction of total wheel load on the side which is raised, show that the height of the C.G. of the vehicle from the ground before raising is given by $h = b \,.\, n \cot \alpha$.

33. A motor-driven machine tool, which absorbs power at the rate of 10 h.p. during the backing stroke and 20 h.p. during the working stroke, absorbs 60 h.p. during reversal, which take place in 2 seconds. A 30 h.p. motor is used, and a flywheel, weighing 450 lb., with radius of gyration 1½ ft., is used to assist the motor at peak load. If the flywheel rotates at 800 r.p.m., but drops in speed 10 per cent during reversal, calculate the overload on the motor during this period. (U.L.)

34. From the indicator diagrams of a double-acting single cylinder vertical steam engine the effective force on the piston in lb. due to steam pressure was calculated for various points in the down stroke as follows—

Percentage of stroke .	0	10	30	50	70	90	100
Effective force . .	6000	6400	5700	3700	1800	0	-3000

Plot this curve and modify it for the effect of the weight and the inertia of the reciprocating parts (it will be sufficient to calculate the values at the beginning and end of the stroke and draw a smooth curve, assuming that the piston acceleration is zero when crank and connecting rod are at right angles). Sketch the turning moment diagram for the down stroke to a scale of 1,000 lb. ft. to 1 in. Weight of reciprocating parts, 200 lb.; stroke, 2 ft.; speed, 240 r.p.m. The connecting rod is four times the crank length. (U.L.)

35. The length of the wheel-base of a motor vehicle, weight W, is B ft. When on a horizontal track the height of the centre of gravity from the ground is H ft., and the horizontal distance of the centre of gravity from a vertical plane through the centre line of the back axle is x ft. If μ is the coefficient of friction between tyres and ground, show that when the vehicle is moving down a gradient inclined at θ to the horizontal, the maximum possible retarding force which can be applied by braking on the front wheels only is

$$F_1 = \mu \,.\, W \,.\, \frac{r + k \,.\, \sin \theta}{1 - \mu \,.\, k}$$

and the maximum possible retarding force which can be applied by braking on the rear wheels only is

$$F_2 = \mu \,.\, W \,.\, \frac{1 - (r + k \,.\, \sin \theta)}{1 + \mu \,.\, k}$$

where $k = \frac{H}{B}$ and $r = \frac{x}{B}$. (cos θ may be taken as 1).

36. The diameter of the rear wheels of a motor vehicle is 32 in., the wheel base is 7 ft. long, and the load on the rear axle is 18 cwt. when the vehicle is stationary on a level surface. If μ between tyres and ground is 0·8, (*a*) calculate the maximum torque that can be exerted on the rear axle, (*b*) state the total vertical load on the rear axle when this torque is being exerted, (*c*) if this maximum torque is exerted when the vehicle is moving at 5 m.p.h., what horse-power is then being delivered to the rear axle ?

CHAPTER III

SIMPLE HARMONIC MOTION—OSCILLATIONS

45. Simple Harmonic Motion. In Art. 8 we dealt with the motion of a particle along a straight line with an acceleration directed towards a fixed point in the line, and proportional to the distance from the particle to the fixed point. This kind of motion is of frequent occurrence in engineering, and we shall now enter into a closer examination of the motion. We saw that the equation of motion is

$$\frac{d^2x}{dt^2} = -k_1^2x \text{ where } k_1 \text{ is constant,}$$

or

$$\frac{d^2x}{dt^2} + k_1^2x = 0 \qquad . \quad . \quad . \quad . \quad . \qquad \text{(III.1)}$$

and that its solution is

$$x = a \cos (k_1t + g_1) \qquad . \quad . \qquad \text{(III.2)}$$

This solution may be verified by substitution.

In Fig. 23, APB is a circle with centre O and radius a. Let BA be a fixed diameter. Let P be a particle moving round the circle in the direction of the arrow, so that OP has constant angular velocity k_1. The particle P will move round the circle with constant speed, k_1a. Let t be the time at which P is in the position shown, the time being reckoned from the instant when P occupied a position for which the angle AOP was g_1. As OP turns through an angle k_1t in time t, we have

$$\widehat{POA} = k_1t + g_1$$

Draw PM perpendicular to OA, meeting it at M. As P moves round the circumference, M moves along BA so that PM is always perpendicular to BA. Let x be the length of OM. Then

$$\begin{aligned} x &= \overline{OP} \cos \widehat{POA} \\ &= a \cos (k_1t + g_1) \end{aligned}$$

which is identical with (III.2) above. The motion of M is therefore that defined above and represented by (III.2).

The acceleration of M is $k_1^2x = k_1^2\overline{OM}$ in the direction M to O. The velocity of M may be found by differentiating (III.2), thus,

$$
\begin{aligned}
V_M &= \text{velocity of } M \\
&= \frac{dx}{dt} \\
&= \frac{d}{dt}\{a\cos(k_1t + g_1)\} \\
&= -ak_1\sin(k_1t + g_1)
\end{aligned}
$$

i.e. $V_M = -k_1\overline{PM}$ (III.3)

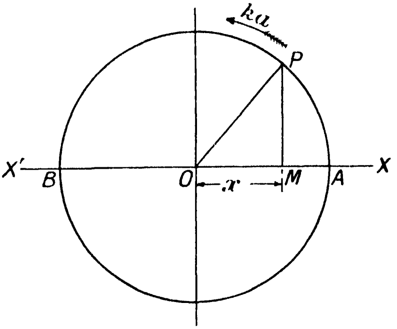


FIG. 23. SIMPLE HARMONIC MOTION

If we call $x = \overline{OM}$, the displacement of the particle equation (III.1) becomes

$$\frac{x}{\frac{d^2x}{dt^2}} = -\frac{1}{k_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(III.4)}$$

$$\text{or} \quad \frac{\text{displacement}}{\text{acceleration}} = \text{constant} \quad . \quad . \quad . \quad . \quad . \qquad \text{(III.5)}$$

It is important to notice that the constant in (III.5) is the reciprocal of the square of the angular velocity of the radius OP, and that the acceleration is always directed towards O.

The motion of M is called *simple harmonic motion.* We shall consider M to be a particle of mass W/g. As P moves continuously round the circle, M will oscillate to and fro along AB. In investigating the motion of an oscillating particle, in order to see if the oscillations are simple harmonic, we write down the equation of motion. If this is of the type (III.1), the motion is simple harmonic. Alternatively, we may use (III.5), and if the ratio on the left is constant, and the acceleration is always directed towards the origin, the motion is simple harmonic.

Time of a Complete Oscillation. For every revolution of P, M makes a complete oscillation ; we have therefore,

$$
\begin{aligned}
t &= \text{time of a complete oscillation of } M \\
&= \text{time of revolution of } P \\
&= \frac{2\pi}{k_1}
\end{aligned}
$$

and from (III.4)

$$t = 2\pi \sqrt{-\frac{x}{\frac{d^2x}{dt^2}}} = \frac{2\pi}{k_1} \qquad . \quad . \quad \text{(III.6)}$$

or

$$t = 2\pi \sqrt{\frac{\text{displacement}}{\text{acceleration}}} \qquad . \quad . \quad \text{(III.7)}$$

which gives the time of an oscillation in all cases of simple harmonic motion. If the displacement is in feet, and the acceleration in feet per second per second, (III.7) will give the time in seconds.

Definitions. a is the distance travelled by the particle on each side of its mid-position, and is called the *amplitude* ; t is the time of a complete oscillation, called the *period* or *periodic time* ; g_1 is the angle turned through by the radius OP at zero time, and is called the *epoch.* The *phase* is a quantity representing at any instant the fraction of the period which has elapsed since the particle was last in some fixed position, such as the extreme position A. Two particles are in the same phase if at any instant they have both made the same fraction of a complete oscillation from corresponding positions. The *frequency* of oscillation is the number of oscillations made in unit time, and is the reciprocal of the period, i.e. $\frac{k_1}{2\pi}$ where k_1 is the angular velocity of OP. k_1 is also known as the *angular frequency.*

The force F acting on the particle M at any time is given by

$$\text{Force} = \text{mass} \times \text{acceleration}$$

or

$$F = \frac{W}{g} \frac{d^2x}{dt^2}$$

substituting from (III.1),

$$F = \frac{W}{g} (-k_1^2 x)$$

i.e.

$$F = -\frac{W}{g} k_1^2 x \qquad . \quad . \quad . \quad . \quad \text{(III.8)}$$

and, substituting the value of x from (III.2),

$$F = -\frac{W k_1^2 a}{g} \cos (k_1 t + g_1) \qquad . \quad \text{(III.9)}$$

(III.8) and (III.9) give the force in terms of the displacement and time respectively. If $\frac{W}{g}$ is the mass in engineer's units, k_1 is in radians per sec., g_1 is in radians, and a in feet, then F is the force in pounds. We shall apply the foregoing to a few simple cases.

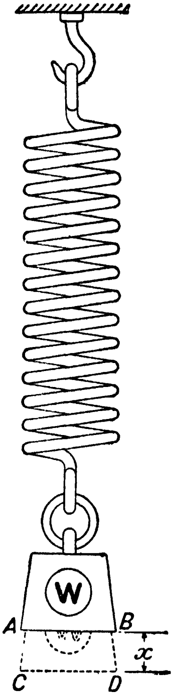


FIG. 24. OSCILLATION OF A MASS SUSPENDED FROM A SPRING

46. **Vertical Oscillations of a Mass, *W* Pounds, attached to the Lower End of a Vertical Spiral Spring of which the Upper End is Fixed.** Let AB, Fig. 24, be the lower edge of the weight in its position of equilibrium. We shall assume that the amplitude of the oscillations is such that the spring is never permanently strained, and is also always under tension. Let En lb. be the force required to stretch the spring through a distance of n ft. within the elastic limit. If the weight is pulled downwards and then released it will begin to oscillate along the vertical axis of the spring. Let CD be the position occupied by the lower end of the weight at time t sec.; and let x ft. be the distance between AB and CD. In the equilibrium position the tension of the spring balances the weight of W. When the weight is in the position CD, there is an additional upward force of Ex lb. This force will produce an acceleration towards the equilibrium position. The equation of motion is

$$\text{Force} = \text{mass} \times \text{acceleration}$$

or

$$-Ex\,. = \frac{W}{g}\frac{d^2x}{dt^2}$$

the negative sign on the left-hand side indicating that the values of x and $\frac{d^2x}{dt^2}$ are of opposite signs. The above equation simplifies to

$$\frac{d^2x}{dt^2} + \frac{gE}{W}x = 0 \quad . \quad . \quad . \quad . \quad (\text{III}.10)$$

Comparing this equation with (III.1), we see that if $\frac{gE}{W} = k_1^2$ the two are the same. Consequently, the motion is simple harmonic, and the results of Art. 45 apply.

Putting $k_1^2 = \frac{gE}{W}$ in (III.2) and (III.6) respectively, we see that

$$\left.\begin{aligned} x &= a \cos\left(\sqrt{\frac{gE}{W}}t + g_1\right) \\ \text{and} \quad t &= 2\pi\sqrt{\frac{W}{gE}} \end{aligned}\right\} \quad . \quad . \quad \text{(III.11)}$$

The quantities a and g_1 depend upon the conditions under which the motion is started, and can be determined in any particular case. The quantity E may be calculated if the dimensions of the spring and the elastic properties of its material are known, or it may be found by determining the ratio of any load placed on the spring to the stretch produced by the load. We have assumed that the mass of the spring is so small as to be negligible. If the mass of the spring is not negligible it is proved in books on mechanics that it may be allowed for by adding one-third the weight of the spring to W in (III.11).

The above relations are true, not only for a weight oscillating at the end of a spring, but in all cases where a weight is attached to a point on an elastic body such that a load at that point produces a vertical deflection proportional to the load. For example, the spring might be replaced by a vertical steel rod, or by a horizontal cantilever with W at the free end. In this latter case the above correction for the mass of the elastic member would not apply, and a new correction would be necessary.

The force F lb. would be given by

$$\text{Force} = \text{mass} \times \text{acceleration}$$

$$F = \frac{W}{g} \times \frac{d^2x}{dt^2}$$

or

$$F = -Ea \cos\left(\sqrt{\frac{gE}{W}}\, t + g_1\right) \quad . \quad \text{(III.12)}$$

47. **Simple Pendulum.** A simple pendulum would consist of a heavy particle suspended from a fixed point by a weightless inextensible string. Such an arrangement is, of course, not possible, but by using a small weight of heavy material, and a light string which is not easily stretched, we can approach fairly closely to the ideal arrangement.

In Fig. 25, O is the fixed point, B is the weight, and OB the string in the equilibrium position. Suppose B to be set swinging along the circular arc ABD, and let C be the position of the weight after time t. Let θ be the magnitude of the angle BOC, l ft. the length of the spring, and W the weight in pounds of B. The forces acting on C are (1) the tension in the string OC, which is at right-angles to the direction of motion and has no effect on the motion, and (2) W lb., the weight of C, acting parallel to OB. The component of W in the direction of motion is $W \sin \theta$ lb., and the equation of motion is

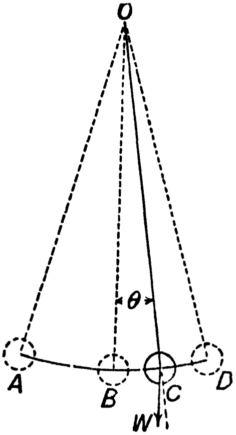


FIG. 25. SIMPLE PENDULUM

$$\text{Force} = \text{mass} \times \text{acceleration}$$

or
$$W \sin \theta = \frac{W}{g} \times \frac{d^2x}{dt^2}$$

where x is the length of the arc BC, i.e. $x = l\theta$.

$$\therefore \frac{d^2x}{dt^2} + g \sin \theta = 0$$

i.e.
$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0$$

or
$$\frac{d^2x}{dt^2} + g \sin \frac{x}{l} = 0 \quad \text{(III.13)}$$

The solution of (III.13) involves the use of elliptic integrals, and is outside the scope of this book. If, however, the angles AOB, DOB are sufficiently small, we may replace $\sin \frac{x}{l}$ by $\frac{x}{l}$. With this substitution (III.13) becomes

$$\frac{d^2x}{dt^2} + \frac{g}{l} x = 0 \quad . \quad . \quad . \quad . \quad \text{(III.14)}$$

which is the same as (III.1) if we write $k_1{}^2 = \frac{g}{l}$.

The solution of (III.14) is therefore by (III.2),

$$x = a \cos \left(\sqrt{\frac{g}{l}}\, t + g_1 \right) \cdot \quad . \quad \text{(III.15)}$$

and by (III.6) the periodic time is

$$t = \frac{2\pi}{k} = 2\pi \sqrt{\frac{l}{g}}$$

The motion of C along the arc AD is thus of the same nature as that of M along BA (Fig. 1), and is on this account considered to be simple harmonic.

The percentage error in writing $\frac{x}{l}$ for $\sin \frac{x}{l}$, or $\sin \theta$ for θ, in the case where $\theta = \cdot 1745$ radians (corresponding to a swing of 10° on each side of the vertical) is about 0·5 per cent. The average error whilst θ varies from 0° to 10° is, of course less than this, and as the percentage error in t is only half the percentage error in the acceleration, we may conclude that the percentage error in the periodic time is not greater than 0·2. For smaller angles of swing the error is less than this.

We apply the term simple harmonic motion to all cases of motion with one degree of freedom in which a co-ordinate X, fixing the position of the moving particle or body, is connected with the time t by the relation

$$X = a \cos (k_1 t + g_1)$$

a, k_1 and g_1 being constants

The force acting on the particle of the simple pendulum is given by

$$\text{Force} = \text{mass} \times \text{acceleration}$$

i.e.

$$F = \text{force in pounds}$$

$$= \frac{W}{g} \times \frac{d^2x}{dt^2}$$

and approximately,

$$F = -\frac{Wx}{l} \quad . \quad . \quad . \qquad \text{(III.16)}$$

We saw above, that the force was $W \sin \theta = W \sin \frac{x}{l}$. The difference between these two expressions is due to the substitution of $\frac{x}{l}$ for $\sin \frac{x}{l}$. Substituting for x in (III.6), the value of x from (III.15),

$$F = -\frac{Wa}{l} \cos \left(\sqrt{\frac{g}{l}}\, t + g_1\right)$$

48. Oscillations of a Rigid Body when Constrained to Turn About a Horizontal Axis. Such an arrangement is called a *compound pendulum*.

OG (Fig. 26) is a rigid body supported by a horizontal pivot at O, and oscillating in the vertical plane. G is the centre of gravity. Let $h = \overline{OG}$, and let K and k be the radii of gyration of the body about O and G respectively. If W is the weight of the body, we have

$$I_o = \text{moment of inertia about } O$$

$$= \frac{W}{g} K^2 \text{ engineers' units}$$

and

$$I_G = \text{moment of inertia about } G$$

$$= \frac{W}{g} k^2$$

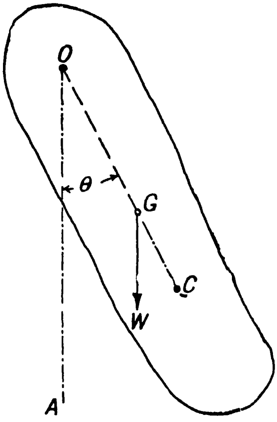


FIG. 26. COMPOUND PENDULUM

But, by (II.9),

$$I_o = I_G + \frac{W}{g} h^2$$

$$\therefore K^2 = k^2 + h^2 \qquad \text{(III.17)}$$

Consider the motion of OG when it is passing through the position shown in Fig. 26, in which OA is vertical, and $\widehat{AOG} = \theta$. The forces acting on OG are (1) the pressure of the axis O on the body; and (2) the weight of the body acting through G. The first force acts through O, and has no moment about O. The moment of the second force about O is $W\overline{OG} \sin \theta = Wh \sin \theta$. We have then,

Turning moment = moment of inertia × angular acceleration

$$-Wh \sin \theta = \frac{W}{g} K^2 \frac{d^2\theta}{dt^2} \quad . \quad . \quad . \qquad \text{(III.18)}$$

the negative sign on the left indicating that the turning moment is in the opposite sense to that of θ increasing. From (III.18) we have,

$$\frac{d^2\theta}{dt^2} + \frac{gh}{K^2} \sin \theta = 0$$

This corresponds to (III.14) if $\frac{K^2}{h} = l$, hence the motion is the same as that in the case of a simple pendulum of length $\frac{K^2}{h}$. For small oscillations θ may be written for $\sin\theta$, and the solution of the equation is,

$$\theta = \theta_0 \cos\left(\sqrt{\frac{gh}{K^2}}\,t + g_1\right) \quad . \quad \text{(III.19)}$$

where θ_0 is a constant equal to the amplitude of the angular swing.

The periodic time t is given by,

$$t = 2\pi\sqrt{\frac{K^2}{gh}} \quad . \quad . \quad . \quad . \quad \text{(III.20)}$$

or substituting from (III.17),

$$t = 2\pi\sqrt{\frac{k^2 + h^2}{gh}} \quad . \quad . \quad . \quad \text{(III.21)}$$

The point O is called the *centre of suspension*. Produce OG to C so that $\overline{OC} = \frac{k^2 + h^2}{h}$. Then $\overline{OC}$ is the length of the equivalent simple pendulum, and C is called the *centre of oscillation.*

49. **To Prove that the Periodic Time is Unaltered if the Centres of Oscillation and Suspension are Interchanged.** Suppose the pendulum to be oscillating with C fixed and O moving below C. Let t' be the time of oscillation. If K' be the radius of gyration about C, we have $(K')^2 = k^2 + \overline{GC}^2$.

From (III.20),

$$t' = 2\pi\sqrt{\frac{(K')^2}{g \,.\, \overline{GC}}}$$

and the length l' of the equivalent simple pendulum is,

$$l' = \frac{(K')^2}{\overline{GC}} = \frac{k^2 + \overline{GC}^2}{\overline{GC}} \quad . \quad . \quad \text{(III.22)}$$

But,

$$\overline{GC} = \overline{OC} - \overline{OG}$$

$$= \frac{k^2 + h^2}{h} - h$$

$$= \frac{k^2}{h}$$

or

$$\frac{k^2}{\overline{GC}} = h \quad . \quad . \quad . \quad . \quad . \quad \text{(III.23)}$$

From (III.22) $l' = \dfrac{k^2}{\overline{GC}} + \overline{GC}$, and substituting from (III.23),

$$\begin{aligned} l' &= h + \overline{GC} \\ &= \overline{OG} + \overline{GC} \\ &= \overline{OC} \end{aligned}$$

Thus, the length of the simple equivalent pendulum is the same as that when O is the centre of oscillation, and the periodic time is unaltered.

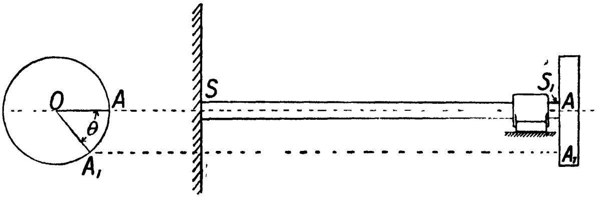


FIG. 27. TORSIONAL OSCILLATIONS

To find the expression for the turning moment, or couple, in terms of the time, we have,

$$\begin{aligned} C &= \text{turning moment} \\ &= -\,Wh \sin \theta \\ &= -\,Wh\theta \text{ for small oscillations.} \end{aligned}$$

Hence, from (III.19),

$$C = -\,Wh\theta_0 \cos\left(\sqrt{\frac{kg}{K^2}}\,t + g_1\right) \quad \text{(III.24)}$$

50. Torsional Oscillations of a Shaft Fixed at One End. Let SS_1 (Fig. 27) be a shaft of circular section, fixed at S and attached to a cylinder at S_1. Suppose the shaft to be supported in the horizontal position by frictionless bearings. Let I be the moment of inertia of the mass of the cylinder about the axis, i.e. $I = \dfrac{W}{g}k^2$, where $\dfrac{W}{g}$ is the mass and k ft. the radius of gyration of the cylinder, and let the moment of inertia of the mass of the shaft be negligible. Suppose the cylinder to be rotated through some angle, not sufficiently large to overstrain the material, and then released. The shaft and cylinder will then oscillate about the position of rest. Consider the motion when the radius originally at OA is passing through the position OA_1, in the direction of θ increasing, where $\widehat{AOA_1} = \theta$. θ is the angle of twist of the shaft, and the turning moment on any cross-section is $C_o\,\theta$, where C_o lb.-ft. is the ratio $\dfrac{\text{couple}}{\text{angle of twist}}$ for the shaft. We have,

Turning moment = moment of inertia × angular acceleration

i.e. $$-C_o\theta = \frac{W}{g}k^2 \times \frac{d^2\theta}{dt^2}$$

or $$\frac{d^2\theta}{dt^2} + \frac{gC_o}{Wk^2}\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad (\text{III}.25)$$

from which, by comparison with (III.1) and (III.2) we obtain

$$\theta = \theta_o \cos\left(\sqrt{\frac{gC_o}{Wk^2}}t + g_1\right)$$

where the amplitude θ_o is a constant, and

$$t = \text{time of oscillation}$$

$$= 2\pi\sqrt{\frac{Wk^2}{C_o g}} = 2\pi\sqrt{\frac{\text{angular displacement}}{\text{angular acceleration}}}$$

or $$t = 2\pi\sqrt{\frac{I}{C_o}} \quad . \quad . \quad . \quad . \quad (\text{III}.26)$$

Let N = frequency = number of oscillations per minute.

Then $$N = \frac{30}{\pi}\sqrt{\frac{C_o}{I}} \quad . \quad . \quad . \quad . \quad (\text{III}.27)$$

It is proved in textbooks on the strength of materials that,

$$C_o = \frac{\pi r^4 G}{24l} \text{ lb.-ft.} \quad . \quad . \quad . \quad (\text{III}.28)$$

where r is the radius, l the length of the shaft (both in inches), and G is the modulus of rigidity of the material of the shaft in pounds per square inch. If r is constant, this may be written

$$C_o = \frac{c}{l} \quad . \quad . \quad . \quad . \quad . \quad (\text{III}.29)$$

where c is a constant. C_o is the couple corresponding to unit twist, i.e. to an angle of twist of one radian.

51. Oscillations of a Rotating Shaft Carrying a Flywheel at Each End. Let AB (Fig. 28) be a shaft carrying flywheels of moments of inertia I_A at A, and I_B at B. When not oscillating, the shaft is assumed to be rotating with uniform speed. If a torsional impulse is applied to one of the flywheels, or some portion of the shaft, torsional oscillations will be set up. The motions due to the oscillations will be superposed on the uniform motion and the relative motions of the parts of the shaft, and the flywheels will be unaffected by the uniform motion. We may, therefore, for simplicity, ignore the uniform motion, and consider the shaft as oscillating about a stationary equilibrium position. The simplest mode of oscillatory motion is that in which some point C in AB remains at rest, and A and B are always rotating in opposite senses. We shall consider only this mode of motion, and shall neglect the moment of inertia of the shaft. During the motion each of the portions CA and CB will be under the same conditions as the shaft in Art. 50. Consequently, if t_A and t_B are their respective times of oscillation we have by (III.26),

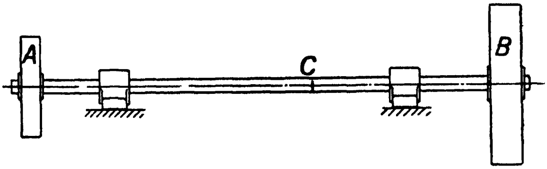


FIG. 28. TORSIONAL OSCILLATIONS OF A SHAFT CARRYING TWO FLYWHEELS

$$t_A = 2\pi\sqrt{\frac{I_A}{{}_AC_0}} \quad . \quad . \quad . \quad . \quad . \qquad \text{(III.30)}$$

and

$$t_B = 2\pi\sqrt{\frac{I_B}{{}_BC_0}} \quad . \quad . \quad . \quad . \quad . \qquad \text{(III.31)}$$

where ${}_AC_0$, and ${}_BC_0$, are the values of C_0 for the portions of shaft CA and CB respectively. Since the times of oscillation are equal,

$$t_A = t_B$$

$$\therefore \frac{I_A}{{}_AC_0} = \frac{I_B}{{}_BC_0} \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(III.32)}$$

But, from (III.29) we see that

$${}_AC_0\,\overline{AC} = {}_BC_0\,.\,\overline{BC}$$

and substituting in (III.32) we have

$$\frac{I_A}{\overline{BC}} = \frac{I_B}{\overline{AC}}$$

Hence
$$\frac{I_A}{I_B} = \frac{\overline{BC}}{\overline{AC}} \qquad . \quad . \quad . \quad \text{(III.33)}$$

which gives the position of C.

To find the time of oscillation, we first find the position of C from (III.33), and then, having found ${}_AC_o$, or ${}_BC_o$, we find the time of oscillation from either (III.30) or (III.31). We may, however, find a single relation for the periodic time. Thus, from (III.33),

$$\frac{I_A}{I_A + I_B} = \frac{\overline{BC}}{\overline{BC} + \overline{AC}}$$
$$= \frac{\overline{BC}}{\overline{AB}} \qquad . \quad . \quad . \quad . \quad \text{(III.34)}$$

Let C_{AB} be the couple needed to produce a twist of one radian in the shaft AB, then from (III.29),

$$C_{AB} \,.\, \overline{AB} = {}_BC_o \,.\, \overline{BC}$$

Hence, from this and (III.34),

$$\frac{I_A}{I_A + I_B} = \frac{C_{AB}}{{}_BC_o}$$ and multiplying both sides by $\frac{I_B}{C_{AB}}$

$$\frac{I_B}{{}_BC_o} = \frac{(I_A I_B)}{C_{AB}\,(I_A + I_B)}$$

Substituting in (III.31),

$$t = 2\pi\sqrt{\frac{I_A I_B}{C_{AB}\,(I_A + I_B)}} \qquad . \quad . \quad \text{(III.35)}$$

or
$$N = \frac{30}{\pi}\sqrt{\frac{C_{AB}\,(I_A + I_B)}{I_A I_B}} \qquad . \quad . \quad \text{(III.36)}$$

52. **Damped Oscillations.** In the above theory we have taken no account of friction. When a body is oscillating in the air, or in any other fluid, there is a frictional resistance to the motion. There is also friction at the point, or points, of support of the body. The principal effect of friction is to diminish gradually the amplitude of the oscillation until it becomes negligible. This effect is known as *damping*. There

is also a change in the periodic time caused by friction. As, however, the friction is seldom very considerable, its effect on the periodic time is only very slight, and is usually neglected. (See Art. 221.)

53. **Forced Oscillations.** The oscillations considered above are all supposed to be caused by some original disturbance, which is not repeated. Such oscillations are called *free oscillation*. Owing to damping these oscillations gradually die away. If, however, the disturbance is continually repeated, the oscillations will continue. Oscillations produced by a continuous succession of disturbances are known as *forced oscillations*. Consider again the case of Art. 46, and assume that an external force of magnitude $P \cos (pt + q)$ acts on the mass W in the sense in which x is positive, p and q being constants. The equation of motion is obtained by adding the quantity $P \cos (pt + q)$ to the left-hand side of the equation of motion.

Thus, $$P \cos (pt + q) - Ex = \frac{W}{g} \frac{d^2x}{dt^2}$$

or $$\frac{d^2x}{dt^2} + \frac{gE}{W} x = \frac{gP}{W} \cos (pt + q) \quad . \quad \text{(III.37)}$$

To find a particular integral of this equation, assume that $x = A \cos (pt + q)$. Substituting in the above, we obtain

$$- Ap^2 \cos (pt + q) + \frac{gEA}{W} \cos (pt + q) = \frac{gP}{W} \cos (pt + q)$$

and dividing through by $\cos (pt + q)$

$$- Ap^2 + \frac{gEA}{W} = \frac{gP}{W}$$

or, $$A = \frac{\frac{gP}{W}}{\frac{gE}{W} - p^2} \quad . \quad . \quad . \quad \text{(III.38)}$$

The period of free oscillation was found to be (III.11),

$$t = 2\pi \sqrt{\frac{W}{gE}}$$
$$= \frac{2\pi}{n}$$

where $n = \sqrt{\frac{gE}{W}}$ and the period of the applied oscillations is

$\frac{2\pi}{p}$. It should be noticed that n is 2π times the frequency. Substituting n^2 for $\frac{gE}{W}$ in (III.38), we have if A = amplitude

$A = \frac{P}{E}\frac{n^2}{n^2 - p^2}$ and the particular integral becomes

$$x = \frac{n^2}{n^2 - p^2} \cdot \frac{P}{E} \cos (pt + q) \quad . \quad \text{(III.39)}$$

The ratio of A to the statical deflection $\frac{P}{E}$ due to the force P is called the *dynamic magnifier* or *magnification factor* and is equal to $\frac{n^2}{n^2 - p^2}$.

The complete solution of (III.37) is obtained by adding this value of x to the value of x given by (III.10), but, as this latter portion of the solution is eliminated by damping, the motion is represented by (III.39). The above solution neglects the effects of friction on the motion. If, however, the friction is small, as is often the case, the above gives a fairly close approximation to the motion. If p is nearly equal to n the amplitude is very large, and if $p = n$ the amplitude becomes infinite. We cannot, however, make use of the above relation for large amplitudes, as our assumptions relating to elasticity and the negligible effects of friction are not justified when the amplitude is large. It is, nevertheless, clear from (III.39), that, if p is sufficiently near to n, the amplitude of the oscillations will become dangerously large, and the elastic material will be overstrained. When $n = p$, i.e. when the two periods are equal, the periods are said to *synchronize*, and there is a state of *resonance*. In machines, or portions of machines, it may occur that disturbing forces are set up by unbalanced forces whose periods are equal, or nearly so, to the free periods of some portions of the mechanism. In such cases overstrain of the elastic members will occur, resulting in a breakdown in the mechanism. To avoid this it is necessary, in designing such appliances as turbine reduction-gearing, to find the value of p for every possible disturbing force or couple, and the value of n for every body capable of oscillating, and, finally, to ensure that no value of n approaches too closely to any value of p. If such close approach occurs, the design should be altered so as to separate more widely the two values. Separation is usually effected by varying n. This is easily varied by changing one or more dimensions of the elastic member, or by changing the moment of inertia of the oscillating mass. The amplitude is proportional to $\frac{n^2}{n^2 - p^2}$. Let $n = p + x$, then

$$\frac{n^2}{n^2 - p^2} = 1 + \frac{p^2}{n^2 - p^2}$$

$$= 1 + \frac{p^2}{2px + x^2}$$

and, if x is a small fraction of p,

$$\frac{n^2}{n^2 - p^2} = 1 + \frac{p}{2x} \text{ approx.}$$

For a given numerical value of x, this is numerically greater when x is positive than it is when x is negative. In altering the dimensions of a shaft in a reduction gear, or other mechanism, so as to make the free period differ from an existing forced period with which it agrees, it is better, therefore, to reduce n by a small amount than to increase it by the same amount. In practice this means that it is better to increase the mass or moment of inertia of the mass, rather than to increase the stiffness of the elastic member.

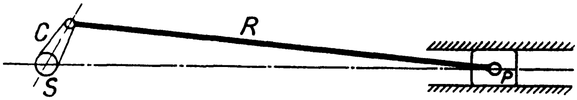


FIG. 29

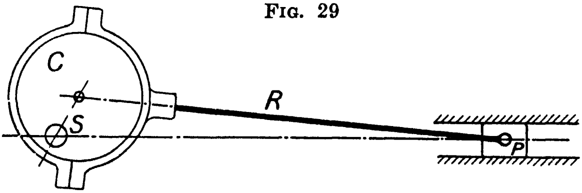


FIG. 30
EQUIVALENCE OF CRANK AND ECCENTRIC

54. **Equivalence of the Crank and the Eccentric.** Figs. 29 and 30 show respectively a crank C, and an eccentric C, each mounted on a crank-shaft S, and connected by means of a connecting-rod R to a crosshead P, which is constrained to move along the straight line passing through the centre of the

shaft S. On comparison of the two figures it will be seen that the two mechanisms are the same and the eccentric and crank are equivalent. The eccentric is virtually a crank of which the crank-pin envelops the crank-shaft. An eccentric is used when the distance from the centre of S to the centre of C is less than the sum of the radii of the crank-shaft and that of a crank-pin of sufficient strength. The distance between the centre of S and the centre of the circular piece C is called the throw, or half-travel, of the eccentric.

The motion of P in the case in which C rotates with uniform angular velocity has been investigated in Chapter II, and the more general case in which the angular velocity is variable is treated by the method of velocity and acceleration images (Chapter V). If R is long compared with the radius of the crank or the throw of the eccentric, as in the case of slide valves, the motion of P is approximately simple harmonic when the angular velocity of C is constant.

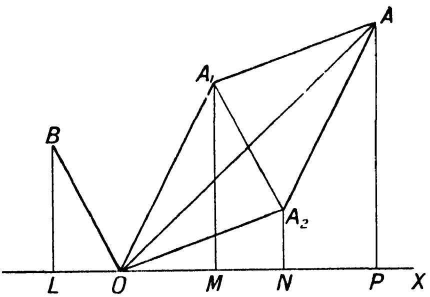


FIG. 31. SUM AND DIFFERENCE OF TWO SIMPLE HARMONIC MOTIONS

55. Sum and Difference of Two Simple Harmonic Motions of the same Period, but of different Phases, along the same Straight Line. Let the motion be represented by

$$x_1 = a_1 \cos (\omega t + g_1) \quad . \quad . \quad . \quad . \quad \text{(III.40)}$$

$$x_2 = a_2 \cos (\omega t + g_2) \quad . \quad . \quad . \quad . \quad \text{(III.41)}$$

also let X_1 be the sum of these, and X_2 the difference.

Then, $X_1 = a_1 \cos (\omega t + g_1) + a_2 \cos (\omega t + g_2)$

and $\quad X_2 = a_1 \cos (\omega t + g_1) - a_2 \cos (\omega t + g_2)$

In Fig. 31, draw a horizontal line OX and draw two lines OA_1, OA_2, making angles A_1OX and A_2OX respectively, equal to the values of $\omega t + g_1$ and $\omega t + g_2$ for any fixed value of t. Complete the parallelogram OA_1AA_2 and join OA and A_1A_2. Draw OB equal and parallel to A_1A_2. Draw BL, A_1M, A_2N and AP all perpendicular to OX. Let $\overline{OA_1} = a_1$ and $\overline{OA_2} = a_2$.

Then,
$$\begin{aligned} X_1 &= x_1 + x_2 \\ &= a_1 \cos(\omega t + g_1) + a_2 \cos\ \omega t + g_2) \\ &= \overline{OA}_1 \cos \widehat{A_1OX} + \overline{OA}_2 \cos (\widehat{A_2OX}) \\ &= \overline{OM} + \overline{ON} \\ &= \overline{NP} + \overline{ON} \\ &= \overline{OP} \\ &= \overline{OA} \cos \widehat{AOX} \end{aligned}$$

i.e.
$$X_1 = \overline{OA} \cos(\omega t + \alpha) \quad . \quad . \quad . \quad \text{(III.42)}$$

where $\alpha = g_2 + \widehat{AOA_2}$

also
$$\begin{aligned} X_2 &= x_1 - x_2 \\ &= \overline{OM} - \overline{ON} \\ &= -\overline{MN} \\ &= \overline{OL} \\ &= \overline{OB} \cos(\omega t + \beta) \quad . \quad . \quad . \quad \text{(III.43)} \end{aligned}$$

where $\beta = g_2 + \widehat{A_2OB}$.

Thus, the sum and difference of two simple harmonic motions are themselves simple harmonic motions of the same period, but of different phases. The lengths $\overline{OA}$ and $\overline{OB}$ are best found by drawing and measurement. If it is required to calculate their values, then, from the triangle OAA_2,

$$\overline{OA} = \sqrt{OA_2^2 + AA_2^2 - 2\overline{OA}_2 \,.\, \overline{AA}_2 \cos \widehat{OA_2A}}$$

or
$$\overline{OA} = \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos(g_1 - g_2)} \quad . \quad \text{(III.44)}$$

and
$$\tan \widehat{AOA_2} = \frac{a_1 \sin(g_1 - g_2)}{a_2 + a_1 \cos(g_1 - g_2)} \quad . \quad . \quad . \quad \text{(III.45)}$$

also
$$\begin{aligned} \overline{OB} &= \overline{A_1A_2} \\ &= \sqrt{a_1^2 + a_2^2 - 2a_1a_2 \cos(g_1 - g_2)} \quad . \quad \text{(III.46)} \end{aligned}$$

and
$$\tan \widehat{A_2OB} = \frac{-a_1 \sin(g_1 - g_2)}{a_2 - a_1 \cos(g_1 - g_2)} \quad . \quad . \quad . \quad \text{(III.47)}$$

The motions represented by (III.40) and (III.41) are the motions which would be given to valves on the line OX if driven by cranks OA_1 and OA_2 respectively, rotating with angular velocity ω, and operating the valves through connecting rods of such lengths that the effects of their changing obliquities may be neglected. OA_1 and OA_2 are called *representative cranks.* The representative crank for the sum of x_1 and x_2 is OA, and that for their difference is OB.

56. **Combination of Two Simple Harmonic Motions of the Same Period along Two Lines at Right-angles.**

Let $$x = a \cos (\omega t + \alpha) \quad . \quad . \quad . \quad . \quad (\text{III}.48)$$
and $$y = b \cos (\omega t + \beta) \quad . \quad . \quad . \quad . \quad (\text{III}.49)$$
be two simple harmonic motions along OX and OY respectively, OX and OY being rectangular axes of co-ordinates, and let P be the point (xy). If in either of the above relations we increase t by $\frac{2\pi}{\omega}$ sec. the angle will increase by 2π radians, and the value of the cosine will be unaltered. Thus the point P occupies the same position at the end of any period of time $\frac{2\pi}{\omega}$ sec. as it did at the beginning. The point therefore moves along a closed curve, which it traces out once in every $\frac{2\pi}{\omega}$ sec. By choosing a suitable instant from which to measure the time, we can make $\alpha = 0$, in which case $\frac{x}{a} = \cos \omega t$. From (III.49),

$$\frac{y}{b} = \cos \omega t \cos \beta - \sin \omega t \sin \beta$$

and substituting $$\frac{y}{b} = \frac{x}{a} \,.\, \cos \beta \,.\, - \sin \beta \sqrt{1 - \frac{x^2}{a^2}}$$

$$\therefore \frac{y}{b} - \frac{x}{a} \cos \beta = - \sin \beta \sqrt{1 - \frac{x^2}{a^2}}$$

and squaring $$\frac{y^2}{b^2} - \frac{2xy \cos \beta}{ab} + \frac{x^2}{a^2} \cos^2 \beta = \sin^2 \beta - \frac{x^2}{a^2} \sin^2 \beta$$

that is $$\frac{y^2}{b^2} - \frac{2 \cos \beta}{ab} \,.\, xy + \frac{x^2}{a^2} = \sin^2 \beta \quad . \quad (\text{III}.50)$$

which is the equation to an ellipse. If $\beta = \frac{\pi}{2}$, the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad (\text{III}.51)$$

which represents an ellipse having its axes along OX and OY. If $b = a$, the ellipse becomes a circle of radius a. Thus, uniform circular motion may be considered as made up of two simple harmonic motions along perpendicular lines, the motions having equal amplitudes and periods, but having phases differing by a quarter period.

If in (III.50) we put $\beta = 0$ the equation reduces to

$$\left(\frac{y}{b} - \frac{x}{a}\right)^2 = 0$$

or

$$\frac{y}{b} - \frac{x}{a} = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(III.52)}$$

the equation to a straight line through the origin of slope $\frac{b}{a}$. Thus if α and β are equal, the particle having the sum of the motions will oscillate along the line $y = \frac{b}{a}x$ with simple harmonic motion of period $\frac{2\pi}{\omega}$ and amplitude $\sqrt{a^2 + b^2}$.

All the ellipses represented by (III.50) lie inside of the rectangle formed by the lines $x = \pm a$, $y = \pm b$.

EXAMPLE.

The end S of the shaft in Art. 50, is given an oscillatory motion represented by $\theta' = \theta_1 \sin qt$, where θ' is the angle turned through when the time is t sec.; find the motion of the disc S_1.

We have

Turning moment = I × angular acceleration

and, using the above symbols and those of Art. 50, the turning moment is $-C_o(\theta - \theta')$. The equation of motion is therefore

$$-C_o(\theta - \theta') = \frac{W}{g} k^2 \frac{d^2\theta}{dt^2}$$

$$\therefore \frac{W}{g} k^2 \frac{d^2\theta}{dt^2} = -C_o(\theta - \theta_1 \sin qt)$$

or

$$\frac{d^2\theta}{dt^2} + \frac{gC_o}{Wk^2}\theta = \frac{gC_o\theta_1}{Wk^2} \sin qt$$

θ_1 is, of course, a constant. To find a particular integral of this, substitute

$$\theta = A \sin qt$$

We have then the identity

$$-Aq^2 \sin qt + \frac{gC_o}{Wk^2} A \sin qt = \frac{gC_o\theta_1}{Wk^2} \sin qt$$

Dividing through by sin qt, and solving for A, we find that

$$A = \frac{\dfrac{gC_o\theta_1}{Wk^2}}{\dfrac{gC_o}{Wk^2} - q^2}$$

and the particular integral is

$$\theta = \frac{\dfrac{gC_o\theta_1}{Wk^2}}{\dfrac{gC_o}{Wk^2} - q^2} \sin qt$$

Put n^2 for $\dfrac{gC_o}{Wk^2}$, and the particular integral becomes

$$\theta = \frac{n^2\,\theta_1}{n^2 - q^2} \sin qt \quad . \quad . \quad \text{(III.53)}$$

To this must be added the solution of (III.25), which is given in (III.26). The full solution is, then,

$$\theta = \theta_o \cos (nt + g_1) + \frac{n^2\,\theta_1}{n^2 - q^2} \sin qt \quad . \quad . \quad \text{(III.54)}$$

The values of g_1 and θ_o depend upon the initial displacement and motion of the free end, and θ_1 and $\dfrac{q}{2\pi}$ are the amplitude and frequency respectively of the forced oscillation. If the initial motion is such that $\theta = 0$ when $t = 0$, then from (III.54), $0 = \theta_o \cos g_1$, or $g_1 = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$. If at the same instant, i.e. $t = 0$, the free and forced oscillations are in the same sense, g_1 must have the value $\dfrac{3\pi}{2}$, and (III.54) becomes

$$\theta = \theta_o \cos \left(nt + \frac{3\pi}{2}\right) + \frac{n^2\,\theta_1}{n^2 - q^2} \sin qt$$

or

$$\theta = \theta_o \sin nt + \frac{n^2\,\theta_1}{n^2 - q^2} \sin qt$$

After a time, friction will damp out the free oscillations and the motion will then be represented by (III.53).

EXAMPLES 3.

1. Define simple harmonic motion and show that when this occurs the time of oscillation is $2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$.

2. Prove formulae for the time of oscillation of (1) a simple pendulum, (2) a compound pendulum, and (3) a weight oscillating along a vertical line when suspended from the end of a vertical spiral spring, and show that the time of oscillation for (3) is that of a simple pendulum whose length is equal to the static extension of the spring due to the weight.

3. A spring carries a mass of 40 lb. weight at its lower end. The spring stretches 1 in. when under a tension of 20 lb. Neglecting the mass of the spring, find the time of oscillation of the arrangement. If the amplitude of the oscillation is 1·25 in., find (1) the maximum velocity, and (2) an expression for the force acting on the mass in terms of the time t sec.

4. A sphere 1 ft. diameter oscillates about a horizontal axis through a point on its surface. If the sphere is homogeneous, find its time of oscillation, also equivalent dynamical system, one mass being on surface of sphere (Art. 41).

5. A uniform circular disc of mass 10 lb. weight and radius of gyration about its axis of 5 in. is attached to one end of a shaft, the other end of which is fixed, the axes being coincident. The shaft turns through an angle of 4° when a turning moment of 20 lb.-in. is applied. Find the period of elastic torsional oscillations.

6. The shaft in the last example is supported in frictionless bearings, and has both ends free, one carrying the given mass, and the other a mass of twice the moment of inertia of the given one about the axis of the shaft. Find the time of oscillation of the system, assuming that the two masses rotate in opposite senses. If the shaft is 3 ft. long, where is the node, or stationary point ?

7. Explain what is meant by (1) the centre of suspension, (2) the centre of oscillation of a compound pendulum, and show that the two may be interchanged without effect on the time of oscillation of the pendulum.

8. K and h are respectively the radius of gyration about the centre of oscillation, and the distance between this centre and the centre of gravity of a compound pendulum. Show that the length of the equivalent simple pendulum is $l = \frac{K^2}{h}$.

9. If in the last example k is the radius of gyration about a horizontal axis through the centre of gravity, what is l as a function of k and h ? If k is kept constant, and h varied, find $\frac{dl}{dh}$, and show that the latter is negative if h is less than k. Hence show that if h is less than k, an increase in the value of h leads to a decrease in the periodic time, whereas if h is greater than k, the contrary is the case.

10. Write a short note on forced oscillations. Find the amplitude of oscillation of the mass in example 3 if the upper end of the spring is given a vertical motion represented by $y = 2 \cos 3t$ in. The amplitude is required some time after the free oscillations have been damped out.

11. Show how to find equivalent cranks for the sum and the difference of the motions of two cranks, assuming the motion due to each crank to be simple harmonic and of the same period. Apply your method to find equivalent cranks for $x_1 + x_2$ and $x_2 - x_1$ having given $x_1 = 4 \cos 3t$ and $x_2 = 5 \cos (3t + 2)$.

12. A particle has motions represented by $x = 2 \cos 3t$, and $y = 3 \sin 3t$ along the co-ordinate axes OX and OY respectively, $\widehat{XOY}$ being a right-angle.

Find the equation of its locus. Find the slope of the tangent to the locus at the time when $t = t_0$.

13. Find the equation to the locus of a particle which has equal periods of oscillation along rectangular axes OX and OY, the amplitudes being 3 in. and 2 in. respectively, having given that $y = 1$, when $x = 3$.

14. Two ends of a long horizontal rod receive simple harmonic motions along lines which are approximately vertical. The amplitudes of these oscillations are 5 in. and 3 in. respectively, and the phase of the smaller oscillations is 90° in front of those of the larger. Find the amplitude and phase of the vertical oscillations of a point (*a*) halfway, and (*b*) three-quarters way along the rod from the end making the largest oscillations.

15. A horizontal beam of elastic material supported or fixed at the ends carries a load at the centre of the span. Show that (neglecting the mass of the beam) the period of free vibration of the load on the beam is the same as that of a simple pendulum whose length is equal to the static deflection of the beam under the load.

16. A connecting-rod weighs 500 lb and is 6½ ft. long. It is hung from a knife-edge at its small end. The centre of gravity of the rod is 47 in. from the point of suspension, and the time taken for the rod to complete an oscillation to and fro is 2·6 sec. Determine its radius of gyration and the moment of inertia about an axis through the centre of gravity and perpendicular to the plane of oscillation. (U.L.)

17. Find an expression for the natural frequency of vibration of a system of two unequal flywheels mounted on the ends of the same shaft. Calculate the frequency in the case of two equal wheels each 1,000 lb. in weight with a radius of gyration of 30 in., mounted at the ends of a 3½ in. shaft, 30 in. long. (U.L.A.)

18. A uniform circular plate is suspended with its plane horizontal by means of three fine threads, which are vertical and are attached to the ceiling and to three equidistant points in the rim of the plate. If l ft. is the length of each string, r ft. the radius of the plate, k ft. its radius of gyration about its axis, and W the weight of the plate in pounds, find the time of oscillation of the system for small angular displacement about its axis. If $l = 12$, $r = 1$, $k = {\cdot}71$, $W = 20$, find the time of oscillation in seconds.

19. The upper end of the spring in example 3 above begins to oscillate when the weight is at rest, the motion of the upper end being given by $\cdot 3 \sin 7t$ ft. Find the displacement of the weight after time t sec., neglecting friction.

20. Find an expression for the displacement of the bob of a simple pendulum of length l ft., when its upper end executes small oscillations in a horizontal line of magnitude $x = A \sin \alpha t$, where A is small, and α^2 is not nearly equal to $\frac{g}{l}$. (Assume that friction has damped out the free oscillations.)

21. In order to find the moment of inertia of a wheel which is free to rotate about its own axis, a weight w lb. is fixed to the wheel at a distance r ft. from the axis of rotation. If T secs. is the time of a complete small oscillation, show that the moment of inertia of the wheel is given by

$$I = w \, . \, r \left\{ \frac{T^2}{4\pi^2} - \frac{r}{g} \right\}$$

22. A turbine disc has a central hole 4 in. in diameter. It is supported on a knife-edge at one end of a diameter of the hole and parallel with the axis, and in this position the time of a small double oscillation is 3·2 secs. Calculate the radius of gyration of the disc.

If the centre of gravity is not exactly in the centre of the disc, show how by four observations similar to the above the exact position of the C.G. may be found.

CHAPTER IV

PAIRS—ELEMENTS—MECHANISMS CONTAINING FOUR LOWER PAIRS—INVERSION

57. **Pairs, Elements, Lower and Higher Pairs.** Having examined the functions of a machine, we shall proceed to study the relative motions of its parts. A *machine* is made up of a number of bodies, one of which is fixed, connected together so as to be capable of determinate relative motion. Two parts of a machine which are in contact and between which there is relative motion are known as a pair, the two parts being *elements* of the pair. As the motions of all the bodies composing the machine are constrained, each body must form an element of at least two pairs. The nature of the contact between the elements of a pair must be such as to permit of the required relative motion. Kinds of relative motion which are commonly required are sliding motion, rotational motion or turning, and screw motion. Pairs which permit of these three kinds of motion are respectively called *sliding pairs*, *turning pairs*, and *screw pairs*. These three pairs form an exclusive class known as *lower pairs*. All other pairs are *higher pairs*, the relative motion between the elements of such pairs being partly turning and partly sliding. Examples of higher pairs are a belt and the pulley with which it is connected, a pair of teeth gearing together one on each of two toothed wheels, two wheels in rolling contact, etc.

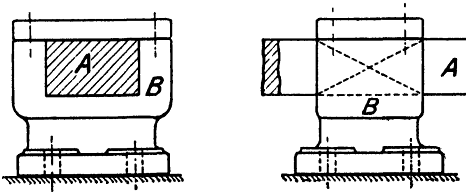


FIG. 32. SLIDING PAIR

58. **Sliding Pairs—Constraint—Closure.** In Fig. 32, *A* and *B* form a sliding pair, *A* is a prism, or a non-circular cylinder closely fitting a recess in *B*. The only possible relative motion of *A* and *B* is that of sliding in the direction of the length of *A*.

The relative motion is thus completely constrained. When the connection between the two elements is such as to allow only the required kind of relative motion to occur the pair is *self-closed.* In many cases the elements of a pair are kept in contact by the action of external forces as, for example, in the cases of a cylindrical shaft in a half-cylindrical bearing, the cross-head and single-slide bar of a steam engine whose crank-shaft rotates only in one sense, etc. Such pairs are said to be *force-closed.* In the examples given, the geometrical constraint is incomplete, but the constraint is completed by *force-closure.*

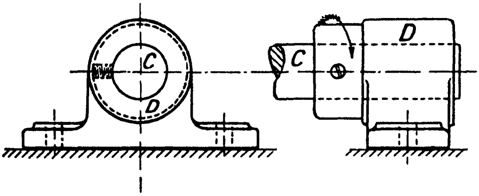


Fig. 33. Turning Pair

59. **Turning Pairs.** *C* and *D* (Fig. 33) form a turning pair. *C* is a circular cylinder fitting closely a cylindrical hole in *D*. The common surface of contact of *C* and *D* may be that of any solid of revolution having the same axis as *C*. If the common surface is not cylindrical, or if there are two flanges on *C*, one on each side of *D*, and the surface is that of a circular cylinder, the closure is complete, and the only possible kind of relative motion is that of rotation about the axis of *C*, the pair being thus self-closed. If *D* does not completely surround *C*, but only covers one portion not greater than half the circumference of *C*, or if *C* can move laterally, as in Fig. 33, the geometrical constraint is incomplete, and force-closure is necessary for complete constraint.

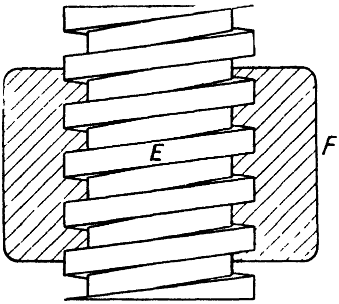


Fig. 34. Screw Pair

60. **Screw Pairs.** *E* and *F* (Fig. 34) form a screw pair. *E* is a circular cylinder with a helical projection called the thread. The inner surface of *F* closely envelops the outer surface of *E*. The elements *E* and *F* are capable of relative motion made up of (1) rotation about the axis of *E*, and (2)

displacement along that axis, the angle of rotation being proportional to the length of the displacement. In Fig. 34, the surface of contact is made up of two helical surfaces (one on each side of the threads) and a cylindrical surface. When the threads are V-shaped (Fig. 127), there is no cylindrical portion in the common surface. For complete geometrical constraint both the helical surfaces must be in contact for all relative positions of the elements, and the surface of contact must extend over more than half the circumference.

It should be noticed that with lower pairs the contact between the elements takes place over a common surface. The area of this surface is fixed by practical considerations, such as the need for preventing wear and for the reduction of friction by means of lubrication, etc. In higher pairs the contact is usually line contact, though in some cases, e.g. a belt and a pulley, the contact is surface contact. A body forming an element of each of two pairs is known as a *link*.

61. **Kinematic Chain, Mechanism, Machine.** An assemblage of links and pairs is called a *kinematic chain* if, and only if, its parts are capable of relative motion and it can be made into a rigid frame by joining any two elements of the pairs by a single link. A *closed kinematic chain* is one in which all the pairs are closed (self-closed or force-closed) and the motion is fully constrained. When one of the links of a kinematic chain is fixed, the chain becomes a *mechanism*, and if the mechanism performs the functions described in Art. 1 it is a *machine*. As any link of a kinematic chain may be fixed, we can obtain from the kinematic chain as many mechanisms as the chain has links, though two or more of these mechanisms may be alike. This method of obtaining different mechanisms, by fixing in turn different links in a kinematic chain, is known as *inversion*, and is dealt with in Art. 65.

62. **Number of Parts in a Kinematic Chain.** We usually represent each link of a kinematic chain by a straight line, and a turning pair joining two links by a dot at the intersection of the two corresponding lines. A rectangle with a dot inside of it represents a link which forms with its two adjacent links a sliding pair and a turning pair respectively. *ABCD* (Fig. 35) represent a kinematic chain, known as the four-bar crank chain, or the quadric cycle chain. *AB*, *BC*, *CD*, and *AD* are four bars, adjacent bars being joined by turning pairs at the four points *A*, *B*, *C*, and *D*.

Fig. 36 represents the simple slider crank chain from which the simple steam engine mechanism (Fig. 42) is derived by fixing the link *OR*. *OC* and *OR* are joined by a turning pair at *O*, *OC* and *CP* by a turning pair at *C*. The rectangle *Q* represents a link which forms a sliding pair with *OR* and a turning pair with *CP*. In each of these cases there are four links and four pairs, each link is an element of only two pairs, and each pair joins only two elements.

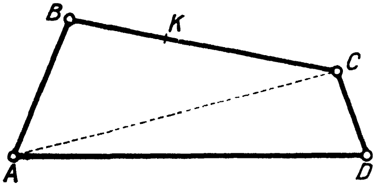


Fig. 35. Four-bar Crank Chain

If in Fig. 35 the elements of the turning pair at *B* are supposed to become one solid piece, the portion *ABC* is equivalent to a single link *AC*, shown dotted, and the linkwork *ACD* is a rigid frame and not a kinematic chain. On the other hand, if, when *B* is a turning pair, we introduce a turning pair at some point *K* in *BC*, joining *BK* and *KC*, the linkwork is deformable even when the positions of two adjacent links are fixed. Thus, if a turning pair is introduced at *K*, *ABKCD* is not a kinematic chain.

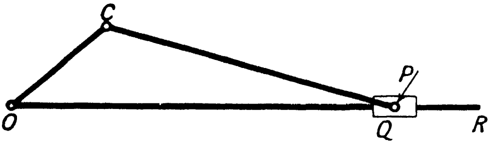


Fig. 36. Single-slider Crank Chain

We shall now obtain a formula connecting the number of links with the number of pairs in a kinematic chain, which formula will enable us to examine a complicated arrangement of links and pairs so as to determine whether or not it is a kinematic chain. A kinematic chain may be made into a rigid assemblage of links by the addition of an extra link connecting any two points, the distance between which varies when the links are in relative motion. Suppose two such points to be joined by a link and let them be points at which turning pairs already exist. The linkwork is now undeformable. Three links connected in the form of a triangle with turning pairs at their intersections are the simplest undeformable linkwork. By connecting two more links by turning pairs each to a point in the original triangular linkwork and then connecting the two bars by a turning pair, we obtain an undeformable linkwork of five links.

We may go on increasing the number of links in the undeformable linkwork by two at a time, and thus increasing the number of turning pairs by one at a time. This assumes that when we connect two new links we connect them to the old links at the points where turning pairs already exist. If this is not the case, we leave out of our reckoning all pairs in excess of two on any one link. We require, then, three links and three pairs to form the original triangular linkwork, and afterwards for each additional pair we require two extra links. Thus, if L = number of links, and p = number of pairs, we have

$$\begin{aligned} L &= 3 + 2\,(p - 3) \\ &= 2p - 3 \end{aligned}$$

This gives the number of links to form a rigid frame. If now we remove a link, the rigid frame becomes a kinematic chain and

$$L = 2p - 4 \quad . \quad . \quad . \quad . \qquad \text{(IV.1)}$$

This formula only applies to cases where p is not less than 4.

If L is greater than $2p - 4$, the linkwork is rigid, and if L is less than $2p - 4$, the linkwork is capable of deformation when two adjacent links are fixed in space. In using (IV.1) two, and only two, pairs for each single bar must be counted, and each pair must be counted only once. A sliding pair is equivalent to a turning pair at infinity, since a straight line may be considered as a circle with its centre at infinity. The relative motion between elements of a higher pair is, as we have previously stated, a combination of sliding and turning, a relative motion which could be produced by a sliding pair and a turning pair connected by a link. On this account, a higher pair must be considered to be equivalent to two lower pairs and one link. In the table on page 92 we give a list of some of the simplest linkages and typical mechanism derived from them.

The reader should draw diagrams of each mechanism and identify the elements and pairs. As an example, Fig. 37 represents the wedge. A, B, and C are the links, and the sliding pairs are A and B, B and C, and C and A. A force

Number and Nature of Pairs	Number of Links	Typical Derived Mechanism
1 turning	2	Simple lever. Wheel and axle
1 sliding	2	Inclined plane
1 screw	2	Screw jack
2 lower	3	No possible kinematic chain, linkage not closed
3 turning	3	No possible kinematic chain, linkage rigid
2 turning, 1 sliding . .	3	No possible kinematic chain, linkage rigid
1 turning, 2 sliding . .	3	No possible kinematic chain, linkage rigid
3 sliding	3	Wedge
1 screw, 1 turning, 1 sliding .	3	Screw press

P is shown acting on the wedge and causing C to move against a resistance W.

63. Possible Mechanisms of Four Links containing only Turning and Sliding Pairs. Each pair in such a mechanism may be either a sliding pair or a turning pair. Four pairs may therefore be arranged in different orders in $2^4 = 16$ ways. As, however, a mechanism is not altered by rearranging the pairs in the reverse order round the linkage, all arrangements in which the order of the pairs is the same in both ways round the linkage, starting from, and finishing at, a fixed link, will be counted twice in the above number. The arrangements which will not be counted twice will be those consisting of (1) four turning pairs, (2) four sliding pairs, (3) two sliding pairs connecting to the fixed link and two turning pairs, and (4)

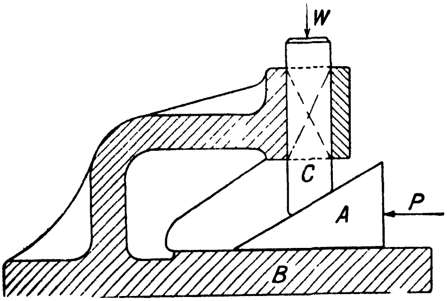


FIG. 37. THE WEDGE

two turning pairs connecting to the fixed link and two sliding pairs. Thus, all but four arrangements will occur in duplicate. The number of distinct arrangements is, then, $4 + \frac{16-4}{2} = 10$.

These are not necessarily possible mechanisms. We cannot, for instance, have either of the two arrangements consisting of four elements connected by three sliding pairs and one turning pair, for this would necessitate a link turning relative to another link, to which it would be connected by a sliding pair. An arrangement of four sliding pairs (Fig. 38) is of little, if any, practical value for the transmission of energy or force, as owing to the effects of friction the arrangement is not easily deformable.* The practical mechanisms of the kind under consideration are, therefore, seven in number, and are limited to (1) those with four turning pairs, or the four-bar mechanism, (2) those with one sliding and three turning pairs, and (3) those with two sliding and two turning pairs. Calling the fixed link 1, and the others in order round the mechanism, 2, 3, and 4, the following are the seven possible arrangements. A dot indicates a turning pair and a dash a sliding pair. The names of some of the derived mechanisms are given in the third column.

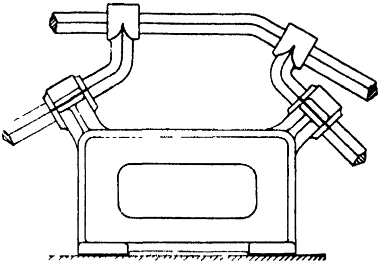


FIG. 38. ARRANGEMENT OF FOUR SLIDING PAIRS

No	Arrangement of Pairs	Names of Mechanisms
1	1 · 2 · 3 · 4 · 1	Four-bar mechanism
2	1—2 · 3 · 4 · 1	Simple engine, pendulum pump
3	1 · 2—3 · 4 · 1	Quick return motion, oscillating cylinder engine Gnome rotary engine
4	1—2—3 · 4 · 1	Double slider crank chain or donkey engine mechanism
5	1—2 · 3 · 4—1	Elliptic trammels
6	1 · 2—3— 4 · 1	Oldham's coupling
7	1—2 · 3—4 · 1	Rapson's slide

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* See Example 24, page 113.

64. Mechanisms obtained by Changes in the Dimensions of the Four-bar Mechanism. Fig. 39 shows the four-bar mechanism (derived from the four-bar chain by fixing a link) as applied for use in a *beam engine*, 1 being the fixed link. The link 2 is extended so as to form the beam ST, pivoted at O. The beam is given an oscillatory motion at S, and the function of the mechanism is to produce rotary motion of the link 4, P being the centre of rotation. The lengths of the links 3 and 4 must be such that with T in its highest position $\overline{TP}$ is the sum of the length of 3 and 4, and with T in its lowest position $\overline{TP}$ is the difference of these lengths.

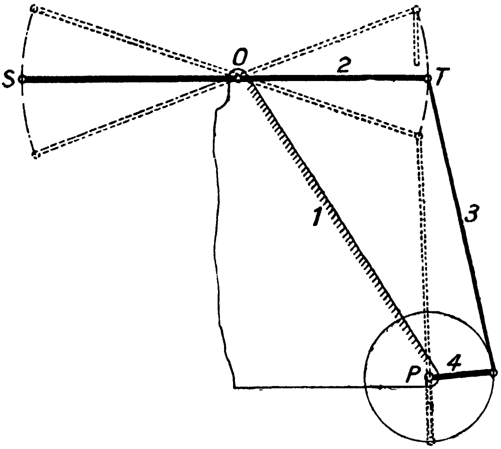

FIG. 39. BEAM ENGINE MECHANISM

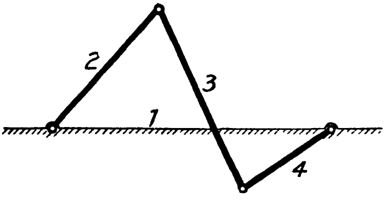

FIG. 40. FOUR-BAR MECHANISM WITH CROSSED LINKS

Fig 40 shows the four-bar mechanism in the form adopted for use in *Watt's parallel motion*. The links 1 and 3 cross, 1 being fixed. The links 2 and 4 are arranged to oscillate about their mean positions. The functions of this mechanism will be dealt with in Art 111. If the four bars are in the form of a parallelogram with one of the long sides fixed, we have the coupling gear for connecting locomotive wheels. The fixed link is the locomotive frame, and the two short links are radii of the coupled wheels. The same arrangement gives a mechanism in which the link opposite to the fixed link keeps its direction unchanged throughout its motion. This mechanism has many applications, as in Roberval's balance where it keeps in a vertical position the rods carrying the scale pans, and in ropemaking machinery, where it keeps the bobbin in a vertical position whilst the latter is carried bodily round a fixed horizontal axis.

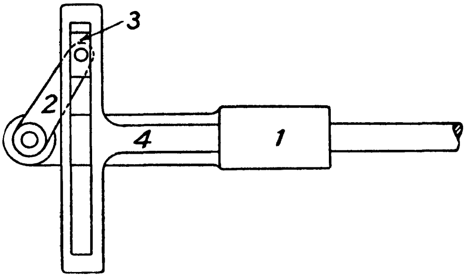

FIG. 41. DOUBLE-SLIDER CRANK CHAIN

By changing the dimensions of the links of the simple engine mechanism (Fig. 42), we can obtain a mechanism known as the pendulum pump (Fig. 45). Though the elements are differently numbered in the two figures, the reader will easily see that the pendulum pump is the simple engine mechanism with the crank longer than the connecting rod.

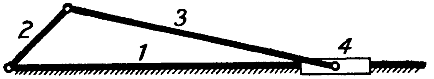

FIG. 42. SIMPLE ENGINE MECHANISM

65. **Inversion of Kinematic Chains.** The mechanisms tabulated in Art. 63 are obtained from only four kinematic chains. These chains are (1) the four-bar chain (Fig. 35), (2) the simple slider crank chain (Fig. 36), (3) the double slider crank chain (Fig. 41), and (4) a chain having alternate sliding and turning pairs. From these chains, by fixing different links in turn, we obtain the tabulated mechanisms. The method of obtaining one mechanism from another by using the same kinematic chain and fixing a different link is known as *inversion*, and the second mechanism is an inversion of the first, or of the kinematic chain.

Figs. 42, 43, 44 and 45 show the results of fixing in order the links 1, 2, 3 and 4 of the simple slider crank chain in which the links are connected by one sliding and three turning pairs. Fig. 42 is the simple steam engine mechanism, dealt with in Art. 72. Fig. 43 gives the mechanism of the Whitworth quick-return motion (Art. 66), and of the Gnome engine used in aircraft, in which the frame carrying the cylinders revolves, and the crank is stationary. Fig. 44 gives the mechanism of the oscillating cylinder engine, which is briefly described in Art. 67. Fig. 45 is the mechanism of the pendulum pump, which is of little practical importance. The mechanisms of Figs. 42 and 45 are kinematically identical, the only difference being in the dimensions of the links (Art. 64).

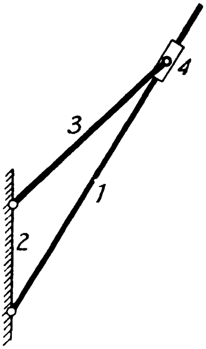

FIG. 43. MECHANISM OF WHITWORTH'S QUICK-RETURN MOTION

66. **Whitworth's Quick Return Motion.** The mechanism of Fig. 43 is reproduced in a practical form in Fig. 46. *S* is a spur-wheel, driven at uniform speed by the pinion *T*. The sliding piece *B* forms a turning pair, with a pin projecting from *S*, and slides in a slot in the slotted lever *PA*, which is pivoted at a fixed point *A*. *PR* is a connecting-rod which connects *P* to a slider carrying the cutting tool, which is constrained to move along the line *CD* produced. As *B* moves from *C* to *D* in the clockwise sense, the tool moves from the right-hand end of its stroke to the left-hand end through a distance $2\overline{AP}$; the tool moves back again whilst *B* moves from *D* to *C* in the same sense. The second stroke is the cutting stroke, and the first stroke is the quick return stroke. Thus, since *OB* has uniform angular velocity, we have the relation

$$\frac{\text{time of cutting stroke}}{\text{time of return stroke}} = \frac{360 - \alpha}{\alpha} \quad . \quad . \quad . \quad (\text{IV}.2)$$

where α is the angle *COD*. As the distance *AP* is variable, it is possible to vary the travel of the tool.

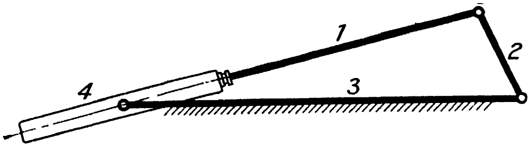

FIG. 44. OSCILLATING CYLINDER ENGINE

67. **The Oscillating Cylinder Engine and the Crank and Slotted Lever Quick Return Motion.** The mechanism of Fig.

44 may be put into two practical forms. The first of these is the oscillating cylinder engine, which differs from the simple engine mechanism by having the piston-rod connected directly to the crank-pin, there being no connecting-rod. The cylinder is carried on two trunnions, one on each side of the cylinder, which form a turning pair with the supporting framework. The second mechanism is the crank and slotted-lever quick-return motion (Fig. 47). This mechanism is similar to the Whitworth's quick-return motion, differing from it only in one respect; in Whitworth's motion the link *AP* revolves continually in one sense, whereas, in the mechanism of Fig. 47, as *A* is outside of the circle described by *B*, the link *AP* oscillates between two extreme positions AP_1 and AP_2. In these extreme positions AP_1 and AP_2 are tangential to the circle. The ratio of the times of the two strokes is given by (IV.2). The angle through which *AP* rocks is 180° − α. The cutting tool is constrained to move along a line perpendicular to the line *AO* produced. The travel of the tool is $\overline{P_1P_2}$ where

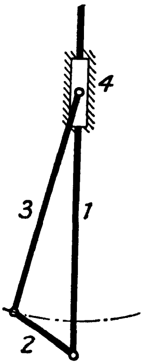

FIG. 45. PENDULUM PUMP

$$\begin{aligned}\overline{P_1P_2} &= 2\,.\,\overline{AP}\sin\widehat{P_1AO}\\ &= 2\,.\,\overline{AP}\cos\frac{\alpha}{2}\\ &= 2\,.\,\overline{AP}\,.\,\frac{\overline{OB}}{\overline{OA}}\end{aligned}$$

The reader will have no difficulty in finding the greatest and least cutting speeds of the tool in terms of the speed of the crank pin *B*. In this mechanism the ratio of the times given in (IV.2) may be varied by varying the distance $\overline{OA}$.

68. **Inversions of the Double-slider Crank Chain.** In Fig. 41 we have a diagram of the double-slider crank chain. Figs. 48, 49, and 50 show the inversions obtained by fixing in turn the links 1, 2, and 4.

The *double-slider crank mechanism* (Fig. 48) is well known. If the crank 2 rotates with uniform angular velocity, the piston-rod 4 executes simple harmonic motion.

Oldham's coupling (Fig. 49) is a mechanism for connecting

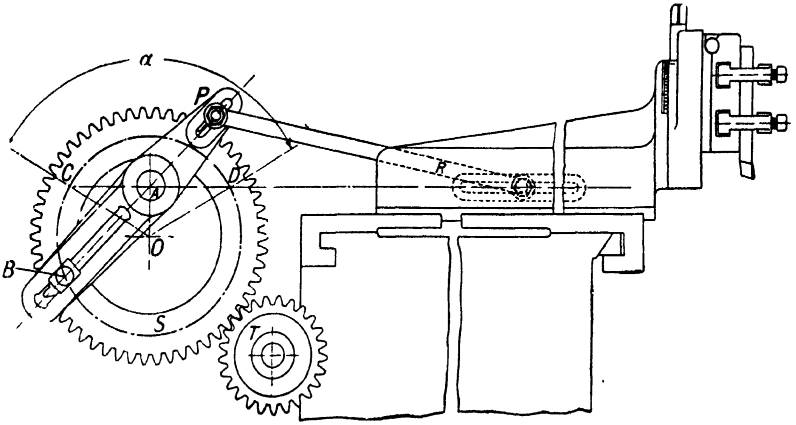

FIG. 46. WHITWORTH'S QUICK-RETURN MOTION

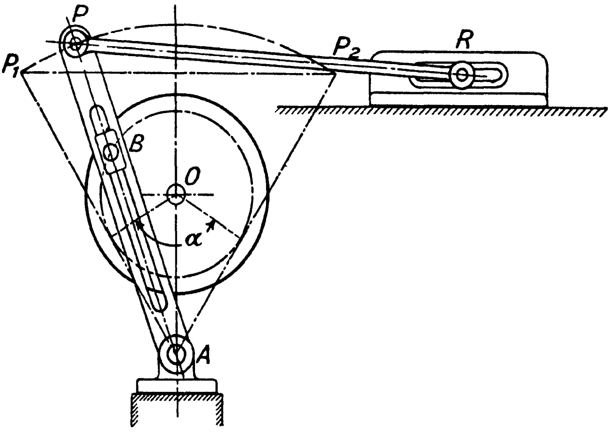

FIG. 47. CRANK AND SLOTTED LEVER QUICK-RETURN MOTION

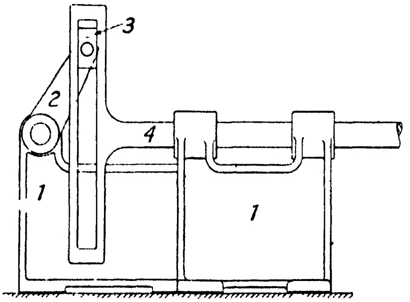

FIG. 48. DOUBLE-SLIDER CRANK MECHANISM

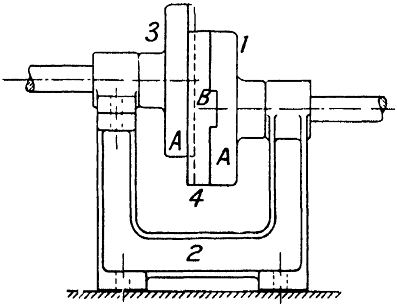

FIG. 49. OLDHAM'S COUPLING

two parallel shafts 1 and 3 whose axes are at a small distance apart. The shafts 1 and 3 form turning pairs with 2. Each shaft has a circular disc *A* fastened rigidly to it. These discs have diametrical grooves cut in their inner faces, and a disc *B* fits closely between the two discs *A*. *B* has a diametrical projection on each face, their directions being at right-angles. The projections on *B* closely fit the grooves in the discs *A*. If 1 is rotated, one disc *A* will cause *B* to rotate, which will in turn cause the other disc *A* and the shaft 3 to rotate. The velocity ratio of the shafts is unity in all positions. There is, of course, sliding motion between *B* and each of the other discs.

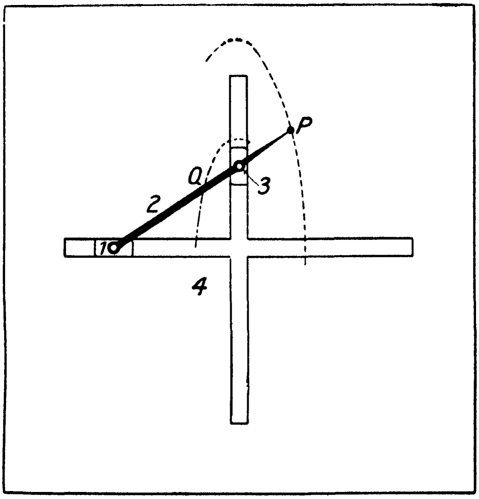

Fig. 50. Elliptic Trammel

The mechanism of Fig. 50 is called the *elliptic trammels*, and is an instrument for drawing ellipses. 4 is a fixed plate, having two straight grooves cut in it, the two grooves being at right-angles. 1 and 3 are sliding pieces, which slide in the grooves. 2 is a bar which forms turning pairs with 1 and 3. Any point *Q* traces out an ellipse on the surface of 4 when relative motion occurs. The proof of this is given in Art. 71, and a similar proof shows that a point *P*, which is on the extension of 2, also traces out an ellipse.

If the link 3 (Fig. 48) is fixed instead of the link 1, the mechanism is unchanged kinematically.

69. **Rapson's Slide** (Fig. 51) is the mechanism represented by arrangement 7 of the table of Art 63. 3 is the fixed link, and 4 and 1 are two sliding pieces, 4 sliding in a slot in 3, and 1 forming a sliding pair with 2, which last forms a turning pair with 3 at *R*. The links 4 and 1 are joined by a turning pair. The mechanism is used for steering ships. Link 2 is the tiller,

and R is the rudder-head. In steering, 4 is pulled either to right or left, along the slot, by means of a chain. There are two chains attached to 4, one running to the right and the other to the left.

70. Plane Motion of a Rigid Body. Instantaneous Centre of Rotation. Consider the motion of the rigid body shown in Fig. 52. Let A and B be any two points in the body and let the arrows at A and B represent their velocities. In any motion of the body the distance AB must remain constant on account of the rigidity of the body. Suppose AI and BI to be drawn perpendicular to the directions of the respective velocities at A and B. The motion of A and B, and therefore that of the whole body, may be considered at the instant as being one of rotation about I, for this would cause A and B to move in the required directions and would keep the length AB invariable. The point I is called the *virtual centre*, or the *instantaneous centre of rotation*, of the body. If v_A, the magnitude of the velocity of A, is known, we have

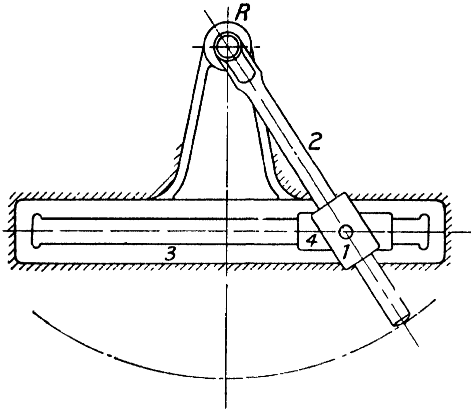

FIG 51. RAPSON'S SLIDE

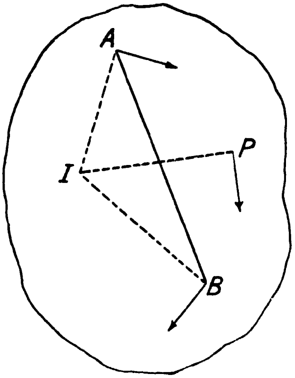

FIG. 52. POSITION OF INSTANTANEOUS CENTRE

$$\omega = \text{angular speed of rotation}$$

$$= \frac{v_A}{IA}$$

The magnitude of the velocity of any point P in the body is then given by $\omega . IP$, and its direction is perpendicular to IP. Thus, if we know the directions of motion of two points in the body, and the magnitude of the velocity of one of them, we can determine the velocity of any other point in the body. From the above we see that

$$v_P = \omega . \overline{IP} = \frac{v_A}{\overline{IA}} . \overline{IP}$$

therefore
$$\frac{v_P}{v_A} = \frac{\overline{IP}}{\overline{IA}} \quad . \quad . \quad . \quad . \quad . \qquad \text{(IV.3)}$$

that is, the velocities of any two points are proportional to their respective distances from the instantaneous centre of rotation.

71. **Space Centrode and Body Centrode.** In Art. 70 we saw that at any instant a rigid body in plane motion relative to a second rigid body supposed fixed in space, may be assumed to be rotating about an instantaneous centre. In other words, the instantaneous centre is a point in the body which may be considered fixed at any particular instant. The locus of the instantaneous centre in space during a finite motion of the body is called the "space centrode." The locus of the instantaneous centre relative to the body itself is called the "body centrode." These two centrodes have, at any instant, the instantaneous centre as a common point, and, during the motion of the body, the body centrode rolls without slipping over the space centrode.

EXAMPLE.

A line AB (Fig. 53) of fixed length slides with its ends A and B moving along the lines $X'OX$, $Y'OY$ respectively, the angle XOY being a right angle. Find the space and body centrodes.

The instantaneous centre for the position of AB shown in the figure is at I, where AI and BI are respectively perpendicular to OX and OY. As AB moves the point I moves in space and $IAOB$ is a rectangle for all positions of AB. As the diagonals of a rectangle are equal, $OI = AB =$ constant, and therefore the locus of I is a circle centre O and radius AB. This circle is the space centrode. Now consider the motion of I relative to the line AB. The angle AIB is a right-angle for all positions of I relative to AB, and consequently I lies on a circle having AB as diameter. This circle is the body centrode. Fig. 53 shows the two centrodes touching at I, and the path in space of AB is the same as if AB were attached to the smaller circle and carried with it in rolling without slipping on the inside of the larger circle, the latter being fixed in space. This arrangement is equivalent to the elliptic trammels (Art. 53). To prove that any point in AB describes an ellipse relative to the axes OX and OY, let P be any point in AB, and let θ be the angle OAB. Then if x and y are the co-ordinates of P and $\overline{AP}$ and $\overline{PB}$ are respectively equal to b and a.

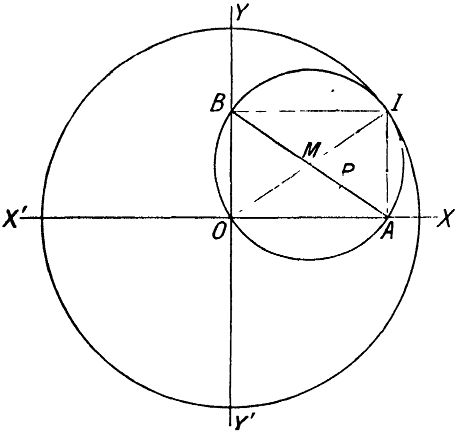

FIG. 53. ROLLING CIRCLE

$$x = \overline{BP} \cos\theta = a \cos\theta$$

$$y = \overline{AP} \sin\theta = b \sin\theta$$

from which

$$\cos\theta = \frac{x}{a} \text{ and } \sin\theta = \frac{y}{b}$$

Squaring these and adding, we have, since $\sin^2\theta + \cos^2\theta = 1$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(IV.4)}$$

which is the equation to an ellipse whose centre is at O and whose semi-axes are of lengths a along OX and b along OY. If $a = b$, the locus is a circle with O as centre. Thus the locus of M, the mid-point of AB, is a circle of radius $\tfrac{1}{2}\overline{AB}$.

72. **The Simple Steam Engine Mechanism.** This mechanism is shown again in Fig. 54. We leave out the piston and piston-rod because they are attached to, and have exactly the same motion as, the crosshead. The crank is usually assumed to rotate with uniform angular velocity. Let ω be this velocity. The instantaneous centre I is found as follows: I is on the perpendicular to the path of P, that is on IP, and is also on the perpendicular to the path of C, that is, on the radius OC. Thus I is the intersection of OC produced with

PI the perpendicular at P to OP. If ω is in radians per second, and r the length of OC in feet, we have,

$$V_C = \omega r \text{ ft. per sec.}$$

From the figure by (IV.3),

$$\frac{V_P}{V_C} = \frac{\overline{IP}}{\overline{IC}} \quad . \quad . \quad . \quad . \quad . \qquad \text{(IV.5)}$$

Let K be the point of intersection of PC produced and OK the perpendicular to OP at O.

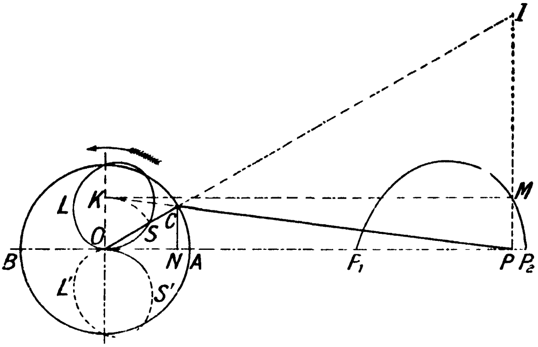

FIG. 54. CURVES OF PISTON VELOCITY

Then, by similar triangles,

$$\frac{\overline{IP}}{\overline{IC}} = \frac{\overline{OK}}{\overline{OC}}$$

$$\therefore \frac{V_P}{V_C} = \frac{\overline{OK}}{\overline{OC}}$$

and substituting $\dfrac{V_P}{\omega r} = \dfrac{\overline{OK}}{r}$

$$\therefore V_P = \omega \, . \, \overline{OK} \quad . \quad . \quad . \quad . \qquad \text{(IV.6)}$$

It should be noticed that $\overrightarrow{OC}$ represents V_C turned through an angle of 90°, also that $\overrightarrow{OK}$ represents the piston velocity turned through 90°; consequently $\overrightarrow{CK}$ represents the velocity of P relative to C, also turned through 90°.

In practice, ω varies owing to the fact that the turning moment applied to the crank is not constant. In many cases the resistance to motion of the machine or machines driven by the engine is also variable. It was shown in Chapter II that by fitting a suitable flywheel to the engine the variation in ω may usually be kept within certain limits. We shall, therefore, assume that ω is constant.

In this case, V_P is proportional to $\overline{OK}$, which therefore represents, to some scale, the speed of the crosshead and the piston to which this is attached. If, then, we draw a circular arc KS with O as centre, $\overline{OK}$ represents the piston speed for the crank position OC. If the point S is found for a series of values of the crank-angle $\widehat{POC}$ say, 0°, 10°, 20°, 30°, 40°, up to 180°, and these points have a smooth curve drawn through them, we obtain a *polar diagram of piston speeds* such that the intercept on any line representing the crank position represents to scale the speed of the piston for that position of the crank. OLS is such a diagram. If the construction be repeated for crank-angles from 180° to 360°, we shall obtain another curve $OL'S'$ below OP, and symmetrical about OP with OLS. These two loops form a complete polar diagram of piston speeds.

Rectangular Diagram of Piston Speeds. Let a line be drawn through K parallel to OP, and cutting PI in M (Fig. 54). Then $\overline{PM}$ represents the piston speed for the position P of the crosshead. As P may be considered to be part of the piston, if we find a point M for each of a series of positions of P, a smooth curve through these points will give us a curve of piston speeds such that the height at any point P represents the speed for the corresponding position of the piston. Such a curve is P_1MP_2.

It will be noticed from each curve, and may be seen from the method of construction, that the maximum speed of the piston occurs before the piston has travelled half-way from the right-hand end to the left-hand end of its stroke; the maximum speed occurs very approximately when the crank and connecting-rod are perpendicular. If the connecting-rod is so long that we may assume it to be practically horizontal all the time, the motion of P is the same as that of N, the projection on OP of the point C, and $\overline{PM} = \overline{OK}$ will be practically the same as $\overline{CN}$. Thus, P_1MP_2 will become a copy of the semicircle

ACB. It is easy to see also that the polar diagrams become circles on the vertical radii of the crank-pin circle as diameters. If the scale of the diagram *OCP* (Fig. 23) representing the mechanism is 1 in. to *a* ft., the scale of velocity is 1 in. to ωa ft. The velocity and acceleration of the piston are found analytically in Chapter II.

73. Line of Connection. In the four-bar mechanism (Fig. 55), in which *DC* is the fixed link, the axis of the link *AB* (called the coupler) is called the *line of connection*, and the point *K*, in which it meets the axis of the fixed link, is the *point of connection*. Suppose ω_1 and ω_2 to be the instantaneous angular velocities of the links *DA* and *CB* respectively, and let *DN* and *CM* be drawn perpendicular to the axis of *AB*. Then,

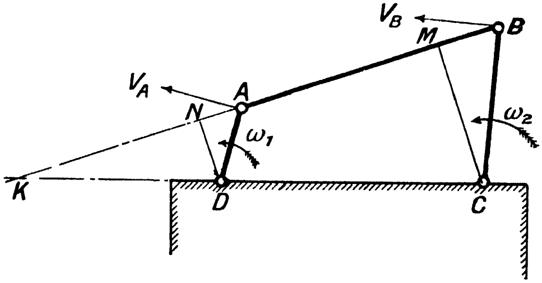

FIG. 55. LINE OF CONNECTION

$$\begin{aligned} V_A &= \text{velocity of } A \\ &= \omega_1 \, . \, \overline{DA} \end{aligned}$$

and the component of this along the line *BK* is $\omega_1 \overline{DA} \cos \widehat{ADN} = \omega_1 \, . \, \overline{DN}$. Similarly, the component of V_B, the velocity of *B*, along the line *BK*, is $\omega_2 \, . \, \overline{CM}$. Since the bar *AB* is of fixed length, the above two components are equal. Hence,

$$\omega_1 \, . \, \overline{DN} = \omega_2 \, . \, \overline{CM}$$

$$\therefore \frac{\omega_1}{\omega_2} = \frac{\overline{CM}}{\overline{DN}}$$

or, since *KND* and *KMC* are similar triangles,

$$\frac{\omega_1}{\omega_2} = \frac{\overline{KC}}{\overline{KD}} \quad . \quad . \quad . \quad . \quad . \quad \text{(IV 7)}$$

That is, *the point of connection divides the fixed link in the* *inverse ratio of the angular velocities of the cranks.* If the link AB crosses the link DC, K lies between D and C, otherwise K lies on the prolongation of DC.

The relation (IV.7) holds also in the case of belt or rope connections, the axis of the straight portion of the belt or rope being the line of connection, and in the case of two surfaces in rolling and sliding contact (Art. 152), provided that the surfaces do not separate during the motion; the line of connection in this last case is the common normal to the surfaces at the point of contact, for it is in the direction of this line that the surfaces have a common component of velocity.

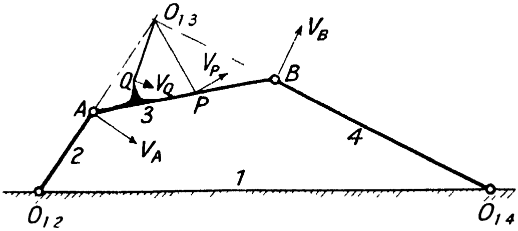

FIG. 56. USE OF INSTANTANEOUS CENTRE

74. Virtual Centres in the Four-bar Crank Mechanism. The four-bar mechanism (Fig. 56) has the link 1 fixed. The bars 2 and 4 rotate about the fixed points $O_{1\cdot2}$ and $O_{1\cdot4}$ respectively. The link 3 moves in the plane of the figure, and we have seen in Art. 70 that at any particular instant it may be considered to be rotating about an instantaneous or virtual centre. As A is a point in the link 2 which is permanently hinged at $O_{1\cdot2}$, A moves at the instant along a perpendicular to the link 2, and the point A in the bar AB, which moves with it, may therefore be considered as rotating about any point in the line of 2. Similarly, the point B in the link 3 may be considered as rotating about any point in the line of the link 4. Thus, the virtual centre of the link 3 is at the point $O_{1\cdot3}$, in which the lines of the links 2 and 4 intersect. If V_A, V_B, and V_P are the velocities of the points A, B, and P in the link 3, we have from (IV.5),

$$\frac{V_B}{V_A}=\frac{\overline{O_{1\cdot3}B}}{\overline{O_{1\cdot3}A}} \quad . \quad . \quad . \quad . \quad \text{(IV.8)}$$

and

$$\frac{V_P}{V_A}=\frac{\overline{O_{1\cdot3}P}}{\overline{O_{1\cdot3}A}} \quad . \quad . \quad . \quad . \quad \text{(IV.9)}$$

P being any point in the link 3. Similarly, V_Q may be found for any point Q carried by 3. The motions of A and B being known, those of any points in the links 2 and 4 may be readily found, as these links rotate about the fixed centres $O_{1 \cdot 2}$ and $O_{1 \cdot 4}$ respectively.

75. **Virtual Centres in the Four-bar Kinematic Chain.** If no one of the links in Fig. 57 is fixed, the points $O_{1 \cdot 2}$, $O_{1 \cdot 4}$, and $O_{1 \cdot 3}$ are still virtual centres, but the motion to which they now refer is in each case the relative motion of the two links denoted by the two figures in the suffix. Thus, $O_{1 \cdot 2}$ is the centre of relative rotation of the two links 1 and 2, $O_{1 \cdot 4}$ is that of the links 1 and 4, and $O_{1 \cdot 3}$ is that of the two links 1 and 3.

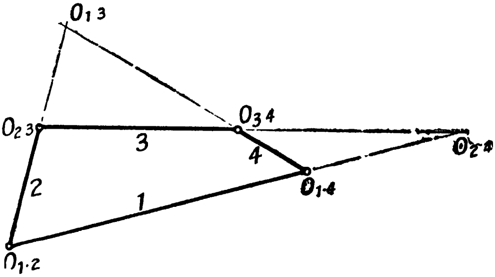

FIG. 57. VIRTUAL CENTRES

In Fig. 57, the virtual centres of relative rotation of all possible pairs of links are marked. Each letter O marks a virtual centre, and the two figures in the suffix denote the two links to which the centre refers. There are six virtual centres, in all, and on the line of each link there are three virtual centres. The virtual centres on the line of any one link are those relating to only three links out of the four, i.e. the link on which the centre occurs and the two adjacent links.

When three bodies are in relative motion, the three virtual centres of relative motion must lie on a straight line. Consider the three bodies represented by the links 2, 3, and 4 (Fig. 57). If now we consider $O_{3 \cdot 4}$ as a point in the link 3 its velocity, relative to the link 2, is directed along the perpendicular to the link 3. But $O_{3 \cdot 4}$ may also be considered as a point in the link 4, and its motion relative to 2 as one of rotation about $O_{2 \cdot 4}$. Hence, the velocity of the point $O_{3 \cdot 4}$, relative to the link 2, is directed along a line perpendicular to the line joining $O_{2 \cdot 4}$ and $O_{3 \cdot 4}$. This direction must agree with that obtained by the previous consideration, which agreement can only occur if the point $O_{2 \cdot 4}$ lies on the line passing through the points $O_{2 \cdot 3}$ and $O_{3 \cdot 4}$. Thus, any line passing through two virtual centres passes also through a third centre. This theorem is often useful in enabling us to find virtual centres when the linkwork is not so simple as that of Fig. 57.

76. **Virtual Centres in the Double-slider Crank Mechanism.** We shall now determine the virtual centres of rotation of the links of the double-slider crank mechanism (Fig. 58). $O_{1\cdot 2}$ is the turning pair between links 1 and 2, and $O_{2\cdot 3}$ is that between 2 and 3, hence $O_{1\cdot 3}$ lies somewhere on the line passing through $O_{1\cdot 2}$ and $O_{2\cdot 3}$. Since the relative motion between 1 and 4 is that of sliding, $O_{1\cdot 4}$ is at infinity along a vertical line. This vertical line may be drawn through any point, say $O_{1\cdot 2}$, and since $O_{1\cdot 2}$ and $O_{1\cdot 4}$ are on this line, so also is $O_{2\cdot 4}$. Again, the relative motion between 3 and 4 is that of sliding, hence $O_{3\cdot 4}$ is at infinity along a horizontal line. But $O_{2\cdot 3}$, $O_{2\cdot 4}$, and $O_{3\cdot 4}$ are on the same line, hence $O_{2\cdot 4}$ is at the intersection of a horizontal line through $O_{2\cdot 3}$ and a vertical line through $O_{1\cdot 2}$, and $O_{3\cdot 4}$ is at infinity along the line through $O_{2\cdot 4}$ and $O_{2\cdot 3}$. We have now found all the centres except $O_{1\cdot 3}$, which lies on the line through $O_{1\cdot 2}$ and $O_{2\cdot 3}$. $O_{1\cdot 4}$ $O_{1\cdot 3}$, and $O_{3\cdot 4}$, lie on the same line, and since the first and last of these points are at infinity, so also is the other point. Thus, 1·3 is at infinity along the line of the crank. When a point is at infinity, it may be assumed to be on any one of a series of parallel lines.

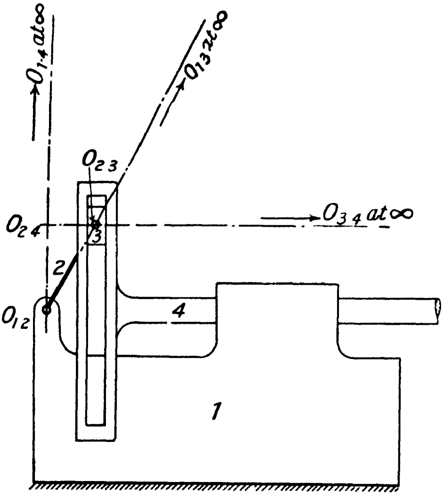

Fig. 58. Virtual Centres

77. **Relative Angular Velocities of Links—Rubbing Speeds in Turning Pairs.** We indicated in Art. 74 the use of the virtual centre for determining the velocities of various parts of a machine. We shall now show how to find the relative angular velocities of links in a mechanism. Suppose that in Fig. 57 the angular velocities of the links 1, 2, 3 and 4 relative to some frame of reference are ω_1, ω_2, ω_3 and ω_4 respectively.

The angular velocities of the links 2, 3, and 4, relative to the link 1, are $\omega_2 - \omega_1$, $\omega_3 - \omega_1$, and $\omega_4 - \omega_1$ respectively, and we have from (IV.7),

$$\frac{\omega_4 - \omega_1}{\omega_2 - \omega_1} = \frac{\overline{O_{1\cdot2}O_{2\cdot4}}}{\overline{O_{1\cdot4}O_{2\cdot4}}} . \qquad \text{(IV.10)}$$

Similarly,

$$\frac{\omega_4 - \omega_3}{\omega_2 - \omega_3} = \frac{\overline{O_{2\cdot3}O_{2\cdot4}}}{\overline{O_{3\cdot4}O_{2\cdot4}}} . \qquad \text{(IV.11)}$$

$$\frac{\omega_3 - \omega_2}{\omega_1 - \omega_2} = \frac{\overline{O_{1\cdot2}O_{1\cdot3}}}{\overline{O_{2\cdot3}O_{1\cdot3}}} . \qquad \text{(IV.12)}$$

and

$$\frac{\omega_3 - \omega_4}{\omega_1 - \omega_4} = \frac{\overline{O_{1\cdot4}O_{1\cdot3}}}{\overline{O_{3\cdot4}O_{1\cdot3}}} . \qquad \text{(IV.13)}$$

The expressions in the numerators and denominators of the fractions on the left-hand side of the above relations are the relative angular velocities of the various pairs of links which are connected by turning pairs. If the relative angular velocity between two links is multiplied by the radius of the common surface of the turning pair connecting them, the result is the rubbing speed between the surfaces of the elements of the pair.

78. **Virtual Centre of a Circular Cylinder Rolling on a Fixed Plane.** In Fig. 59, the circle *PQR* represents the end view of a cylinder rolling without slipping towards the right on the plane whose edge view is the line *PT*. It is sufficient to consider the motion of the circle *PQR* on the line *PT*. The centre *O* of the circle moves along a line parallel to *PT* and the virtual centre of the wheel is therefore on the line *OP*. The velocity of *P* is at right-angles to the line joining *P* to the virtual centre, i.e. at right-angles to *OP*. As, however, there is no sliding between the cylinder and the plane, the velocity of *P* is zero, and *P* is the virtual centre of rotation. The velocity of *Q* is in a direction perpendicular to *PQ*, and since the angle *PQR* is a right-angle, *PR* being a diameter of the circle, *Q* is moving along the line *QR*. Thus, all points of the circumference of the circle move instantaneously along lines directed towards or from the highest point *R*. Also $\frac{V_R}{V_o} = \frac{\overline{PR}}{\overline{PO}} = 2$, or the point

R has twice the linear velocity of the point O, which is $\omega \, . \, \overline{PO}$ where ω is the angular velocity. Also

$$\frac{V_Q}{V_R} = \frac{\overline{PQ}}{\overline{PR}} = \cos \widehat{RPQ}$$

$$\therefore \; V_Q = V_R \cos \widehat{PRQ}$$

or
$$V_Q = 2V_o \cos \widehat{RPQ} \quad . \quad . \quad . \qquad \text{(IV.14)}$$

If there is slip between the cylinder and the plane, so as to reduce the velocity of O for a given angular velocity ω of rotation of the cylinder, then P on the cylinder will move to the left relative to the point P on the plane. O still moves to the right, and P moves to the left, so that there is some point S on OP between O and P, which moves neither one way nor the other. S is the virtual centre. If ω is the angular velocity of the cylinder, we have

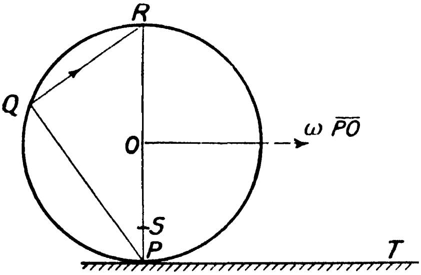

Fig. 59. Cylinder Rolling on Plane

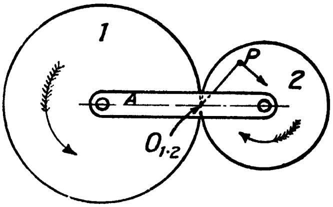

Fig. 60

$$\text{Velocity of } O \text{ without slip } = \omega \, . \, \overline{OP}$$

and
$$\text{velocity of } O \text{ with slip} = \omega \, . \, \overline{SO}$$

$$\therefore \text{ velocity of slipping} = \omega \, . \, \overline{OP} - \omega \, . \, \overline{SO}$$

$$= \omega \, . \, \overline{SP} \quad . \qquad \text{(IV.15)}$$

The velocity of any point in, or on, the circle, is ω multiplied by its distance from S. The percentage slip is $\frac{\omega \, . \, \overline{SP}}{\omega \, . \, \overline{OP}} \times 100$ $= 100 \frac{\overline{SP}}{\overline{OP}}$. If the slip is in the other sense so as to increase the velocity of O for a given angular velocity, S is in OP produced.

79. **Two Cylinders in Pure Rolling Contact.** In Fig. 60, 1 and 2 are circular cylinders in contact. The axes on which the cylinders rotate are fixed to an arm A. The arrangement is a kinematic chain of three links connected by two turning pairs and one higher pair, the relative motion between 1 and 2 being that of pure rolling. Let ω_1 and ω_2 be the magnitudes of the angular velocities of rotation relative to the link A of 1 and 2 respectively, their senses being those of the arrows. $O_{1.2}$ is the virtual centre of relative rotation of 1 and 2. The angular velocity of the wheel 2 relative to 1 is $\omega_1 + \omega_2$ in the clockwise sense, and that of 1 relative to 2 is of the same amount in the counter-clockwise sense. The velocity of a point P in the wheel 2 relative to the wheel 1 is $(\omega_1 + \omega_2)\overline{O_{1.2}P}$ in the direction perpendicular to $\overline{O_{1.2}P}$.

EXAMPLES 4.

1. Describe the three kinds of lower pairs, giving a sketch of each kind, and state the types of relative motion of which the pairs permit.

2. What is meant by "force closure." Define the terms: pair, link, kinematic chain, mechanism, and machine.

3. Prove the relation $L = 2p - 4$, where L is the number of links, and p the number of pairs in a kinematic chain.

4. Test the formula of example 3 by applying it to the cases of (*a*) Stephenson's link motion (Fig. 70), and (*b*) Joy's valve gear (Fig. 76).

5. What do we mean by the term "inversion" as applied to a kinematic chain. What mechanisms are obtained by inversion of the single-slider crank chain ?

6. Give diagrams of the different kinematic chains of four pairs containing only sliding and turning pairs.

7. Describe the mechanism employed in a shaping machine having a quick-return motion, and show how you would obtain a diagram giving the speed of the tool at any point in the stroke. Answer in writing. (I.C.E.)

8. Explain what is meant by the "inversion" of a kinematic chain and indicate with sketches how many operations of this kind are possible with a double-slider crank chain. Show clearly how a quick-return motion is derived from a single-slider crank chain, and obtain the proportions of such a motion that will give a maximum return velocity of 2½ times the minimum forward velocity. Compare the times of outward and return strokes. (L.U.)

9. What is meant by the instantaneous centre of rotation of a body ? The wheel of a vehicle is 4 ft. in diameter. When the vehicle is in motion the wheel slips uniformly at a rate of 4 in. per revolution. Determine the instantaneous centre of rotation of the wheel at any time, (*a*) relative to the vehicle, and (*b*) relative to the ground. (I.C.E.)

10. A circle rolls without slipping on a curve in its own plane. Assuming

the curve to be fixed in space, find the space and body centrodes What are the centrodes if there is slip of amount r per cent ?

11. Draw the space and body centrodes for the connecting-rod of a simple engine in which the crank is 1 ft. long and the connecting-rod 3 ft. 6 in. long.

12. Show that the centre of the disc B in Oldham's coupling (Fig. 49) moves in a circular path, and that it makes a complete revolution whilst the disc A makes half a revolution. Find also the maximum relative sliding velocity between the disc B and each of the discs A, if the shafts make 60 revs. per min., and the axes of the shaft are 2 in. apart.

13. What is the "line of connection" ? Indicate by sketches which is the line of connection in each of the following cases : (1) a belt drive, (2) two circular cylinders in pure rolling contact, (3) two toothed wheels gearing together.

14. Show that in any kinematic chain the virtual centres are such that the line joining any two of them, produced if necessary, will pass through a third virtual centre. Draw diagrams showing all the virtual centres of (*a*) the single-slider crank chain, and (*b*) the double-slider crank chain.

15. A quadrilateral $ABCD$ represents the centre line diagram of a four-bar kinematic chain in which AD is fixed and is 8 ft. long. The link AB rotates about A with uniform speed of 100 revs. per min., and by means of the link BC drives the link CD, which rotates about D. AB is 2 ft., BC 9 ft., and CD 4 ft. long respectively.
Determine the magnitude of the angle through which the link CD moves, and also find the angular speeds of the links CD and BC in one of the positions in which the link BC is perpendicular to the link AB. (I.C.E.)

16. Assuming that the bearing surfaces of the turning pairs in the mechanism of the last example are the surfaces of circular cylinders of 2 in. diameter, find the rubbing speed for each turning pair when the mechanism is in the position given.

17. Taking the simple engine mechanism of example 11, find the angular velocity of the connecting-rod relative to the crank when the piston is at the middle of its stroke. Crank angle 30°. Crank pin speed 10 ft. sec.

18. $\omega\overline{CK}$ (Fig. 54) has been shown to represent the magnitude of the velocity of the crosshead relative to the crank-pin, where ω is the angular velocity of the crank. Hence show that the angular velocity of the connecting-rod is $\dfrac{\omega \, . \, \overline{CK}}{\overline{CP}}$, and that the maximum angular velocity of the connecting-rod occurs when the crank is at a dead centre, its magnitude then being : angular velocity of crank $\times \dfrac{\text{length of crank}}{\text{length of connecting-rod}}$.

19. Define what is meant by "centrodes." Centrodes are said to roll on one another. Show that this is true in the case of a 4 link motion in which the opposite links are equal but set so as not to be parallel. (U.L.)

20. With the aid of sketches, define the terms : (*a*) Instantaneous centre of rotation ; (*b*) space centrode ; (*c*) body centrode.

21. Sketch two types of quick-return motion suitable for slotting or shaping machines. Give the proportions so that the return stroke will only occupy half the time of the working stroke.

22. Two cylinders of radii r_1 and r_2, and angular velocities relative to the line joining their centres of ω_1 and ω_2 in opposite senses, are in pure rolling

contact. Find, relative to the first cylinder, the velocities of points on the second cylinder which are (1) half-way round, and (2) two-thirds of the way round the cylinder from the point of contact.

23. The flanges of an Oldham's coupling are 9 in. diameter. The distance between the axes of the parallel shafts is ½ in. Calculate the kinetic energy in the central piece when the shafts rotate at 240 revs. per min. Also obtain an expression for the velocity of the connecting-piece in one of the flanges. The central disc is 1 in. thick, and each projecting strip upon it ½ in. wide and ¼ in. deep. The weight of the disc not including the connecting strips is 16·5 lb. and the two connecting strips ·3 lb. each. (U.L.A.)

24. Show that the arrangement of Fig. 38 is not a mechanism even though it satisfies the condition (IV.1). What mechanism does it become when the two elements of any one sliding pair are firmly fixed together ?

25. The tool box of a shaping machine is driven through a pin and slot mechanism. If the cutting stroke is to take twice as long as the return stroke, determine the relative dimensions of the mechanism. What will be the ratio of the maximum to the minimum angular velocity of the swinging link ? (U.L.)

26. Indicate the kinematical relationship between the mechanism of the Oldham's coupling and of the elliptic trammels. Prove that the latter mechanism enables a true ellipse to be drawn mechanically. (U.L.)

27. In Rapson's slide (Fig. 51) show that, if T lb. is the force urging 4 along its path, the couple on the rudder-head is $Th \sec^2 \alpha$ lb.-ft., where h feet is the distance from the axis of R to the path of 4 and α is the angle turned through by 2 from its mid-position.

28. Define "centre of rotation" for a given displacement of a rigid body in plane motion and "instantaneous centre." The mechanism of the reciprocating steam engine is represented by a triangle OAB, in which OA is the crank, AB the connecting rod and OB the line of stroke. If BA produced meets the perpendicular at O to OB in N, prove that if AB represents the angular velocity of the crankshaft, BN represents, on the same scale, the angular velocity of the connecting rod relative to the crank. (U.L.)

29. Give sketches and a brief description of each of the three mechanisms derived from the double-slider crank chain by fixing different links.

30. Sketch and describe an Oldham's coupling for connecting two parallel shafts not in the same straight line. If the driving shaft is rotating uniformly will the driven shaft rotate with uniform angular velocity throughout a revolution, or not ? (L.U.)

31. Indicate the kinematical relationship between the mechanism of the Oldham's coupling and of the elliptic trammels. Prove that the latter mechanism enables a true ellipse to be drawn.

CHAPTER V

VELOCITIES AND ACCELERATIONS OF POINTS IN MECHANISM —IMAGES—LINK-MOTION AND VALVE GEARS— THE SLIDE VALVE

80. **Relative Velocities and Accelerations of Points in a Straight Rigid Bar Rotating about One End.** Let PA (Fig. 61) be a bar rotating about P with angular velocity ω and angular acceleration α. The velocity of the point A is $\omega \,.\, \overline{PA}$ along a perpendicular to PA. If B is any other point in the straight bar, the velocity of B is $\omega \,.\, \overline{PB}$, also along a perpendicular to PA. Dividing one of these by the other, we have

$$\frac{V_B}{V_A} = \frac{\text{velocity of } B}{\text{velocity of } A} = \frac{\omega \,.\, \overline{PB}}{\omega \,.\, \overline{PA}}$$

$$\therefore\ V_B = \frac{\overline{PB}}{\overline{PA}} \,.\, V_A \qquad . \quad . \quad . \quad . \qquad (\text{V}.1)$$

Now consider the acceleration of the point A. This is the resultant of two components (1) one of magnitude $\dfrac{V_A{}^2}{\overline{PA}}$ $= \omega^2 .\, \overline{PA}$ along AP (see Art. 18) and (2) one of magnitude $\alpha \,.\, \overline{PA}$ perpendicular to AP. We have then

$$\begin{aligned} A_A &= \text{acceleration of } A \\ &= \sqrt{\omega^4 \,.\, \overline{PA}^2 + \alpha^2 \,.\, \overline{PA}^2} \\ &= \overline{PA} \,.\, \sqrt{\omega^4 + \alpha^2} \end{aligned}$$

Dividing component (2) by component (1) we see that the direction of A_A makes an angle $\tan^{-1} \dfrac{\alpha}{\omega^2}$ with AP. Similarly,

$$\begin{aligned} A_B &= \text{acceleration of } B \\ &= \sqrt{\omega^4 \,.\, \overline{PB}^2 + \alpha^2 \,.\, \overline{PB}^2} \\ &= \overline{PB} \sqrt{\omega^4 + \alpha^2} \end{aligned}$$

The direction of A_B is clearly the same as that of A_A. Dividing, we have

$$\frac{A_B}{A_A} = \frac{\overline{PB}}{\overline{PA}}$$

Hence, $$A_B = \frac{\overline{PB}}{\overline{PA}} A_A \quad . \quad . \quad . \quad . \quad . \qquad (V.2)$$

81. **Straight Rigid Bar in Plane Motion.** If the bar of Fig. 61 has both ends moving, the results of the previous section will apply to the relative motions with respect to P. Writing relative velocities and accelerations, with respect to P, in place of the above velocities and accelerations we have

$${}_PV_B = \frac{\overline{PB}}{\overline{PA}} {}_PV_A \quad . \quad . \quad . \qquad (V.3)$$

and $${}_PA_B = \frac{\overline{PB}}{\overline{PA}} {}_PA_A \quad . \quad . \quad . \qquad (V.4)$$

Now, $$V_A = V_P + {}_PV_A \text{ (see Art. 17)} \qquad (V.5)$$

and $$V_B = V_P + {}_PV_B \quad . \quad . \qquad (V.6)$$

Substituting in the latter equation from (V.3) we have

$$V_B = V_P + \frac{\overline{PB}}{\overline{PA}} {}_PV_A \quad . \quad . \qquad (V.7)$$

and similarly $$A_B = A_P + \frac{\overline{PB}}{\overline{PA}} {}_PA_A \quad . \quad . \qquad (V.8)$$

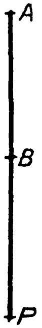

FIG. 61
ROTATING BAR

From (V.7), $$V_B = V_P + \frac{\overline{PB}}{\overline{PA}} (V_A - V_P)$$

or $$V_B = \frac{\overline{AB}}{\overline{PA}} V_P + \frac{\overline{PB}}{\overline{PA}} . V_A \quad . \quad . \quad . \qquad (V.9)$$

In a similar manner, we may prove that

$$A_B = \frac{\overline{AB}}{\overline{PA}} A_P + \frac{\overline{PB}}{\overline{PA}} A_A \quad . \quad . \quad . \qquad (V.10)$$

Equations (V.7) and (V.8), or (V.9) and (V.10), enable us to determine the motion of any point in a rigid link when the motions of its two ends are given.

EXAMPLE.

A straight link in a mechanism has at a given instant its two ends moving in the same direction with speeds of 3 ft. per sec. and 15 ft. per sec. respectively. Find the velocity of a point one-third of the distance along the rod from the slower-moving end.

Here $$\frac{\overline{PB}}{\overline{PA}} = \tfrac{1}{3} \qquad \frac{\overline{AB}}{\overline{PA}} = \tfrac{2}{3}$$

$$V_P = 3 \text{ ft. per sec.} \quad V_A = 15 \text{ ft. per sec.}$$

From (V.9), $$\begin{aligned} V_B &= \tfrac{2}{3} \times V_P + \tfrac{1}{3} V_A \\ &= \tfrac{2}{3} \times 3 + \tfrac{1}{3} \times 15 \\ &= 7 \text{ ft. per sec.} \end{aligned}$$

that is, the point is moving at 7 ft. per sec. in the same direction as that in which the ends are moving.

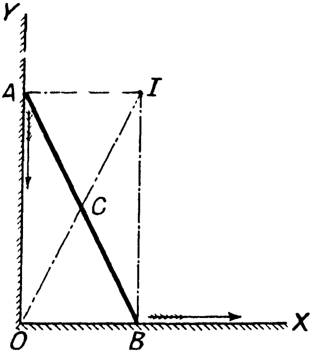

FIG. 62 SLIDING LINK

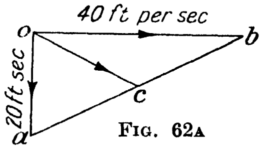

FIG. 62A

EXAMPLE.

A straight, rigid link in a mechanism has, at a given instant, one end moving along a line OX at 40 ft. per sec., and the other end along OY at 20 ft. per sec., $\widehat{XOY}$ being a right-angle. Find the velocity of the mid-point of the link.

Method 1. *Use of the Instantaneous Centre.* Let AB (Fig. 62) be the rod and AI and BI perpendiculars to OY and OX respectively. Then I is the instantaneous centre of rotation; $IAOB$ is a rectangle.

By the properties of the instantaneous centre,

$$\frac{\overline{IA}}{\overline{IB}} = \frac{V_A}{V_B} = \frac{20}{40} = \tfrac{1}{2}$$

Hence $\dfrac{\overline{AO}}{\overline{OB}} = 2 = \tan \widehat{OBA}$, and the angle OBA is thus fixed by the conditions of the problem.

Again, if $AB = 2a$, and C is the mid-point of AB,

$$\frac{V_C}{V_B} = \frac{\overline{IC}}{\overline{IB}} = \frac{a}{2a \sin \widehat{OBA}} = \frac{1}{2 \times \frac{2}{\sqrt{5}}} = \frac{\sqrt{5}}{4}$$

$$\therefore V_C = \frac{\sqrt{5}}{4} \cdot 40 = 10\sqrt{5} = 22{\cdot}4 \text{ ft. per sec.}$$

the direction of V_C being along the perpendicular to OI.

Method 2. *Use of the Velocity Diagram.* In Fig. 62, draw oa parallel to YO to represent a velocity of 20 ft. per sec., and ob parallel to OX to represent a velocity of 40 ft. per sec. Join ab. Then, from (V.7), writing C for B, B for A, and A for P,

$$V_C = V_A + \frac{\overline{AC}}{\overline{AB}} {}_A V_B$$

$$= \overrightarrow{oa} + \tfrac{1}{2}\, \overrightarrow{ab}$$

and if c is the mid-point of ab,

$$V_C = \overrightarrow{oa} + \overrightarrow{ac}$$

$$= \overrightarrow{oc}$$

By scale measurement the velocity of C is found to be 22·4 ft. per sec., as above.

82. **Velocity Diagram for a Rigid Bar.** The method of the preceding example can be applied to the general problem of finding the velocity or acceleration of a point in a rigid bar if the velocities or accelerations respectively of two points in the bar are known.

Let $\overrightarrow{oa}$ and $\overrightarrow{op}$ (Fig 63) represent to scale the velocities of the two points A and P in Fig. 61. Join pa. Then, from (V.7),

$$V_B = V_P + \frac{\overline{PB}}{\overline{PA}} {}_P V_A$$

$$= \overrightarrow{op} + \frac{\overline{PB}}{\overline{PA}} \overrightarrow{pa}$$

Now, take a point b in pa, so that

$$\frac{\overline{pb}}{\overline{pa}} = \frac{\overline{PB}}{\overline{PA}}$$

and we have

$$\begin{aligned} V_B &= \overrightarrow{op} + \frac{\overline{pb}}{\overline{pa}}\overrightarrow{pa} \\ &= \overrightarrow{op} + \overrightarrow{pb} \\ &= \overrightarrow{ob} \end{aligned}$$

That is, the velocity of any point B (Fig. 61) is given by $\overrightarrow{ob}$ (Fig. 63) if the point b divides the line pa in the same ratio as the point B divides the line PA. The above is also true in the case of accelerations. If $\overrightarrow{oa}$ and $\overrightarrow{op}$ represent the accelerations of A and P respectively, then $\overrightarrow{ob}$ represents the acceleration of the point B.

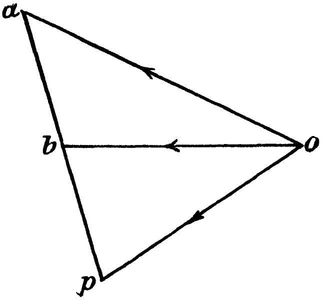

FIG. 63. VELOCITY DIAGRAM FOR RIGID BAR

83. Velocity Image. Acceleration Image. Comparing Figs. 61 and 63, we notice that for each point A, P, or B of the former there is a corresponding point a, p, or b respectively of the latter such that the vectors $\overrightarrow{oa}$, $\overrightarrow{op}$, and $\overrightarrow{ob}$ in the latter diagram give the velocities of the points A, P, and B in the former. Fig. 63 is called the *velocity image* of the bar PBA (Fig. 61). If the vectors $\overrightarrow{oa}$, $\overrightarrow{op}$, $\overrightarrow{ob}$ represent accelerations, the figure is called the *acceleration image* of the bar PAB.

A diagram giving the velocity images, or the acceleration images, of all the bars constituting a mechanism, is called the velocity image, or the acceleration image, respectively, of the mechanism.

A point and its image divide a link and the image of the link respectively, in the same ratio. The fixed point from which the vectors are drawn in an image is called the pole.

Points in the mechanism diagram will be represented by capital letters, A, B, C, etc.; points in the velocity image by corresponding small letters, a, b, c, etc.; and points in the acceleration image by corresponding small accented letters a', b', c', etc. The pole of the velocity image will be denoted by o, and that of the acceleration image by o'. For any point P in the mechanism, $\overrightarrow{op}$ will represent the velocity, and $\overrightarrow{o'p'}$ the acceleration of the point. If a point in a mechanism is fixed, its image will coincide with the pole of the image diagram.

84. **Method of Procedure in Drawing Images.** In general the velocity and acceleration of one point in a mechanism are known and from these the images for a particular configuration of the mechanism are to be drawn. The method consists of combining with the known velocity, or acceleration, the relative velocity or acceleration respectively of a second point in the mechanism. This relative motion is not always completely known, but we are usually able to draw a locus of the image of the second point. By considering the motion of this point relative to some other point, we are able to obtain a second locus of the image and thus to fix its position. It is essential that we should see clearly that *the relative velocity of two points in a rigid body is directed along the perpendicular to the straight line joining them.* This follows from the fact that one point describes a circle relative to the other. The relative acceleration of the two points has two components, one of magnitude $\frac{v^2}{r}$ directed along the line joining them (where v is the relative velocity, and r is the distance between the points), and the other directed along the perpendicular to the line joining the points (Art. 80). When the velocity image has been drawn,

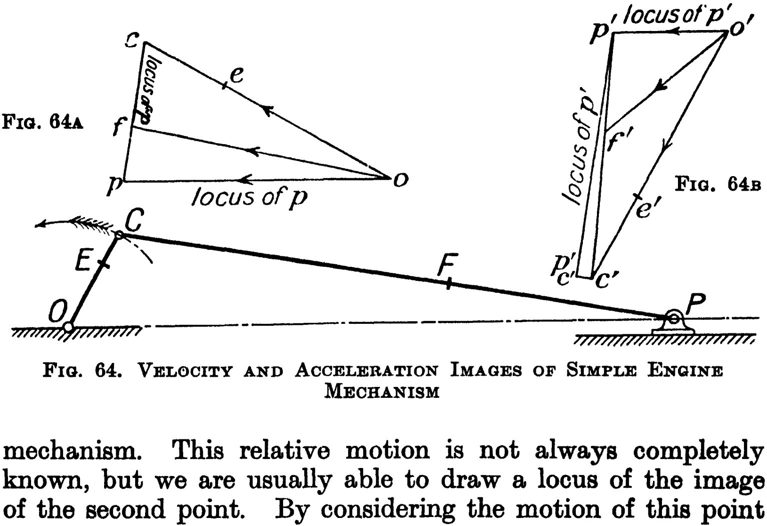

FIG. 64. VELOCITY AND ACCELERATION IMAGES OF SIMPLE ENGINE MECHANISM

the quantity $\frac{v^2}{r}$ may be found, and so the first component of the acceleration is known completely, and the direction only of the second component is known.

85. **Velocity and Acceleration Images of the Simple Engine Mechanism.** Let OCP (Fig. 64) represent the mechanism.

Velocity Image. Draw oc (Fig. 64A) perpendicular to OC, to represent V_c the crank-pin velocity. We now make use of two considerations respecting the motion of P.

1. P is moving along OP, hence draw through o a line parallel to OP. This is a locus of p.

2. P moves relative to C along the perpendicular to CP. Hence, draw through c a line perpendicular to CP. This is a second locus of p, and p is the point of intersection of the two loci. ocp is the complete velocity image and $\overrightarrow{op} = V_P$, the velocity of the crosshead and piston.

Acceleration Image. We begin by calculating the component accelerations along the bars of the mechanism. These calculations are shown in the table below. We have assumed the crank-pin to be 9 in. long, the connecting-rod 4 ft. long, and the angular velocity ω of the crank 10 radians per sec., ω being constant throughout the motion. The relative velocities are obtained from the velocity image. The crank-pin speed is $\frac{9}{12} \times 10 = 7{\cdot}5$ ft. per sec.

TABLE OF CENTRIPETAL ACCELERATIONS

Bar or Link	v = Relative velocity of ends of bar in ft. per sec.	r = Length of bar in ft.	Acceleration along bar $= \frac{v^2}{r}$ ft. sec.^{2}
OC	$\overline{oc} = 7{\cdot}5$	$\frac{3}{4}$	$\frac{7{\cdot}5^2}{\frac{3}{4}} = 75$
CP	$\overline{cp} = 3{\cdot}80$	4	$\frac{3{\cdot}80^2}{4} = 3{\cdot}61$

In Fig. 64B, draw $o'c'$ parallel to CO (Fig. 64) to represent an acceleration of 75 ft. per sec. per sec. Since ω is constant, C has no component acceleration perpendicular to OC, hence $\overrightarrow{o'c'} = A_C$, the acceleration of C. We now make use of the two following considerations.

1. The resultant acceleration of P is along PO. Hence, draw a line through o' parallel to PO. This is one locus of p'.

2. The acceleration of P relative to C has two components: (a) 3·61 ft. per sec. per sec. along PC (see table on page 120), and (b) one perpendicular to this. Hence, draw $c'p'_{c'}$ parallel to PC to represent an acceleration of 3·61 ft. per sec. per sec., and through $p'_{c'}$ draw a line perpendicular to PC. This is a second locus of p', and p' is the point of intersection of the two loci. Join $p'c'$, then $o'p'c'$ is the complete acceleration image.

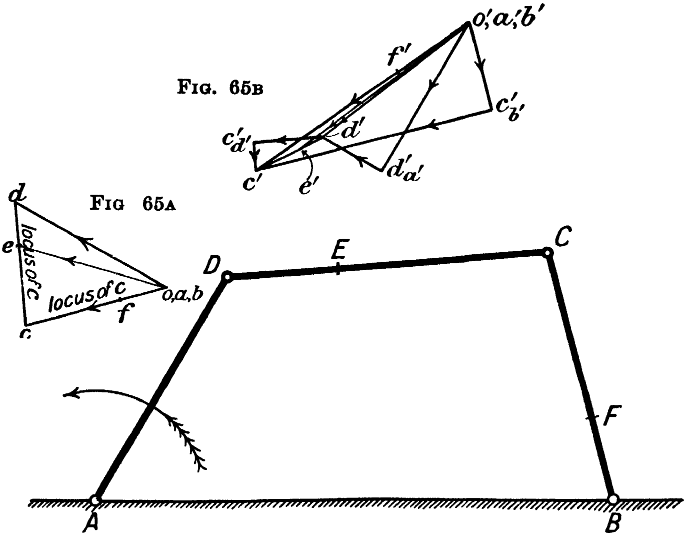

Fig. 65B

Fig 65A

Fig. 65. Images of the Four-bar Mechanism

For any point E on OC, $\overrightarrow{oe}$ gives the velocity, and $\overrightarrow{o'e'}$ the acceleration, if $\frac{\overline{oe}}{\overline{oc}} = \frac{\overline{o'e'}}{\overline{o'c'}} = \frac{\overline{OE}}{\overline{OC}}$

For any point F on PC, $\overrightarrow{of}$ gives the velocity and $\overrightarrow{o'f'}$ the acceleration if $\frac{\overline{pf}}{\overline{pc}} = \frac{\overline{p'f'}}{\overline{p'c'}} = \frac{\overline{PF}}{\overline{PC}}$

A point in the acceleration image indicated by a letter with a suffix is not an image, but is only a point used in construction;

the suffix indicates that the point was obtained by drawing a line of fixed length from the point having that suffix as letter.

By measurement we have—

Velocity of piston $= \overline{op}$ (Fig. 64A), $\overline{op}$ being measured to scale

$= 7{\cdot}2$ ft. per sec.

and

Acceleration of piston $= \overline{o'p'}$ (Fig. 64A), $\overline{o'p'}$ being measured to scale

$= 31{\cdot}3$ ft. per sec. per sec.

86. **Velocity and Acceleration Images of the Four-bar Mechanism.** Let *AB* (Fig. 65) be the fixed link, and let the lengths of the links be $\overline{AD} = 1$ ft., $\overline{DC} = 1{\cdot}25$ ft., $\overline{CB} = 1$ ft., and $\overline{AB} = 2$ ft. Suppose *AD* to be rotating as indicated by the arrow with an angular velocity of 10 rads. per sec., and an angular acceleration in the same sense of 40 radians per sec. per sec. Fig. 65A is the velocity image. The speed of *D* is $10 \times 1 = 10$ ft. per sec., and *od* is drawn to represent this magnitude in a direction perpendicular to that of *AD*. *oc* and *dc* are drawn perpendicular to *BC* and *DC* respectively. These two lines are the loci of *c*, which is, therefore, their point of intersection. Since *A* and *B* are fixed points, their images *a* and *b* coincide with the pole *o*. By scale measurement from Fig. 65A, and calculation, the component accelerations along the bars are obtained and tabulated.

TABLE OF RELATIVE VELOCITIES AND RELATIVE CENTRIPETAL ACCELERATIONS

Bar	v = Relative velocity of ends of bar in ft. per sec.	r = Length of bar in ft.	Acceleration along bar $= \frac{v^2}{r}$ ft. per sec. per sec.
AD	$\overline{ad} = 10$	1·00	$\frac{10^2}{1} = 100$
DC	$\overline{dc} = 7{\cdot}25$	1·25	$\frac{7{\cdot}25^2}{1{\cdot}25} = 42{\cdot}0$
CB	$\overline{cb} = 8{\cdot}50$	1·00	$\frac{8{\cdot}5^2}{1} = 72{\cdot}25$

The component acceleration of D perpendicular to $AD = ar$ $= 40$ ft. per sec. per sec.

In Fig. 65B we draw $o'd'_{a'}$ parallel to DA of length to represent 100 ft. per sec. per sec., and $d'_{a'}d'$ perpendicular to AD to represent 40 ft. per sec. per sec. $\overrightarrow{o'd'}$ represents the acceleration of D. We next draw $d'c'_{d'}$ parallel to DC, representing in length an acceleration of 42·0 ft. per sec. per sec., and we draw a locus of c' through $c'_{d'}$ perpendicular to DC. $o'c'_{b'}$ is drawn parallel to CB of length to represent 72·25 ft. per sec. per sec., and a second locus of c' is drawn through $c'_{b'}$ perpendicular to CB. c' is the point of intersection of the loci.

Fig. 65B is the complete acceleration image. The velocity and acceleration of any other point, such as e and f, on any of the links, are represented by the lines joining the poles to the images of the points, the senses being from the poles to the images. By measurement we have

$$\text{Velocity of } C \qquad = \overrightarrow{oc} = 8{\cdot}50 \text{ ft. per sec.}$$

$$\text{Acceleration of } C = \overrightarrow{o'c'} = 152 \text{ ft. per sec. per sec.}$$

87. **Acceleration of Piston of Simple Engine Mechanism. Klein's Construction.** In Fig. 54, when dealing with the above mechanism we saw that if $\overline{OC}$ represents the crank-pin speed, then $\overline{OK}$ represents the piston speed on the same scale, and $\overline{KC}$ represents the magnitude of the velocity of P relative to C. Thus we have a diagram showing the parts of the mechanism as well as their speeds. If we can superpose on this a diagram giving the accelerations, we shall then have a ready means of finding the motions or relative motions for any position of the mechanism. *Klein's construction* is one of several methods of doing this.

In Fig. 66, OCP is the mechanism, OC being the crank and P the crosshead. Let PC, produced if necessary, meet in K a line OK, perpendicular to OP. With C as centre and CK as radius, draw a circle cutting the circle on CP as diameter in L and N. Let LN cut CP in M, and let LN, produced if necessary, meet OP in p'. Then, $\omega^2\overline{op'}$ is the magnitude of the acceleration of P where ω is the constant angular velocity of the crank and $\overline{op'}$ is measured on the scale of the mechanism diagram.

In Art. 85, we saw that the acceleration image has a side $o'c'$ parallel to CO of length $\omega^2 . \overline{OC}$, and one parallel to CM of length $\frac{\overline{cp}^2}{\overline{CP}}$ (Figs. 64, 64A and 64B), the other two sides being respectively perpendicular to CP and parallel to OP. As the sides of the quadrilateral $OCMp'$ are drawn in these four directions, it only remains to show that $\frac{\omega^2\overline{OC}}{\frac{\overline{cp}^2}{\overline{CP}}} = \frac{\overline{OC}}{\overline{CM}}$ in order to prove the quadrilateral similar to the acceleration image. The result follows, for

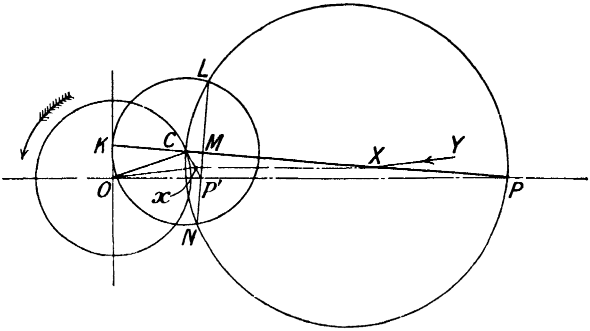

FIG. 66. KLEIN'S CONSTRUCTION

$$\frac{\omega^2 . \overline{OC}}{\frac{\overline{cp}^2}{\overline{CP}}} = \frac{\omega^2\overline{OC}}{\frac{\omega^2 . \overline{CK}^2}{\overline{CP}}}$$

$$= \frac{\overline{OC} . \overline{CP}}{\overline{CL}^2}$$

$$= \frac{\overline{OC} . \overline{CP}}{\overline{CM} . \overline{CP}} \text{ since } \overline{CL}^2 = \overline{CM} . \overline{CP}$$

$$= \frac{\overline{OC}}{\overline{CM}}$$

Hence, $OCMp'$ is similar to the acceleration image, and since

$\omega^2 . \overline{OC}$ = magnitude of acceleration of C

$\therefore \omega^2 . \overline{Op'}$ = magnitude of acceleration of P

TABLE SHOWING THE QUANTITIES REPRESENTED BY THE LINES IN FIG. 66

LINE	MOTION REPRESENTED	REMARKS
OC	$\omega \,.\, \overline{OC}$ = crank-pin speed $= 10 \times \frac{3}{4} = 7 \cdot 5$ ft. per sec.	OC is perpendicular to the direction of motion
OC	$\omega^2 \,.\, \overline{OC}$ = crank-pin acceleration $= 100 \times \frac{3}{4} = 75$ ft. sec.^{2}	The sense of the acceleration is from C to O
OK	$\omega \,.\, \overline{OK}$ = piston speed $= 10 \times 3 \cdot 2 = 32$ ft. per. sec.	Direction of motion perpendicular to OK, i.e. from P to O
KC	$\omega \,.\, \overline{KC}$ = magnitude of velocity of piston relative to crank-pin $= 10 \times \cdot 74 = 7 \cdot 4$ ft. per sec.	Direction of relative motion perpendicular to CP
CM	$\omega^2 \,.\, \overline{CM}$ = radial component of acceleration of P relative to C $= 100 \times \cdot 156 = 15.6$ ft. sec.^{2}	Direction of acceleration from P to C
Mp'	$\omega^2 \,.\, \overline{Mp'}$ = transverse component of the acceleration of P relative to C $= 100 \times \cdot 248 = 24 \cdot 8$ ft. sec.^{2}	Direction of acceleration perpendicular to CP, the sense being that of $\overrightarrow{p'M}$
Op'	$\omega^2 \,.\, \overline{Op'}$ = acceleration of piston $= 100 \times \cdot 878 = 87 \cdot 8$ ft. sec.^{2}	Direction along OP, if Op' is drawn to right the acceleration of P is towards the left
Cp'	$\omega^2 \,.\, \overline{Cp'}$ = acceleration of P relative to C $= 100 \times \cdot 292 = 29 \cdot 2$ ft. sec.^{2}	Sense from p' to C
Ox	$\omega^2 \,.\, \overline{Ox}$ = acceleration of X	Direction along Ox or XY which is parallel to Ox

Hence the piston acceleration may be found readily for any crank position, its sense being that of $\overrightarrow{p'o}$. If $\overline{OC} = \frac{3}{4}$ ft., and $\omega = 10$ radians per sec.

$$\text{Acceleration of piston} = \omega^2 \,.\, \overline{Op'}$$

$$= 100 \,.\, \overline{Op'}$$

and measuring to scale we find $Op' = \cdot 878$ ft. Hence, acceleration of piston $= 87 \cdot 8$ ft. per sec. per sec.

Fig. 66 contains lines which convey quite a fund of information relating to the motions of the parts of the mechanism. We give on page 125 a table showing what the lines represent and give numerical values for the position and values given above. The last result in this table has not yet been proved; the proof is given below—

$$\begin{aligned} A_X &= \text{acceleration of any point } X \text{ in } CP \\ &= A_P + {}_PA_X \\ &= A_P + \frac{\overline{PX}}{\overline{PC}}\, {}_PA_C \end{aligned}$$

But ${}_PA_C = \overrightarrow{p'c}$, and $A_P = \overrightarrow{op'}$, and if Xx is drawn parallel to OP,

$$\frac{\overline{PX}}{\overline{PC}} = \frac{\overline{p'x}}{\overline{p'C}}$$

$$\begin{aligned} \therefore A_X &= \overrightarrow{op'} + \frac{\overline{p'x}}{\overline{p'C}} \cdot \overrightarrow{p'C} \\ &= \overrightarrow{op'} + \overrightarrow{p'x} \\ &= \overrightarrow{ox} \end{aligned}$$

Klein's construction often gives a small acceleration diagram, and the necessary accuracy cannot always be obtained by its use.

88. **Motion of the Point of Crossing of Two Links Rotating in a Plane about Fixed Centres.** Let O_1P and O_2P (Fig. 67) be two links rotating about their ends O_1 and O_2, with uniform angular velocities ω_1 and ω_2 respectively. P is the point at which the bars cross, and we require to find the velocity and acceleration of P relative to the fixed link O_1O_2. The point P considered as moving along the rotating link O_1P has, relative to the fixed link O_1O_2, component velocities, (1) $\omega_1\overline{O_1P}$ perpendicular to O_1P, and (2) one of unknown magnitude along O_1P. In Fig. 67A, $\overrightarrow{op_1}$ represents the first component and a line through p_1 perpendicular to op_1 represents the second. This latter line is a locus of p, where $\overrightarrow{op}$ is the velocity of P.

Again, considering the motion of P as a moving point on the rotating link O_2P, we obtain the two components $\overrightarrow{op_2}$ of magnitude $\omega_2 \,.\, \overline{O_2P}$ perpendicular to O_2P, and a component in a direction perpendicular to the former and of unknown magnitude. A line drawn through p_2 perpendicular to op_2 is a second locus of p. p is at the intersection of the two loci $\overrightarrow{op}$ represent the velocity in space of the point P, and $\overrightarrow{p_2p}$ and $\overrightarrow{p_1p}$ represent the velocities of the point P relative to the links O_2P and O_1P respectively.

To find the acceleration of P we proceed in a similar manner. First, consider P to be moving along the rotating bar O_1P. The motion is that considered in Art. 19, and by (I.38), the component acceleration along the perpendicular to O_1P is the Coriolis acceleration $2\omega_1 v_1$, the term $r\dfrac{d\omega_1}{dt}$ vanishing, because ω_1 is constant. v_1 is the velocity of P relative to the bar O_1P, and is given by $\overrightarrow{p_1p}$. The sense of the component acceleration is opposite to the sense of the displacement of points in the bar O_1P because $\overrightarrow{p_1p}$ shows that the relative motion of P along O_1P is directed towards the centre of rotation O_1.

FIG. 67A

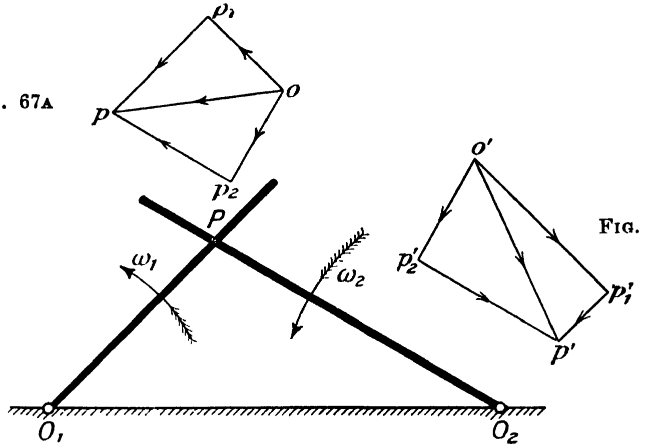

FIG. 67B

FIG. 67. ROTATING LINKS

In Fig. 67B, draw $o'p_1'$ perpendicular to O_1P, and of length $2\omega_1 \cdot \overline{p_1p}$, and through p_1' draw a line perpendicular to $o'p'_1$; this line is a locus of p'. Next, consider P as a point moving along the rotating link O_2P. Draw $o'p_2'$ perpendicular to the direction of O_2P, and of length $2\omega_2\overline{p_2p}$, and draw a line through p'_2 perpendicular to $o'p_2'$. This line is a second locus of p' and p' is at the point of intersection of the two loci. $\overrightarrow{o'p'}$ represents the acceleration of the point of intersection P of the links. The sense of the component acceleration vector $\overrightarrow{o'p_2'}$ is the same as that of the displacements of the points in O_2P because $\overrightarrow{p_2p}$ shows that the motion of P relative to O_2P is outwards from the centre O_2. Though we have used the same notation as in velocity and acceleration images, the two figures 67A and 67B are *not images*. The points in an image are images of points fixed on the links of the mechanism, whereas the points p and p' refer to a point P which is moving along both links.

89. **To Find the Resultant Accelerating Force Acting on a Body in Plane Motion, having given the Angular Acceleration and the Linear Acceleration of the Centre of Gravity.** Let B (Fig. 68) be a body in plane motion, and let a be the linear acceleration of its centre of gravity G, and α the angular acceleration of the body; the senses of these accelerations are indicated by the arrows.

If C is the accelerating couple necessary to produce the angular acceleration α, and P the force necessary to produce the linear acceleration a, we have,

$$C = I\alpha \quad . \quad . \quad . \quad . \quad (V.11)$$

and

$$P = \frac{W}{g} . a \quad . \quad . \quad . \quad (V.12)$$

where W is the weight of the body, and I is its moment of inertia about an axis through G perpendicular to the plane of motion. By the method of Art. 29 we can replace the force P and the couple C by a single force P parallel to the former and at a distance from it equal to $\frac{C}{P}$; this force P is shown in the figure.

EXAMPLE.

Suppose the body to be a uniform bar 6 ft. long weighing 25 lb. Let a be 4 ft. per sec. per sec. along a line, making an angle of 60° with the length of the bar (Fig. 69), and let α = 1 radian per sec. per sec. in the counterclockwise sense.

Here $$I = \frac{25}{g}\,(\text{radius of gyration})^2$$

$$= \frac{25}{32{\cdot}2}\left(\frac{6^2}{12}\right) = \frac{75}{32{\cdot}2} = 2{\cdot}329 \text{ engineers' units}$$

and $$C = I\,.\,\alpha$$

$$= 2{\cdot}329 \times 1$$

$$= 2{\cdot}329 \text{ lb. ft. units}$$

Also, from (V.12),

$$P = \frac{W}{g}\,a$$

$$= \frac{25}{32{\cdot}2}\,.\,4 = \frac{100}{32{\cdot}2}$$

$$= 3{\cdot}105 \text{ lb. weight}$$

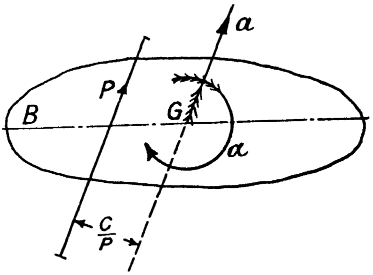

FIG. 68

From these, $\frac{C}{P} = \frac{2{\cdot}329}{3{\cdot}105} = \frac{3}{4}$ ft. Hence the accelerating force is of magnitude 3·105 lb. weight, and its line of action is inclined at 60° with the axis of the rod and is distant $\frac{3}{4}$ ft. from the line FG. If the line of action of the force cuts AB in C, then

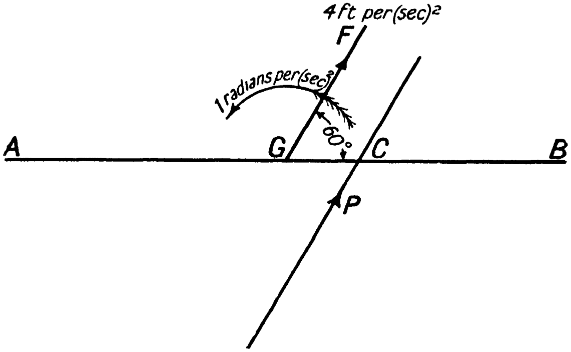

FIG. 69

$$\overline{GC}\,\sin 60^\circ = \frac{3}{4}$$

$$\therefore\ GC = \frac{3}{4} \times \frac{2}{\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$= {\cdot}866 \text{ ft.}$$

It should be noticed that the moment of P about G is in the same sense as that of the couple C.

In applying the methods of this article to the problem of finding the force necessary to produce the acceleration of a link in a mechanism, the method is to find (1) from the acceleration image, the acceleration a of the centre of gravity of the link, and (2) the angular acceleration α of the link; if A and B are two points in the link, the value of ${}_{A}A_{B}$ (the acceleration of B relative to A) is found from the acceleration image. From this, the component acceleration perpendicular to the link AB may be found.

Let this be ${}_{A}A'_{B}$. The value of α is then calculated from the relation $\alpha = \dfrac{{}_{A}A'_{B}}{AB}$. From these values of a and α, the accelerating force may be found by the above method. As an example, suppose it is required to find the resultant accelerating force on the connecting-rod of the simple engine. If, in Fig. 66, X is the centre of gravity of the connecting-rod, then $a = \omega^2.\overrightarrow{xO}$ is the acceleration of the centre of gravity. Also, since $\omega^2.\overrightarrow{p'M}$ is the transverse acceleration of P relative to C, the value of α is $\dfrac{\omega^2 . \overline{p'M}}{CP}$, hence the accelerating force may be found as above. See also Art. 223.

90. **Reversing Gears. Link Motions.** In locomotives, winding engines, etc., where the direction of motion is to be variable, rotation being first in one sense and then in the other, it is necessary to provide some appliance for reversing the motion. Such engines are usually supplied with steam through a slide valve (Art. 98), driven by an eccentric mounted on the crank-shaft. This eccentric is equivalent to a crank (Art. 54), and the angular position of the eccentric is in front of the engine crank by an angle 90° + α where α is an acute angle and is called the *angle of advance* (Art. 98). One obvious method of providing for motion in the reverse direction is to shift the eccentric round until it lags 90° + a behind the crank. The engine will then reverse its motion, so that the lag becomes a lead, and the eccentric will again be 90° + α in front of the crank with the crank-shaft rotating in the reverse direction.

This method was adopted at an early period in the development of the steam-engine, but is now obsolete. A method of obtaining the same result in a much more efficient manner is that in which there are two eccentrics on the crank-shaft, each making a fixed angle 90° + α with the crank on opposite sides of the crank position, so that one is always leading and the other lagging (Fig. 70). The eccentrics are connected by eccentric rods to a link whose mean position is roughly perpendicular to the line of stroke of the engine. The link forms a sliding pair with a sliding piece or block. This sliding piece forms a turning pair with one end of the rod which operates the slide valve. If the sliding block is at one end of the link the motion of the valve is almost entirely produced by one eccentric. If the block is moved to the other end of the link, the motion is then almost entirely produced by the other eccentric, and so the engine is reversed. By placing the block in intermediate positions, the valve is given a motion which is intermediate between the full forward motion and the reverse motion, and we shall see that this is equivalent to using an eccentric of smaller throw with a different value of α.

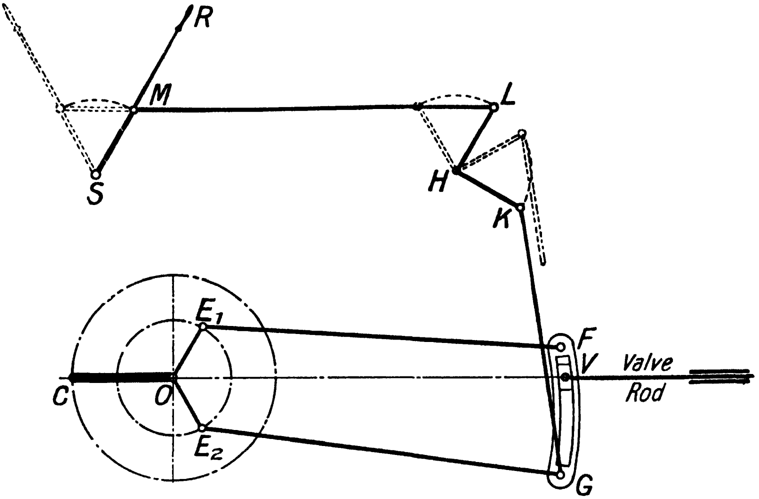

FIG. 70. STEPHENSON'S LINK MOTION

The position of the sliding block on the link may be varied by (1) moving the block along the link which remains fixed, (2) moving the link, the block remaining in a fixed position, and (3) moving both the block and the link. The method (1) is used in Gooch's link motion, (2) is used in Stephenson's link motion, and (3) is used in Allan's link motion. The only differences in these three mechanisms are those due to the method of varying the relative positions of the link and block. All the pairs are turning pairs with the exception of the block and link, and the valve-rod and the framework, which are sliding pairs.

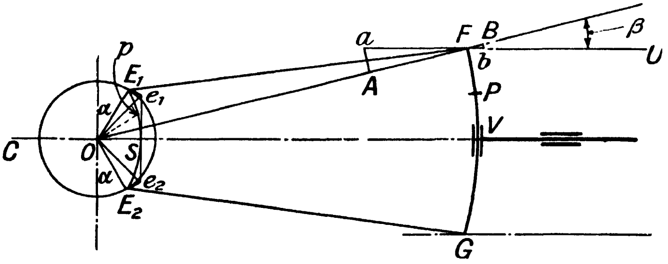

FIG. 71. STEPHENSON'S LINK MOTION

In Fig. 70, the series of links *RS*, *ML*, *LHK*, which is a cranked lever, and *KG*, provide a control arrangement. In the position shown, *F* is near the sliding block *V*, and the valve-rod is operated mainly by the eccentric OE_1. If the control handle *R* is thrown over to the dotted position, *G* is moved near to the block and the valve-rod is then operated mainly by OE_2. In intermediate positions the valve-rod is partly operated by each of the eccentrics, so that variations of steam distribution are obtained by varying the position of the block in the link.

The accurate determination of the actual motion of the valve-rod is very tedious, and is outside our scope. We shall, however, give an approximate method which is in general use for the determination of the motion of the valve-rod and valve. The method is that of finding a single eccentric, and its position relative to the crank, which would, if directly operating the valve, produce the same distribution as that which is produced by the link motion with the handle *RS* in a given position.

91. Equivalent Eccentric for Stephenson's Link Motion. The link motion is shown diagrammatically (Fig. 71) with the block *V* in a position half-way between *F* and *G* (Fig. 72 is an enlargement of part of Fig. 71). The point *G* is constrained to move on a small arc of a circle of large radius, the centre of the circle being vertically above the mean position of G. The error in assuming G to move along a horizontal line is therefore only small; also, as the link FG rocks through only a small angle during the revolution of the crank, we may assume that F is always at a fixed vertical height above G, i.e. that F also moves on a horizontal straight line FU.

If circular arcs be drawn with O as centre and $\overline{E_1F} + \overline{OE_1}$, and $\overline{E_1F} - \overline{OE_1}$ respectively as radii cutting FU in b and a respectively, the points a and b are the extreme positions of F. If the same arcs cut OF in A and B, we have $AB = (\overline{E_1F} + \overline{OE_1}) - (\overline{E_1F} - \overline{OE_1}) = 2\overline{OE_1}$. The short arcs aA and bB may be treated as straight lines perpendicular to AB. Hence,

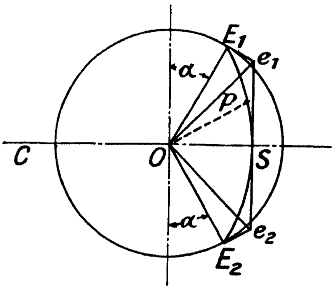

FIG. 72

$$\text{if } \widehat{BFb} = \beta,\ \overline{ab} = \overline{AB} \times \frac{1}{\cos\beta}$$

The travel of F along the horizontal line through F is therefore $\dfrac{2\overline{OE_1}.}{\cos\beta}$ F will be in the position b when OE_1 has turned through the angle E_1OF. If, however, OE_1 were directly connected to the block by a connecting-rod E_1V, V would be at the end of its travel when OE_1 had turned through the angle E_1OV. Hence, the effective inclination of OE_1 to the line of stroke, when it is operating the point K, is reduced by the angle $\widehat{FOV} = \beta$. Thus, if $\widehat{COE_1} = \widehat{COE_2} = 90° + \alpha$, the effective angle of advance is increased from α to $\overline{\alpha + \beta}$. If now we draw Oe_1 so that $\widehat{E_1Oe_1} = \beta$, and $\widehat{OE_1e_1} = 90°$, Oe_1 is the equivalent eccentric because its angle of advance is $\alpha + \beta$ and $\overline{Oe_1} = \dfrac{\overline{OE_1}}{\cos\beta}$.

Similarly, Oe_2 is the equivalent eccentric for the horizontal motion of G. Since V is in the middle of the link FG, we have, using (V.9), and neglecting the curvature of the link,

$$V_V = \text{Velocity of } V = \frac{\overline{VG}}{\overline{FG}} \cdot V_F + \frac{\overline{FV}}{\overline{FG}} \cdot V_G$$
$$= \tfrac{1}{2}(V_F + V_G)$$

or, the velocity of the valve is half the sum of the velocities of the points F and G. Similarly, the acceleration of the valve is half the sum of those of F and G. Thus, the motion of the valve is made up of half of that due to the eccentric Oe_1, together with half that due to the eccentric Oe_2. If S is the mid-point of the line e_1e_2, then OS is the single equivalent eccentric for the motion of V, for

$$\overrightarrow{OS} = \overrightarrow{Oe_1} + \overrightarrow{e_1S}$$

and $$\overrightarrow{OS} = \overrightarrow{Oe_2} + \overrightarrow{e_2S}$$

and, since $$\overrightarrow{e_1S} = -\overrightarrow{e_2S}$$

we have on adding $$2.\overrightarrow{OS} = \overrightarrow{Oe_1} + \overrightarrow{Oe_2}$$

$$\therefore \overrightarrow{OS} = \tfrac{1}{2}(\overrightarrow{Oe_1} + \overrightarrow{Oe_2})$$

If the link is in its lowest position with F coinciding with V, OE_1 is the equivalent eccentric; if the link is in its highest position, OE_2 is the equivalent eccentric; and we have seen that OS is the equivalent eccentric when the link is in the mid-position as in Fig. 71. The approximate determination of the equivalent eccentric for any other position of the link is made in the following manner. Draw a circular arc through the three points, E_1, S, and E_2. Let P be the point in the link which coincides with V in the given position. Divide the arc E_2sE_1 at p, so that $\frac{\text{arc } E_1p}{\text{arc } E_1E_2} = \frac{\overline{FP}}{\overline{FG}}$, the distances on the right being measured along the curved line FPG. Then Op is the equivalent eccentric.

92. **Displacement of a Slide-valve.** In Fig. 73, a single eccentric OE, having an angle of advance α, operates a slide valve through a valve-rod VR, to which it is connected by a connecting-rod, or eccentric-rod EV; OC is the crank. The dotted lines show the mechanism in the position in which the crank has turned through an angle θ from the outer dead centre. Neglecting the obliquity of the eccentric-rod, the valve has a displacement x from its mid-position, given by

$$x = \overline{OE'} \sin \overline{\theta + \alpha}$$
$$= \overline{OE} (\sin \theta \cos \alpha + \cos \theta \sin a)$$
$$= \overline{OE} \cos \alpha \sin \theta + \overline{OE} \sin \alpha \cos \theta$$

or, if $\alpha = \overline{OE} \sin \alpha$, and $b = \overline{OE} \cos \alpha$,

$$x = \alpha \cos \theta + b \sin \theta \quad . \quad . \quad . \quad (V.13)$$

In Fig. 74, let $\overline{OA} = \alpha$ and $\overline{OB} = b$, COA being a straight

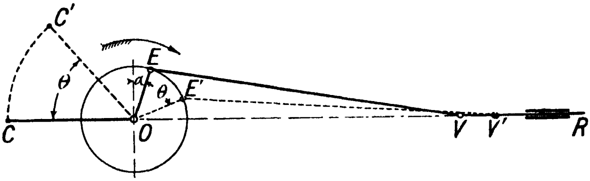

FIG. 73. DISPLACEMENT OF SLIDE VALVE

line, and OB a line perpendicular to COA. Then, if OC is the crank, and OA and OB two eccentrics, and the dotted lines OC', OA', OB' respectively show their positions when the crank has turned through an angle θ, we have

x_1 = displacement of valve if operated separately by eccentric OA

$$= \overline{OA'} \cos \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (V.14)$$
$$= \alpha \cos \theta$$

Similarly, x_2 = displacement of valve if operated separately by eccentric OB

$$= \overline{OB'} \sin \theta$$
$$= b \sin \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (V.15)$$

From (V.13) we have,

$$x = x_1 + x_2$$

If, then, we can, by means of a mechanism known as a valve gear, give to a valve the sum of the motions which would be given to it separately by the two eccentrics OA and OB, the valve will move as if operated by a single eccentric OE (Fig. 74). If the valve gear permits of variation in the magnitude of $\overline{OB}$ the equivalent eccentric may thus be varied. If, further, it is possible to reverse the sense of OB so that it becomes OB_1, the direction of rotation of the crank-shaft may be reversed. We shall see that with all valve gears the motion of the valve is the same as that which would be produced by adding the motions due to OA and OB, and that the sense of OB may be readily changed.

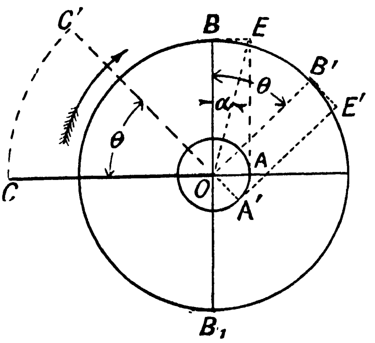

FIG. 74. EQUIVALENT ECCENTRIC

93. **The Hackworth Valve Gear.** Fig. 75 shows the arrangement of Hackworth's valve gear. The rod EF takes the place of the link FG of Fig. 71, and the point P actuates the valve through a connecting-rod PR, which is connected to the valve-rod VR by a turning pair. OE is an eccentric opposite to the crank OC. The motion given to P by the eccentric OE is $\frac{\overline{PF}}{\overline{EF}}$ times the motion of E (Art. 77). Owing to the vertical motion of E, the point F has an equal motion in the vertical direction (we are again neglecting the effect of the varying obliquity of the rod EF and also that of PR). F slides along a straight groove in a link ST, which is inclined at an angle θ to the vertical.

The extent of the horizontal motion of F is therefore tan θ times that which would be produced by an eccentric of radius, or throw, equal to $\overline{OE}$. The horizontal displacement of F varies from $\overline{OE}$ tan θ to $-\overline{OE}$ tan θ. Thus, we may consider the horizontal motion of F to be produced by an eccentric of throw $\overline{OE}$ tan θ. When OE is in the position OE', F will be in the middle of its stroke. If OE is rotating in the sense indicated, F will be moving towards T when OE is in the position OE'. When E is vertically above O, F will be in its extreme right position, and will begin to move to the left.

The equivalent eccentric for the motion of F is therefore a right-angle behind OE in position, and is represented by OG where $\overline{OG} = \overline{OE}$ tan θ. The horizontal motion given to

P by this eccentric would be $\dfrac{\overline{PE}}{\overline{EF}}$ times the motion, which OG would give to P. Thus, comparing with Art. 92, we see that $\overline{OA}$ and $\overline{OB}$ of Fig. 74 are given by

$$\overline{OA} = \frac{\overline{PF}}{\overline{EF}} \cdot \overline{OE} \qquad \text{(V.16)}$$

and

$$\overline{OB} = \frac{\overline{PE}}{\overline{EF}} \overline{OE} \tan\theta \qquad \text{(V.17)}$$

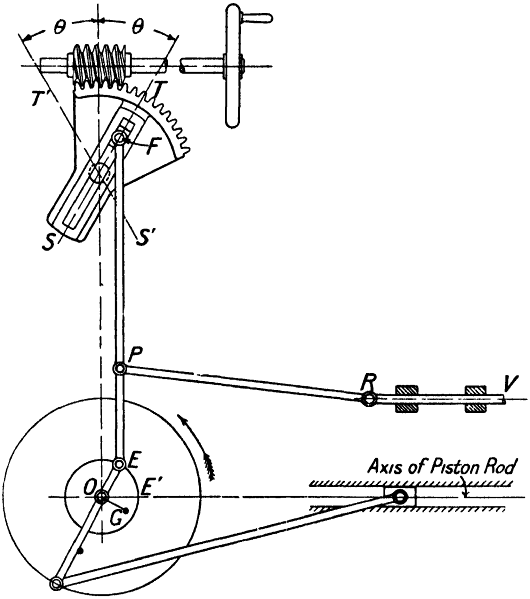

Fig. 75. The Hackworth Radial Valve Gear

The correct sense of rotation for the arrangement of Fig. 75 is the counter-clockwise sense. By varying θ, the value of OB is varied, and on throwing the link ST over into the position $S'T'$, θ becomes negative and the sense of $\overline{OB}$ is reversed. When this is the case the crank-shaft rotates in the clockwise direction. The position of ST is controlled by a handle, so that θ may be varied from $-\theta$ to $+\theta$ where θ is an acute angle. The mechanism thus gives full control over the steam distribution.

94. Joy's Valve Gear. Fig. 76 shows the arrangement of Joy's valve gear. In this mechanism the point *P* is in *EF*, produced (compare Fig. 75); also *F* is guided along a circular groove as shown instead of along a straight link. The point *E* is not moved by an eccentric, as before, but, owing to the motion of the link *MN*, *E* is carried round the closed curve shown dotted. The horizontal motion of *P* due to that of *F* is found exactly as before; the throw of the equivalent eccentric for the motion of *F* is $\frac{1}{2}y \tan \theta$, where y is the distance between the horizontal tangents to the locus of *E*, and θ is the inclination to the vertical of the tangent to the locus of *F* at its mid-point. The equivalent eccentric for the horizontal motion of *P* due to that of *F* is therefore of length $\overline{OB} = \frac{1}{2}\frac{\overline{PE}}{\overline{FE}} y \tan \theta$. Thus, as before, *OB* is the eccentric, which, if directly operating a valve, would give to it the motion which is given to *RV* by *P* owing to the motion of *F*. *OA* is the eccentric which would give the valve motion which is given to *RV* owing to the horizontal motion of *E*. Since *E* has a horizontal travel of amount x, the travel of *P* due to this is $\frac{\overline{PF}}{\overline{FE}} \cdot x$, and the throw of the corresponding eccentric is $\overline{OA} = \frac{1}{2}\frac{\overline{PF}}{\overline{FE}} \cdot x$.

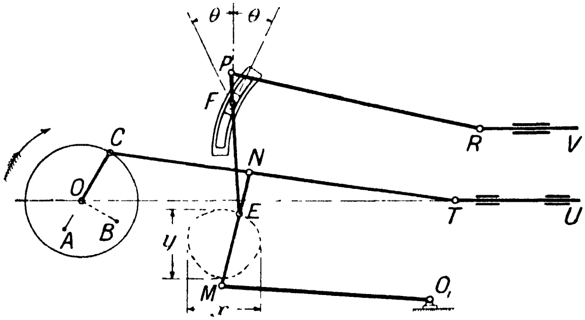

FIG. 76. JOY'S VALVE GEAR

P and *E* move in opposite senses relative to *F*, and as the horizontal motions of *E* and *C* are approximately in the same sense at all times, those of *P* and *C* will be in opposite senses.

Thus, OA is opposite to OC. OB is perpendicular to AOC, and the sense of rotation is clockwise for the position shown. The motion of the valve is varied by changing the position of the slotted link. If the link is moved so that the tangent to the path of F is at the same inclination to the vertical, but on the other side of it, the direction of rotation of the engine will be reversed.

95. **Walschaert's Valve Gear.** This gear is shown in Fig. 77, and requires little explanation. The link MT is rigidly fixed to the crosshead T and the horizontal motion of E is approximately the same as that of the crosshead. The travel of P due to that of E is $\frac{\overline{PF}}{\overline{FE}}$ of that of the piston, but in the opposite sense. Hence, the eccentric OA which would give this motion is of length $\frac{\overline{PF}}{\overline{FE}} \cdot \overline{OC}$, and its position is opposite to that of OC. The slotted link NH is pivoted at H, N is operated by the eccentric OD, which is perpendicular to the crank, and the motion of W is communicated to the point F in the link EP by means of the connecting-rod WF. For the position shown, the motion of W is $\frac{\overline{WH}}{\overline{NH}}$ times that of N. The motion communicated to F is that which would be given to it by an

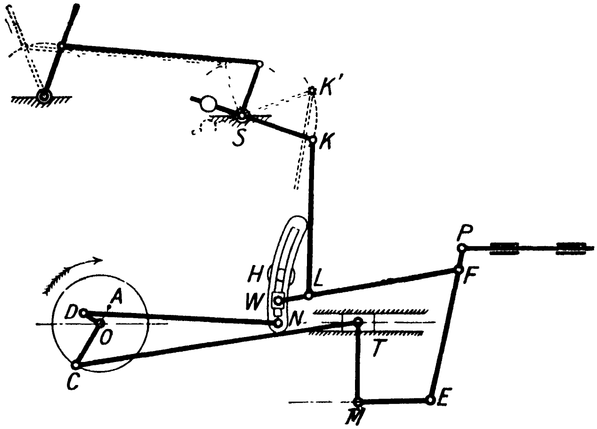

FIG 77. WALSCHAERT'S VALVE GEAR

eccentric of length $\overline{OB} = \frac{\overline{WH}}{\overline{NH}} \cdot \overline{OD}$, its position coinciding with that of OD. The motion of P due to that of F is $\frac{\overline{PE}}{\overline{FE}}$ times the motion of F; the motion of P due to that of D is therefore that due to an eccentric of length $\frac{\overline{PE}}{\overline{FE}} \cdot \frac{\overline{WH}}{\overline{NH}} \cdot \overline{OD}$. If the position of K is moved along the arc KK', W moves along the slot nearer to H, and the length OB is reduced. If the slider is above H, which is the case when the handle is thrown over to the dotted position, the sense of OB is reversed, and B is in DO produced. The direction of rotation for the position shown is clockwise.

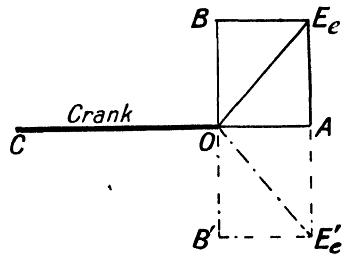

FIG. 78. EQUIVALENT ECCENTRIC FOR RADIAL VALVE GEAR

96. **Equivalent Eccentric for any Valve Gear.** Having found the magnitudes of $\overline{OA}$ and $\overline{OB}$, we reverse the method of Art. 88 so as to find OE_e the equivalent eccentric. OE_e is the diagonal of a rectangle of which OA and OB are sides (Fig. 78). In each of the Figures 76 and 77 the gears have been shown with the controlling link set so as to produce clockwise rotation of the crank-shaft. If the controlling link is thrown over to the position shown dotted in each case, the sense of rotation will be reversed; the effect is produced by changing OB into OB' thus changing OE_e to OE_e'.

In Hackworth's gear (Fig. 75),

$$\overline{OA} = \frac{\overline{PF}}{\overline{EF}}\overline{OE}, \text{ and } \overline{OB} = \frac{\overline{PE}}{\overline{EF}}\overline{OE} \tan\theta$$

In Joy's gear (Fig. 76),

$$\overline{OA} = \tfrac{1}{2}\frac{\overline{PF}}{\overline{EF}} \cdot x, \text{ and } \overline{OB} = \tfrac{1}{2}\frac{\overline{PE}}{\overline{EF}}\, y \tan\theta$$

In Walschaert's gear (Fig. 77),

$$\overline{OA} = \frac{\overline{PF}}{\overline{EF}} \cdot \overline{OC}, \text{ and } \overline{OB} = \frac{\overline{PE}}{\overline{EF}} \cdot \frac{\overline{WH}}{\overline{NH}} \cdot \overline{OD}$$

The reader should notice the essential difference in the principles underlying the actions of (1) a link motion, and (2) a valve gear. In (1) there are two eccentrics of the same throw making equal angles with the crank. These would, if operating the valve separately, cause the engine to rotate in the direct sense, or in the reverse sense, respectively. The mechanism gives to the valve part of the motion due to each eccentric, and adjustment is obtained by altering the relative magnitudes of these parts. In (2) there may be one eccentric (Fig. 77) or none (Fig. 76). The motion given by the mechanism to the valve is the sum of the motions which would be given to it by two eccentrics whose angles of advance are 0° and 90° respectively. The variation is obtained by altering the travel of the former eccentric.

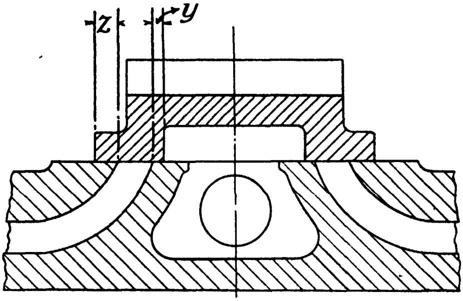

FIG. 79. THE SIMPLE SLIDE VALVE

97. **Velocity and Acceleration Images of Valve Gear Mechanism.** The exact motions of points in these mechanisms may be found by the methods of Art. 84. As an example, if the motion of OC (Fig. 76) is known, the velocity and acceleration of the point N may be determined, as in the case of the simple engine mechanism. Next the velocity and acceleration images of the bars NM and O_1M may be drawn, from which the motion of E is known. As the paths of F and R are known, the velocity and acceleration images of PE and of PR may be drawn. Thus, the motion of any point in the mechanism may be found.

98. **The Slide Valve** controls the supply of steam to, and the discharge of steam from, the cylinder of the steam-engine. It is actuated by an eccentric, link motion, or valve gear. In Art. 53, the eccentric is shown to be equivalent to a crank. Since the length of the eccentric-rod is large compared with the

throw of the eccentric, the varying obliquity of the rod may be disregarded and the motion of the valve taken to be simple harmonic (Art. 65).

Fig. 79 shows the section of a slide-valve and ports, with the valve in its central position (i.e. the eccentric at right-angles to the line of stroke). The distance z is known as the steam lap or outside lap, and the distance y is known as the exhaust lap or inside lap. The opening to steam when the crank is on the dead centre is known as the lead.

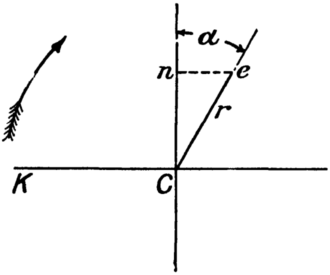

FIG. 80

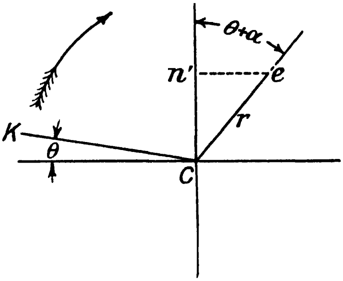

FIG. 81

The radius r of the eccentric is given by

$$r = \text{steam lap} + \text{maximum opening to steam} \qquad \text{(V.18)}$$

In the case of a valve with no lap or lead, the eccentric will be at right-angles to the crank. Where the valve has both lap and lead, the eccentric Ce (Fig. 80) must be advanced by an angle α, so that

$$\sin \alpha = \frac{en}{ec} = \frac{\text{outside lap} + \text{lead}}{r} \quad . \quad \text{(V.19)}$$

α is known as the " angle of advance " of the eccentric.

99. To Find the Opening to Steam (s) of the Valve when the Crank has Moved Forward an Angle θ from the Dead Centre Position. Displacement of valve from said position

$$= en' \text{ (Fig. 81)}$$
$$= r \,.\, \sin(\theta + \alpha)$$

The valve must move a distance z (Fig. 79) before it begins to admit steam, so that the opening to steam is given by

$$s = r \,.\, \sin(\theta + \alpha) - z \quad . \quad . \quad . \quad \text{(V.20)}$$

The values of θ when admission begins and cut-off occurs are given by the solutions of the equation

$$r \, . \sin(\theta + \alpha) - z = o \quad . \quad . \quad . \qquad \text{(V.21)}$$

Example.

Steam lap = $1\frac{1}{4}$ in.; maximum opening = $\frac{7}{8}$ in.; lead = $\frac{1}{8}$ in.

From (V.18) $r = 1\frac{1}{4} + \frac{7}{8} = 2\frac{1}{8}$ in.

From (V.19) $\sin \alpha = \dfrac{1\frac{3}{8}}{2\frac{1}{8}} = \cdot 648$

$$\alpha = 40{\cdot}3^\circ$$

$\therefore$ from (V.20), for zero opening,

$$\begin{aligned} 2{\cdot}125 \sin(\theta + 40{\cdot}3^\circ) &= 1{\cdot}25 \\ \sin(\theta + 40{\cdot}3^\circ) &= \cdot 589 \\ \theta + 40{\cdot}3^\circ &= 36^\circ \text{ or } 144^\circ \\ \therefore \theta &= -4{\cdot}3^\circ \text{ or } 103{\cdot}70 \end{aligned}$$

In this case admission will begin 4·3° before the dead centre, and cut-off will occur 103·7° after the dead centre.

Various graphical constructions are in use for showing the port opening for any position of the crank, the best known and probably the most useful being the diagrams of Zeuner and of Reuleaux. The Reuleaux diagram is the most straightforward and the simplest where the solution of practical problems is concerned, but the construction of the Zeuner diagram is also given since this is widely used.

100. **Zeuner Valve Diagram** (Fig. 82). We shall assume that the line of stroke is horizontal, the cylinder being to the left of the crank. The construction will then be as follows—

With centre C and radius equal to the half-travel (r) of the valve, describe the circle $AYA'Y'$. Draw YOY' perpendicular to AA'. Set back the angle YCB = angle of advance (a) and draw BCB'. On BC and $B'C$ as diameters draw circles. With centre C and radius CD = steam lap, draw the arc LDM. With centre C and radius CE = exhaust lap, draw the arc FEG. Let CP be any position of the crank, making an angle $ACP = \theta$ with the line of dead centres AA'; then RS is the opening of the valve to steam for this position of the crank, and $R'S'$ is the opening to exhaust at the other end of the valve.

PROOF. Join BR. Then, since the angle in a semicircle is a right-angle,

$$\widehat{RBC} + \widehat{BCR} = 90^\circ = \widehat{ACP} + \widehat{BCR} + \widehat{YCB}$$

$$\therefore \widehat{RBC} = \theta + \alpha$$

$$\text{Since } BC = r, \quad \therefore \quad CR = r \,.\, \sin(\theta + \alpha)$$

$$\therefore \quad RS = r \,.\, \sin(\theta + \alpha) - \text{steam lap}$$

$$= \text{opening to steam}$$

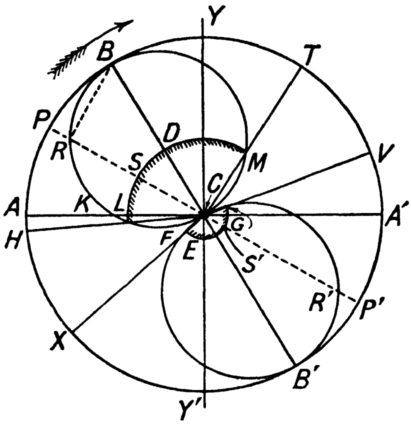

FIG. 82. THE ZEUNER VALVE DIAGRAM

It will be seen that $\overline{KL}$ = lead, and $\overline{BD}$ = maximum opening to steam.

CH = position of crank at beginning of admission

CMT = position of crank at cut-off

CGV = position of crank at beginning of release

CFX = position of crank at beginning of compression

Regarding AA' as the stroke, the corresponding positions of the piston (assuming infinitely long connecting-rod) will be obtained by dropping perpendiculars from H, T, V, and X respectively on to AA'.

101. **Reuleaux Valve Diagram** (Fig. 83). With centre C and radius equal to the half-travel (r) of the valve, describe a circle. Set back the angle ACD equal to the angle of advance (α), and draw DCD'. With centre C and radius CB equal to the steam lap, describe a semicircle. Draw SBS' parallel to DD', to touch this semicircle. On the other side of DD', with centre C and radius CF equal to the exhaust lap, describe a semicircle. Draw EFE' parallel to DD', to touch this semicircle. Let CP be any position of the crank, making an angle $ACP = \theta$ with the line of dead centres AA'. Draw PNM perpendicular to SS'. Then PN is the opening of the valve to steam for this position of the crank, and $P'N'$ is the opening to exhaust at the other end of the valve.

Proof. $\overline{CP} = r$; $\widehat{PCM} = \theta + \alpha$. $\therefore \overline{PM} = r \,.\, \sin(\theta + \alpha)$

Since $\overline{NM}$ = steam lap

$\therefore \overline{PN} = r \,.\, \sin(\theta + \alpha)$ – steam lap

= opening to steam

The *lead* in this case will be the radius AL of the circle with centre A, which touches SS'.

DD' is known as the *displacement line*, SS' as the *steam line*, and EE' as the *exhaust line*.

CS = position of crank at beginning of admission

CS' = position of crank at cut-off

CE' = position of crank at beginning of release

CE = position of crank at beginning of compression

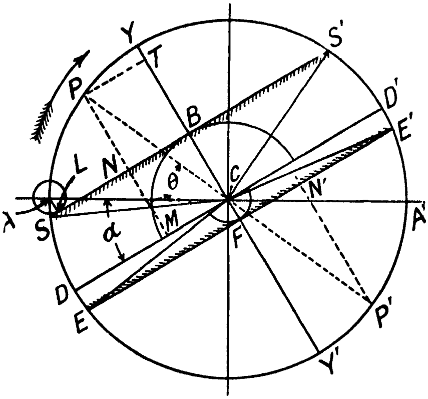

Fig. 83. Reuleaux Valve Diagram

The corresponding points of the stroke may be found as before.

102. Velocity of Valve. Referring to Fig. 81, the velocity of the point e is $r \,.\, \omega$ (ft. per sec.), at right-angles to Ce, where ω is the angular velocity of the crank (radians per sec.) and $Ce = r$ (ft.).

$\therefore$ Velocity of valve = horizontal component of $r \,.\, \omega$

$= r \,.\, \omega \,.\, \cos(\theta + \alpha)$

$= \omega \,.\, \overline{PT}$ (Fig. 83) . . (V.22)

If PT is measured in inches and N = revs. per min. of crank, then

$$\text{Velocity of valve (ft. per sec.)} = \frac{2\pi N}{60} \cdot \frac{\overline{PT}}{12}$$

$$= \frac{\pi}{360} \cdot N \,.\, \overline{PT} \,. \qquad \text{(V.23)}$$

It will be seen that the velocity of the valve during admission is greatest at admission and cut-off, and is proportional, at these points, to the length of the steam line SS'.

103. Travel of Valve Not Given. Ordinary problems in which the travel is given can be solved by the Reuleaux diagram without the use of any special constructions, if it is borne in mind that SS' passes through S' and touches the lead circle, also that SS', DD' and EE' are parallel.

In designing a slide-valve, however, the problem to be solved is usually as follows : Given point of cut-off, maximum opening, and lead, also point of compression, to find the travel, angular advance, steam and exhaust laps, and point of release. As soon as the travel is found, the remaining particulars are easily obtained.

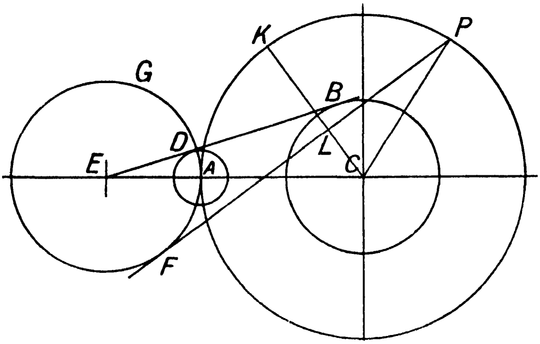

FIG. 84. DETERMINATION OF VALVE TRAVEL

The following method of finding the travel is due to Prof. J. Harrison, and has the merit of being simple and straightforward (Fig. 84).

Assume a travel. With centre C and radius CA equal to half the assumed travel, describe a circle. With centre C and radius CB equal to the maximum opening, describe a circle. With centre A and radius AD equal to the lead, describe a circle. Draw the common tangent BDE, meeting CA produced in E. With centre E and radius EA, describe a circle. Let CP be the crank position at cut-off. From P draw a tangent PF to the last circle drawn. Draw CLK at right-angles to PF and measure KL. Then,

$$\text{Correct travel} = \text{assumed travel} \times \frac{\text{max. opening}}{KL} \qquad (V.24)$$

The Reuleaux diagram may now be drawn, and, as a check on the accuracy, the angle between *PF* and *CA* should agree with the angle of advance found.

104. **Effect of Connecting Rod Obliquity on Actual Points of Cut-off.** *CS′* (Fig. 83) gives the correct position of the crank at cut-off on the outstroke, and the cut-off will take place on the instroke when the crank is opposite to *CS′*. The corresponding points of the stroke at which cut-off takes place will, however, depend upon the ratio *n* of the length of the connecting-rod to that of the crank, the *average* point of cut-off being given by the method described above.

Harrison's false-centre method (Art. 40) will give very closely the actual points of cut-off in the two strokes, the construction being as given below (Fig. 85).

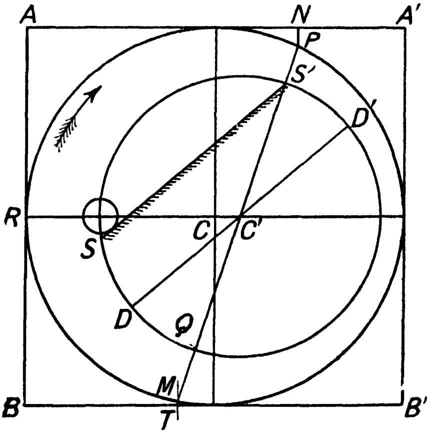

FIG. 85

Choose any convenient radius *CR* (preferably 5 units), and draw the circle *RPM*. Enclose this circle in a square *AA′B′B*. Make $CC' = \frac{CR}{2n}$. Taking *C′* as centre, draw the Reuleaux diagram, and find the crank positions *C′S′* and *C′Q* at cut-off. Produce *C′S′* and *C′Q* to cut the outer circle at *P* and *M* respectively and project *PN* and *MT* vertically to *AA′* and *BB′*.

The actual point of cut-off on the outstroke will be at the fraction $\frac{AN}{AA'}$, of the stroke, and that on the instroke will be at the fraction $\frac{B'T}{B'B}$ of the stroke.

105. **Expansion Valves.** With an ordinary slide valve, very early cut-off is accompanied by very early release and excessive compression, and it is unusual in this case to arrange for a cut-off earlier than 0·4 of the stroke. When an earlier cut-off

is required, and it is not desired to use trip gears, an " expansion " valve, working on the back of the main valve is used, its purpose being to deal with cut-off only, the main valve being set for the points of release and compression required.

If the motion of the expansion valve relative to the main valve is specified by the radius and angular advance of a " relative " eccentric, the point of cut-off can be obtained if the lap of the expansion valve is known. This lap may be positive or negative. In the case of the " Meyer " expansion valve, the lap can be varied by means of a right-and-left-handed screw actuating two separate plates as shown in Fig. 86.

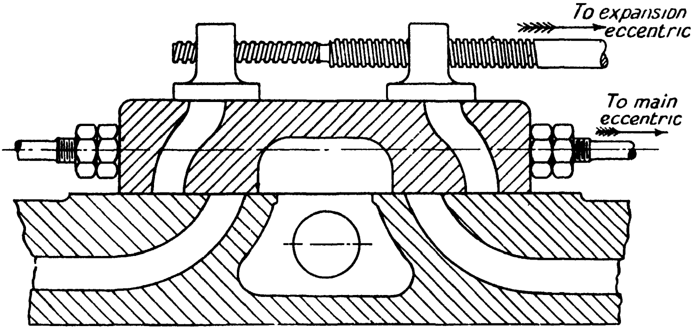

Fig. 86. The Meyer Expansion Valve

The plates are shown in Fig. 86 with positive lap ; if these are moved nearer to each other by rotating the right and left-handed screws, the lap will decrease until it becomes zero. If the distance between the plates is further decreased, each port in the main valve will be open by a definite amount, when the expansion valve is in its central position relatively to the main valve, and the amount of this opening is known as " negative lap."

Let *Cm* (Fig. 87) be the radius of the main eccentric, angle of advance *xCm*, and let *Ce* be the radius of the expansion eccentric, angle of advance *xCe*. Join *me*. Then, since the rotating vector $\overrightarrow{Ce}$ is the sum of the rotating vectors $\overrightarrow{Cm}$ and $\overrightarrow{me}$, the projection of $\overrightarrow{me}$ will give the motion of the expansion valve relative to the main valve. Thus *Cr*, equal and parallel to *me*, is the relative eccentric required, and the motion of the expansion valve relative to the main valve is the same as if the main valve were stationary and the expansion valve driven by an eccentric of radius *Cr*, having an angle of advance *xCr*.

The range of cut-off can now be obtained as follows : With centre *O* and radius *OA* equal to *Cr* (Fig. 88), draw a circle. Set back the angle *AOD* equal to *xCr* (Fig. 87). Draw *SS'* parallel to *DD'* such that *a* is the maximum positive lap of the expansion valve, and draw *RR'* parallel to *DD'* such that *b* is the maximum negative lap. The range of cut-off will then be from N_1 to N_2.

The principle of the above method is also applicable to the analysis of the motions of sleeve valves for internal combustion engines.

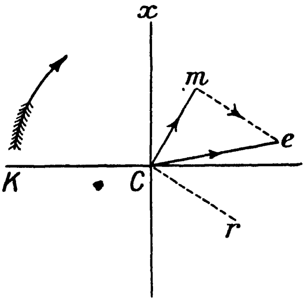

FIG. 87. EQUIVALENT ECCENTRIC FOR RELATIVE MOTION OF TWO VALVES

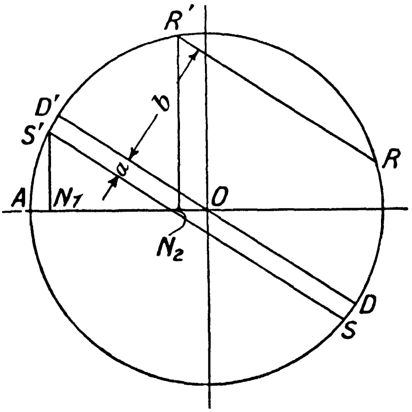

FIG. 88. RANGE OF CUT-OFF

106. Acceleration Centre. A point in a rigid body which has zero acceleration at a given instant is known as the *instantaneous acceleration centre*, or, briefly, as the acceleration centre. At the given instant any point *P* in the rigid body may be considered as having an acceleration

$$A_P = A_C + {}_CA_P$$

where *C* represents the acceleration centre. But $A_C = 0$,

$$\therefore A_P = {}_CA_P$$

In Art. 80 we saw that the direction of ${}_CA_P$ is inclined to the line *CP* at an angle θ, where $\tan \theta = \dfrac{a}{\omega^2}$. The direction of A_P is the same as that of ${}_CA_P$ and so the direction of the acceleration of any point makes an angle θ with the line joining the point to the acceleration centre, where $\tan \theta$ has the given value. If, then, lines be drawn joining two points of the lamina to the acceleration centre, the angle between these lines is the same as that between the directions of the accelerations of the two points. From the acceleration image of the body, the angles between the accelerations may be measured. If, now, on the line joining two points, we draw a segment of a circle containing an angle equal to that between the directions of their accelerations, this segment is a locus of the acceleration centre. By taking one of the two points with a third point in the body a second locus may be drawn and the acceleration centre is at the intersection of the loci. When the acceleration centre is known it is useful for determining the accelerations of points in the body. The numerical value of the acceleration of a point P is proportional to $\overline{CP}$ (Art. 80); hence, if C is known and the acceleration of one other point in the link is also known, the acceleration of all the other points may be determined by proportion.

EXAMPLES 5.

Draw the velocity image and the acceleration image for the following mechanisms in the positions shown, taking your dimensions from the figures:

1. The simple engine mechanism of Fig. 64, assuming the crank to be 9 in. long and rotating in the clockwise sense at 90 revs. per min. Give the numerical values of the velocities and accelerations of E, F, and P.

2. The four-bar mechanism of Fig. 65, assuming that AD is 1 ft. long and is rotating in the clockwise sense with angular velocity 10 radians per sec. Write down the velocity and acceleration of the point C.

3. The link motion of Fig. 70, assuming $\overline{OE}_1 = \overline{OE}_2 = 3$ in., and the speed of rotation to be 60 revs. per min. clockwise. Find the velocity and acceleration of the point V on the link. Find also the velocity of the valve rod.

4. Sketch and describe the action of a Joy valve gear, suitable for use on a locomotive. In your description state clearly why the lever centred on the slipper block and giving motion to the slide valve cannot be pinned directly to the connecting-rod. Explain also how reversal of the motion and how expansive working are effected by the gear. (I.C.E.)

5. Describe, with the aid of sketches, a Stephenson link motion. What do you understand by the term equivalent eccentric as applied to a link motion.

Find the equivalent eccentric corresponding to the mid-gear position of the motion block for a link motion having the following dimensions—

Angular distance between each eccentric and the crank	104°
Length of each eccentric rod	6 ft. 3 in.
Length of the link	$24\frac{3}{4}$ in.
Radius of each eccentric sheave	4·3 in.

(I.C.E.)

6. The slide-valve of a horizontal steam-engine is driven from a block C in a link AB, which can be assumed to remain vertical, as it is very long. The ends of the link AB have horizontal and simple harmonic motions derived from eccentrics each of $2\frac{1}{2}$ in. throw, which lead the crank by angles of 110° and 250° respectively. Find the half-travel and equivalent angle of advance of the slide-valve when the block C is in such a position that $AC = 3 \,.\, CB$. (U.L.)

7. The crank radius of an oscillating cylinder engine is 1 ft. long and the shaft rotates at 90 revs. per min. The piston-rod is 4 ft. long and the distance between the crank-shaft and trunnions is 4 ft. Find the absolute velocity of the piston in magnitude and direction when the crank is at an angle of 60° from the outer dead centre. (U.L.)

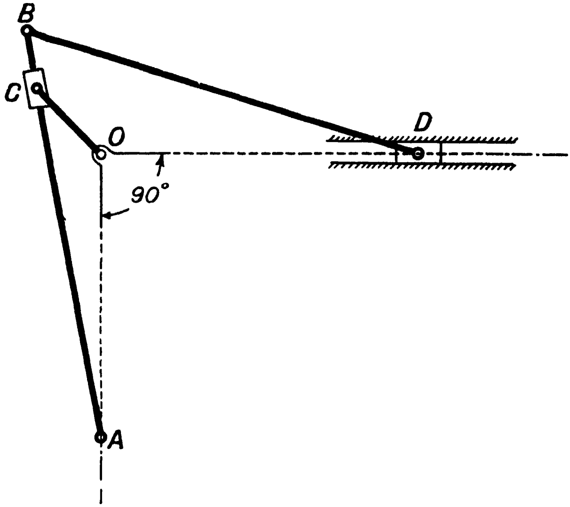

FIG. 89

8. Two slotted links *APC*, *BPD* are pivoted on fixed centres *A*, *B*, 4 in. apart. The links cross and carry a common block *P*. Determine the velocity and acceleration of this block at the instant when it is 34 in. from *A* and 24 in. from *B*, the angular velocity of the link *AC* being 2 radians per sec., and that of *BD* 1·5 radians per sec., both links moving outwards from the line *AB*. (U.L.)

9. *OC* (Fig. 89) rotates uniformly about *O*, the velocity of *C* being 4 ft. per sec. *C* slides along the link *AB*, which is pivoted at *A*, and *D* reciprocates on the line *OD*. If *OC* = 6 in., *OA* = 18 in., *AB* = *BD* = 26 in., find the ratio of the times of the forward and backward strokes of *D*. Also determine the velocity of *D* when the angle *AOC* is 135°. (L.U.)

10. Find the velocity of the slider *E* in the mechanism shown in Fig. 90 when the crank rotates at 1 rev. per sec. (U.L.)

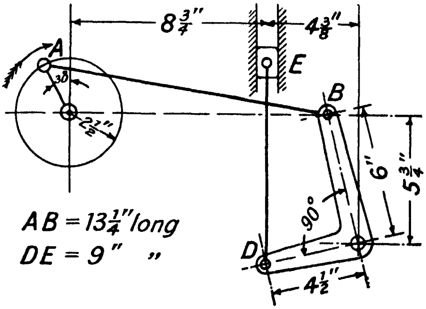

FIG. 90

11. A riveter (Fig. 91) is operated by a piston *F*, acting through the links *EB*, *AB*, *BC*; *D* is the ram carrying the tool. The piston moves in a line perpendicular to the line of motion of *D*, *BC* = 2*AB*. In the position shown,

AB makes 12° with *AC*, and *BE* is at 90° to *AC*. Find the velocity ratio of *E* to *D*. In the same position the total load on the piston is 500 lb.; find the thrust exerted by *D* if the efficiency is 72 per cent. (U.L.)

12. The outline of a portion of the mechanism of a Joy's valve gear is shown in Fig. 92. The crank *OR* is 1 ft. and the connecting-rod *CR* 6 ft. 3 in. The length *RD* is 4 ft. *AB* is a swing link 3 ft. 8 in. long, pivoting at the fixed point *A*. *BD* is 1 ft. 7½ in. in length. The crank revolves at 210 revs. per min. Find the angular velocity of the link *BD* at the instant when the crank angle *ROC* is 150°. (U.L.)

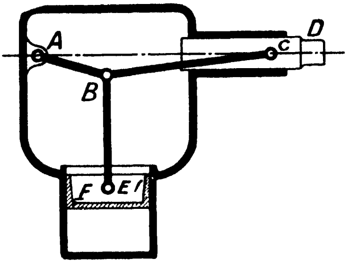

FIG. 91

13. Two cranks of lengths respectively 2 ft. and 1 ft. are pivoted on centres 4 ft. apart, and are coupled by a rod 4 ft. 6 in. long. The small crank rotates at 120 revs. per min. Determine the velocity and acceleration of the end of the large crank when this makes an angle of 60° with the line of centres. (U.L.)

14. Show that a variable cut-off with a *D* slide valve can be obtained with a gear which gives the displacement of the valve, x, from mid position, $x = r \cos(\theta + \phi) = A \cos\theta - B \sin\theta$ where A is constant and B is variable, θ is the crank angle from the line of dead centres. (Note $\phi = 90 +$ angle of advance.)

Describe the Walschaert valve gear, and find approximate expressions for *A* and *B*. How is the gear designed to keep the lead nearly constant? (U.L.A.)

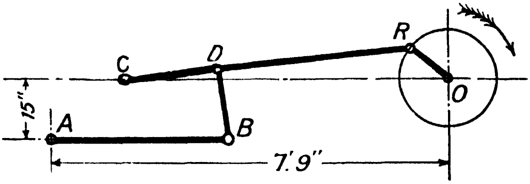

FIG. 92

15. Copy the diagram of Fig. 76 on a larger scale. Assume that the crank is 12 in. long and is making 200 revs. per min. Draw the velocity and acceleration images for the given position; also find the inertia force on the rod *EFP*, which is of uniform section and weighs 25 lb.

16. A mechanism as shown in Fig. 93 has cranks *BC* and *EF* rotating about fixed centres *C* and *F* respectively, and *A* slides in the line *AC*. $AB = ED = FE = 3$ ft., $BC = BD = 1$ ft. The point *B* has a velocity of 50 ft. per sec. When θ is 30° determine: (1) the velocities of the points *A*, *D* and *E*, (2) the accelerations of *D* and *E*. (U.L.A.)

17. A four-bar motion consists of cranks *AB* and *CD*, a fixed link *AC* and a coupler *BD* of the following lengths: *AB* is 1 ft., *CD* is 2 ft., *AC* is 3 ft., and *BD* is 2½ ft. The crank *AB* rotates uniformly at 50 revs. per min. The coupler *BD* is of uniform section and weighs 25 lb. Determine the direction, line of action, and magnitude of the resultant accelerating force acting on the coupler *BD* when the crank *AB* is at right-angles to *AC*.

18. In a simple engine mechanism the length of the crank is 15 in., and that of the connecting-rod is 6 ft. The weight of the connecting-rod is 270 lb., its radius of gyration about the centre of gravity is 20 in., and its centre of

gravity is at a distance of 40 in. from the crank-pin. Find the magnitude and line of action of the accelerating force (or inertia force) acting on the connecting-rod when the crank is rotating at a uniform speed of 100 revs. per min., at the instant when the crank has turned through 30° from the inner dead-centre position.

19. Sketch the velocity and acceleration images of a rod rotating about one end (1) with uniform angular velocity, (2) with angular velocity ω and angular acceleration α Sketch the images of the two bars O_1P and O_2P of Fig. 67, Art. 88.

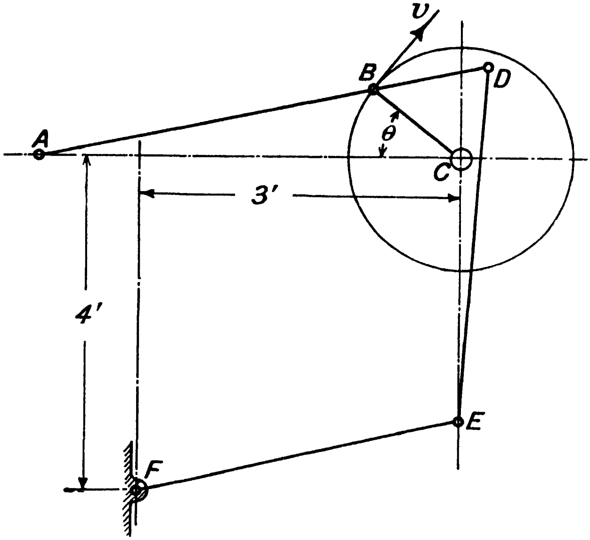

FIG. 93

20. Show that the mechanism of Joy's valve gear gives to the valve, for any crank angle, a displacement which is approximately the sum of those which would be given to it separately by two eccentrics, one of which is 90°, and the other 180°, in front of the crank.

21. Repeat example 20 for the mechanism of the Hackworth radial valve gear.

22. Explain how to find approximately the equivalent eccentric for the mechanism of Stephenson's link motion with the link in any given position.

23. ABC is a simple engine mechanism, with the crank BA equal in length to the connecting-rod BC. Let a be the length of the crank, and θ the crank angle. Find expressions for the velocity and acceleration of a point D in BC where $\overline{BD} = b$. Assume the angular velocity of the crank to be constant. Prove that D traces out an ellipse, and find the radius of curvature r at the end of the major axis, making use of the relation

$$\frac{(\text{velocity})^2}{r} = \text{radial acceleration}$$

24. A particle P moves so that its rectangular co-ordinates x and y are given by $x = bt$ and $y = at^2$ respectively, t being the time. Show that the

path of P is a parabola, and, by the method suggested in example 23, find the radius of curvature of the parabola at the origin.

25. The crank of a steam-engine is 9 in. long, and the connecting-rod is 3 ft. 9 in. long. The crank-pin is rotating at a uniform speed of 100 revs. per min. Find, using Klein's construction—

(1) The piston acceleration,

(2) The acceleration of the piston relative to the crank-pin,

(3) The acceleration of the middle point of the connecting-rod,

when the crank has turned through 30° from the inner dead centre.

26. Show how to determine the angular acceleration of the connecting-rod of a direct-acting engine for any given configuration of the mechanism. (I.C.E.)

27. In the mechanism shown in Fig. 94, the connecting-link BC is perpendicular to OE when the point B falls on E OA, $= 9$ in., $OE = 3$ ft., $AB = 3$ ft., $BC = 4$ ft., and $CD = 1$ ft. The crank A has a speed of 80 revs. per min. Determine the velocity and acceleration of the point D when the crank angle AOE is 30°. (I.C.E.)

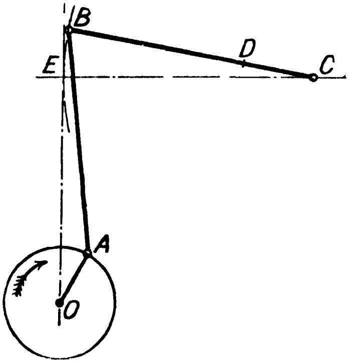

FIG. 94

28. The travel of a slide valve is $3\frac{1}{2}$ in., advance 45°, steam lap $1\frac{1}{8}$ in. Find the points of admission and cut-off. If the exhaust lap is $\frac{5}{16}$ in., find the points of release and compression, neglecting obliquity of connecting-rod.

29. Given travel $3\frac{1}{2}$ in., lead $\frac{3}{16}$ in., cut-off at 0·55, compression at 0·8, find the maximum opening, angle of advance, point of release, steam and exhaust laps.

30. Given travel 3 in., steam lap $\frac{3}{4}$ in., lead $\frac{1}{8}$ in., compression at 0·85 of stroke, find the angle of advance, maximum opening, exhaust lap, and points of cut-off and release.

31. Given travel 4 in., lead $\frac{1}{4}$ in., cut-off at 0·2, compression at 0·8, find the maximum opening, angle of advance, point of release, steam and exhaust laps. What conclusion do you draw from your results ?

32. In each of the questions 28–31 above, find the velocity of the valve at cut-off if the speed of revolution is 300 per min.

33. In question 28, taking crank angle from admission to cut-off—

(*a*) Plot a curve of piston velocity on a crank angle base,

(*b*) Plot a curve showing valve openings on the same base,

(*c*) Plot a curve showing the ratio of (*a*) to (*b*) on a piston travel base.

Take a piston stroke of 2 ft., and speed 200 revs. per min.

(*Note.* (*c*) gives a measure of the rate of drop of pressure in the cylinder due to " wire-drawing.")

34. Given cut-off at 0·65, maximum opening $1\frac{1}{8}$ in., lead $\frac{1}{4}$ in., compression at 0·85, find the travel, angular advance, steam and exhaust laps, point of release.

35. Repeat question 34, taking a cut-off of 0·25.

36. In each of questions 28 to 31, find the actual points of cut-off in each stroke, taking the connecting-rod = 4 cranks in length.

37. The following particulars of a Meyer expansion gear are given: Travel of main valve 3 in., angular advance 35°; travel of expansion valve $3\frac{1}{2}$ in., angular advance 85°. Find the range of lap of the expansion valve if the range

of cut-off is from 0·15 to 0·45. Find also in each case the maximum opening to steam if the lap of the main valve is 1½ in., lead ⅛ in.

(*Note.* *The maximum opening to steam occurs when the main and expansion valves are open by equal amounts.*)

38. A simple slide valve has a stroke of 4 in. The angle of advance of the eccentric is 35°. If connecting-rod = 4 cranks, determine the steam and exhaust laps necessary for cut-off at 65 per cent and for release at 95 per cent on the inward stroke. (U.L.)

39. The displacements in inches of the main and expansion valves of a Meyer gear are $x = 1{\cdot}5 \cos (\theta + \frac{3}{4}\pi)$ and $x = 1{\cdot}5 \cos (\theta + \pi)$. The outside lap of the main valve is 0·8 in. Find the point of cut-off by the main valve on the outward stroke; find also the negative laps for the expansion valve to give cut-off respectively at 0·2 and 0·3 of the outward stroke. Connecting-rod = 5·5 cranks. (U.L.A.)

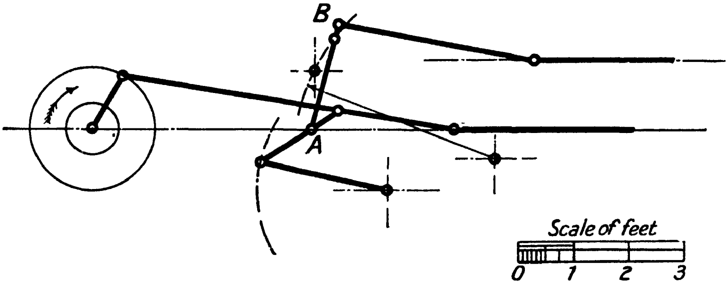

FIG. 95

40. Fig. 95 is an outline sketch of a Joy valve gear. You are required to construct for the position of the crank shown when the crank is making 180 revs. per min.—

(*a*) A velocity diagram,

(*b*) An acceleration diagram.

Crank radius is 13 in. Find the inertia forces on the rod *AB*, which is of uniform section and weighs 24 lb. (U.L.A.)

41. A Meyer variable cut-off gear is required for a range of cut-off from 0·1 to 0·75 of the stroke. The lead of the main valve is ¼ in., maximum port opening 1¼ in., width of passage through main valve 1⅜ in. The throw of the expansion eccentric is 1·1 times the throw of the main eccentric and it is set directly opposite the crank. The width of the steam ports in the steam chest is $1\frac{5}{16}$ in., the bars 1 in., and the exhaust port 2⅞ in., and release occurs at 0·95 stroke. The width of the plates is such that they just touch for latest cut-off.

Make a dimensional sketch of the valves, showing both valves in mid-position and the cut-off plates in position for latest cut-off. Show that re-opening does not occur with the earliest cut-off. (U.L.A.)

42. Find the approximate equivalent eccentric for the gear of the previous example. Give the length and the angle of advance.

43. *AB* is a rigid link whose acceleration centre is a point *C*. Lines are drawn through *A* and *B* respectively representing their accelerations. These lines intersect in a point *P*. Show that the points *A*, *B*, *P*, and *C* are concyclic.

44. Show that if *A*, *B*, *C*, *D*, etc., are points in a rigid body and *a′*, *b′*, *c′*, *d′*, etc., are their acceleration images, the two polygons obtained by joining the

two sets of points in the same order are similar. What point in the body has the pole o' of the acceleration diagram as image ?

45. Find the acceleration centre of the coupling link in example 13, in the given position. Using this, find the acceleration of the middle point of the coupling link. You may use any results given in the answer to example 13. Given the directions and magnitudes of the accelerations of two points A and B in a rigid link, give a geometrical construction for finding the acceleration centre.

46. In the oscillating cylinder mechanism of Fig. 44 the lengths of the elements 1 and 3 are each 4 ft., and that of the crank is 1 ft. If the crank is rotating at a constant speed of 60 r.p.m., find for the instant when the crank is at the inner dead centre position;

(*a*) The velocity and acceleration of the piston relative to the cylinder;

(*b*) The velocity and acceleration of the piston relative to the framework.

47. Solve the previous example for the position in which the crank has turned through 60° from the inner dead centre.

(*Note.* In solving problems on the acceleration of a body sliding along a rotating bar the relative transverse acceleration is sometimes ignored (see I.38). This should be taken into account as in Art. 88. When considering the motion of that point in element 1 (Fig. 44) which instantaneously coincides with the axis of the trunnion, $r = 0$ in I.38, and the second term on the right disappears from the expression.)

48. A single cylinder steam engine has a connecting rod 4 ft. long, the stroke being 2 ft. A simple slide valve is required to cut off steam at 0·6 of the stroke and have a maximum opening to steam of 1 in. The lead is one-tenth of an inch. Find the travel, steam lap, and angle of advance of the valve to satisfy these conditions for the outstroke. (U.L.)

49. In a steam engine the connecting rod is four times as long as the crank. The latter makes 120 r.p.m. and is 1 ft. long. Draw the acceleration diagram for the cross-head when the crank makes an angle of 45° with the inner dead centre. What is the magnitude of the acceleration ? (U.L.)

50. On a petrol engine of 6 in. stroke running at 1,800 r.p.m. the diameters of the main bearings, the crank pins, and the gudgeon pins are respectively 2½ in., 2½ in., and 1¾ in. The connecting rod ratio is 3·5.

Find the rubbing speeds of these bearings when the piston is at one-fifth stroke (*a*) on the down-stroke, (*b*) on the up-stroke, in inches per second. Find also the velocities and accelerations of the piston in these two positions, all in in.-sec. units. (U.L.)

51. Referring to the slotted link mechanism shown in Fig. 89, if $OC = r$, $OA = p \, . \, r$, and $AOC = 180° - \theta$, the angular velocity ω_{AB} of AB is given by

$$\omega_{AB} = \frac{1 + p \, . \cos\theta}{1 + p^2 + 2p\cos\theta} \cdot \omega_{OC}{}^{*}$$

and the angular acceleration α_{AB} of AB is given by

$$\alpha_{AB} = \frac{p(p^2 - 1)\sin\theta}{(p^2 + 1 + 2p \, . \cos\theta)^2} \cdot \omega^2{}_{OC}{}^{*}$$

Plot curves of $\frac{\omega_{AB}}{\omega_{OC}}$ and $\frac{\alpha_{AB}}{\omega^2{}_{OC}}$ for a half revolution of OC, when $p = 1{\cdot}8$.

Show that for maximum angular acceleration of AB

$$\cos\theta = \tfrac{1}{2}\left\{\frac{p^2 + 1}{2p} - \sqrt{8 + \left(\frac{p^2 + 1}{2p}\right)^2}\right\}.$$

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* "Slotted Link Mechanisms," by A. T. J. Kersey, *Engineering,* 11th and 18th April, 1947.

CHAPTER VI

STRAIGHT LINE MECHANISMS—HOOKE'S JOINT

107. **Straight Line Mechanisms.** One of the commonest forms of constraint in mechanisms is that which permits only of relative motion of an oscillatory nature along a straight line. This constraint is usually provided by the use of a sliding pair one element of which is fixed to the frame of the machine, as in the case of the crosshead and slide bars of the simple steam-engine. There are, however, other methods of providing the necessary constraint, and mechanisms for this purpose are called *straight line mechanisms*. These mechanisms are of two kinds: (1) those in which only turning pairs are used, and (2) those in which the required relative motion is obtained by the use of a mechanism in which at least one point is constrained to move in a straight line by the use of a sliding pair, the motion of the point under consideration being a kind of copy of the motion of a point in the sliding pair.

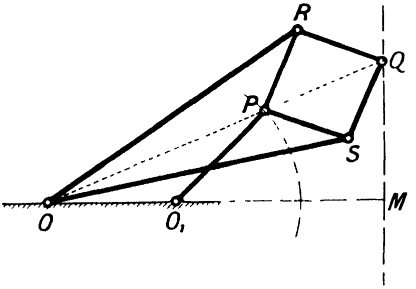

FIG. 96. PEAUCELLIER'S PARALLEL MOTION

There are also two types of each of the above kinds; (*a*) those in which the straight line motion produced is exact, and (*b*) those in which it is only approximate.

108. **Exact Straight Line Mechanisms made up of Turning Pairs.** We shall consider two well-known mechanisms under this heading. These are shown in Figs. 96 and 97, the former being Peaucellier's and the latter Hart's parallel motion.

In each figure, OO_1 is a fixed link and the lines represent straight links connected by turning pairs at their intersections. In Fig. 96, the links *PR*, *RQ*, *QS*, and *SP* are equal, and *OR* and *OS* are equal. In Fig. 97, $\overline{RV} = \overline{ST}$, $\overline{RT} = \overline{SV}$, and the points *O*, *P*, and *Q* divide the links *VR*, *VS* and *TR* respectively in the same ratio. In both mechanisms $\overline{O_1O}=\overline{O_1P}$. In Fig. 96

it follows from symmetry that OPQ is a straight line. In Fig. 97 it is easy to see that $VRST$ is a trapezium and that OP and OQ are respectively parallel to RS and VT, which are themselves parallel. Hence OPQ is a straight line. In both mechanisms the point P is made to describe a circle about O_1 as centre, and because of the equality of $\overline{O_1O}$ and $\overline{O_1P}$, these circles pass through O. We shall prove that Q describes a straight line perpendicular to OO_1.

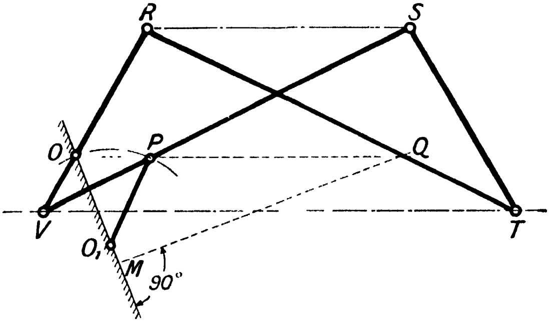

FIG. 97. HART'S PARALLEL MOTION

Let O (Fig. 98) be a point on the circumference of a circle OPA, centre O_1. Let OA be a diameter and OP any chord. Let Q be a point in OP, or OP produced, such that for all chords through O, $OP \cdot OQ = k$, where k is a constant, then shall the locus of Q be a straight line perpendicular to OA. Draw QM perpendicular to OA, or OA produced. Join AP. OAP and OQM are similar triangles. Hence,

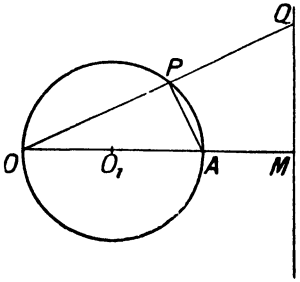

FIG. 98

$$\frac{\overline{OP}}{\overline{OA}} = \frac{\overline{OM}}{\overline{OQ}}$$

or $\overline{OP} \cdot \overline{OQ} = \overline{OM} \cdot \overline{OA}$, and since $\overline{OP} \cdot \overline{OQ} = k$

$$\overline{OM} \cdot \overline{OA} = k$$

$$\therefore \overline{OM} = \frac{k}{\overline{OA}} = \text{constant} \qquad . \quad . \quad . \qquad \text{(VI.1)}$$

Thus, for all positions of Q, OM is fixed so that the locus of Q is the straight line QM perpendicular to OM.

In order to prove that the point Q in either Fig. 96 or Fig. 97 describes a straight line perpendicular to OO_1, it remains only to show that in each mechanism $OP \cdot OQ$ is constant. In Figs. 99 and 100, we have reproduced the parts of the figures which fix the relative positions of the points O, P, and Q.

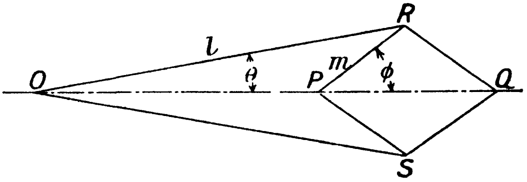

Fig. 99

In Fig. 99, let $OR = l$ and $\overline{RP} = \overline{RQ} = m$. In Fig. 100, let $\overline{VP} = \overline{PS} = \overline{RQ} = \overline{QT} = l$, and let $\overline{OV} = \overline{OR} = m$; i.e. O, P, and Q are the middle points of the rods on which they lie (we are here taking a particular case).

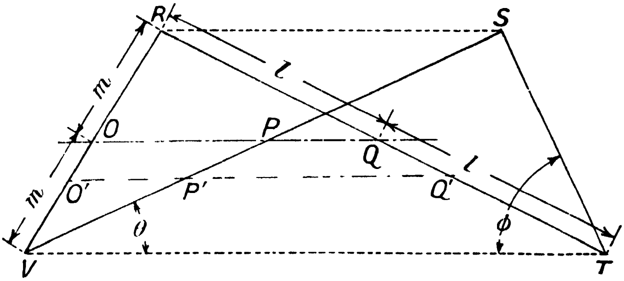

Fig. 100

In both figures it will be seen that $\overline{OP}$ is the difference of the projections on OQ of the lines l and m, and that $\overline{OQ}$ is the sum of these projections. If θ is the acute angle between the lines of length l and OQ, and ϕ that between the lines of length m and OQ, we have,

$$\overline{OP} = l \cos \theta - m \cos \phi$$

$$\overline{OQ} = l \cos \theta + m \cos \phi$$

$$\begin{aligned} \therefore \overline{OP} \,.\, \overline{OQ} &= l^2 \cos^2 \theta - m^2 \cos^2 \phi \\ &= l^2 (1 - \sin^2 \theta) - m^2(1 - \sin^2 \phi) \\ &= l^2 - m^2 + m^2 \sin^2 \phi - l^2 \sin^2 \theta \\ &= l^2 - m^2, \text{ since } m \sin \phi = l \sin \theta \end{aligned}$$

or $$\overline{OP} \,.\, \overline{OQ} = \text{constant} = k \quad . \quad . \quad . \quad . \qquad \text{(VI.2)}$$

Thus, in each case the point Q will describe a straight line perpendicular to OO_1, and at a distance from O given by

$$\begin{aligned} \overline{OM} &= \frac{k}{\overline{OA}} \\ &= \frac{\overline{OP} \,.\, \overline{OQ}}{2 \,.\, \overline{OO_1}} \qquad \text{(See Fig. 98.)} \\ &= \frac{l^2 - m^2}{2 \,.\, \overline{OO_1}} \end{aligned}$$

In Fig. 100, we have assumed O, P, and Q to be the middle points of the bars. If, however, OPQ occupy other positions on the bars, say $O'P'Q'$, then, since in all positions $O'P'Q'$ is parallel to RS and VT, $\frac{O'P'}{OP}$ will be constant for all positions of the mechanism, as will $\frac{O'Q'}{OQ}$, and so the product $O'P' \,.\, O'Q'$ will remain constant because it bears a constant ratio to the constant quantity $OP \,.\, OQ$. In this case the value of $\overline{OM}$ will not be given by the above relation.

We have proved that if the point P in either of the arrangements of Figs. 96 and 97 is made to describe a circle passing through the point O, the point Q will describe a straight line. In the proof, the points P and Q may be interchanged, so that if Q is made to describe a circle passing through O, then P will trace out a straight line.

109. **An Exact Straight Line Mechanism, containing One Sliding Pair.** Fig. 101 shows *Scott Russell's parallel motion.* P is the moving member of a sliding pair, C being the fixed member. PAQ is a straight link connected by turning pairs to the link OA and the link P. OA rotates about O. The mechanism OAP is really the simple engine mechanism. A is

the middle point of PQ, and $\overline{OA} = \overline{AP} = \overline{AQ}$. The instantaneous centre for the bar QAP is at I in OA produced, and is such that IP is perpendicular to OP. Join IQ. Then Q moves along the perpendicular to IQ. By simple geometry it is clear that the figure $OPIQ$ is a rectangle, and IQ is perpendicular to OQ. Hence, for all positions of QP, Q is moving along the vertical line OQ, and traces out the straight line QOQ'. If the crank makes complete revolutions, P will oscillate along the line OP through a distance $2 \times \overline{OA}$ on each side of O, and Q will oscillate along QOQ' through the same distance, $2 \times \overline{OA}$, above and below O. The locus of Q is a copy of the locus of P, and any irregularities in the latter will be reproduced in the former. If the motion of P is limited to a short distance near the right-hand end of its stroke, the crank OA will oscillate about the horizontal, which will be its mean position, and the motion of Q will be correspondingly limited.

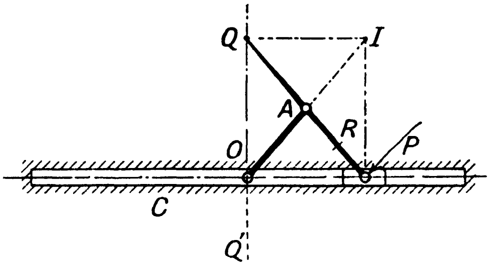

FIG. 101. THE SCOTT RUSSELL PARALLEL MOTION

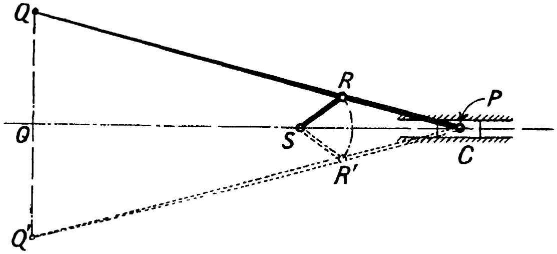

FIG. 102. MODIFIED SCOTT RUSSELL MECHANISM

110. **Mechanisms Derived from Scott Russell's Parallel Motion.** We proved in Art. 71 that any point R in PQ traces out an ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ where } a = \overline{QR}, \text{ and } b = \overline{PR}$$

We may then remove the link OA from the mechanism, if we constrain some point R to move on a suitable ellipse. If PQ rocks through only a small angle about the horizontal position, the point R will move along a short arc at the end of the major axis of the ellipse. This arc will deviate very little from the circle of curvature at the end of the major axis. Thus, there will only be a very slight difference in the motion of Q in the vicinity of O if we constrain R to move, not on the ellipse, but on an arc of the circle of curvature at the end of the major axis.

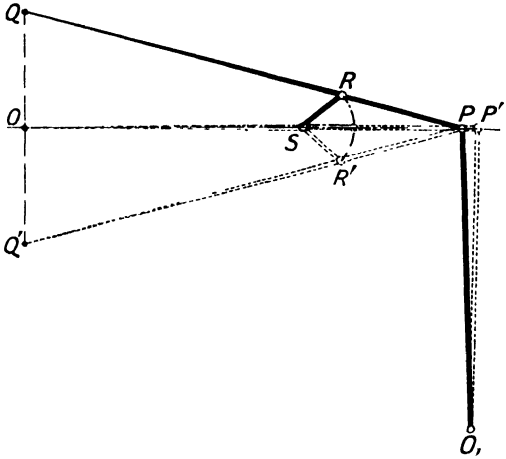

FIG. 103. THE GRASSHOPPER PARALLEL MECHANISM

Fig. 102 shows a mechanism in which the bar OA of Fig. 101 has been removed, and the constraint has again been completed by introducing a bar SR.

The length of the link SR is given by

$$\overline{SR} = \text{radius of curvature at end of major axis}$$
$$= \frac{(\text{semi-minor axis})^2}{(\text{semi-major axis})}$$
$$= \frac{b^2}{a}$$
$$= \frac{\overline{RP}^2}{\overline{QR}}$$

The line traced by Q is approximately straight.

The arrangement of Fig. 102 is an *approximate straight line mechanism* containing one sliding pair and three turning pairs. The difference between the actual locus of Q and a straight line is inappreciable for small angular displacements of the link PQ, but the difference increases with the angular displacement. The point P in Fig. 102 is constrained to move along OP by the sliding pair $P-C$. This sliding pair is equivalent to a turning pair at infinity along a line through P perpendicular to OP. Such an arrangement may be approximated to by making use of a turning pair at O_1 in Fig. 103, where

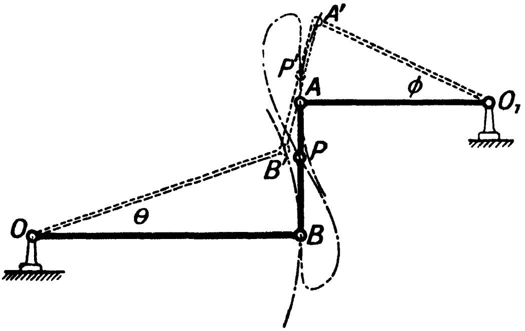

Fig. 104. Watt's Parallel Motion

$\overline{O_1P}$, while not infinite, is large compared with the distance moved by P in one stroke. The motion of P is then limited to a short length of arc, and P may, without serious error, be considered as moving along the tangent OP'. The mechanism thus modified is known as the "*Grasshopper*" *parallel motion.*

In the mechanism, SR oscillates about S as centre through an angle RSR', and thus causes P to oscillate along the circular arc which has O_1 as centre, O_1P being a long rod whose mean position is perpendicular to OS produced. Q oscillates along the line QOQ', which is practically straight. It will be noticed that the mechanism is actually a four-bar crank chain $SRPO_1$, and that all the pairs are turning pairs.

111. **Watt's Parallel Motion.** Various modifications of the four-bar mechanism have been used to give approximate straight line motions. The earliest type was one devised by Watt, and known as *Watt's parallel motion.* This is shown in Fig. 104.

$OBAO_1$ is a crossed four-bar chain with OO_1 fixed. OB and

AO_1 are parallel in the mean position of the mechanism, and in the figure the coupling-rod AB is perpendicular to O_1A and OB. The tracing-point P which traces out the approximate straight line is in AB. In any small displacement of the mechanism the rod AB will remain approximately vertical, and the points A and B will move through approximately equal vertical distances. Let $OB'A'O_1$ be the mechanism in a displaced position, P' being the new position of P. If the motion of P is along a straight line, that line must be PA, because, in the initial position, the instantaneous centre I is at infinity along the lines OB, AO_1, and the motion is in the direction of PA. If the angles BOB', AOA' are small, the arcs BB', AA' are nearly vertical, and neglecting the inclination of $A'B'$ to the vertical, $\overline{BB'} = \overline{AA'}$.

Let $\theta = \widehat{BOB'}$, $\phi = \widehat{AO_1A'}$, then

$$\theta \, . \, \overline{OB} = \phi \, . \, \overline{O_1A}$$

or $$\phi = \frac{\overline{OB}}{\overline{O_1A}} \theta$$

The distance of B' to the left of BA

$$= \overline{OB} \, (1 - \cos \theta)$$

or, since $\cos \theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \ldots$

distance of B' to left of BA

$$= \overline{OB} \, . \, \frac{\theta^2}{2} \text{ approx.}$$

Similarly, the distance of A' to the right of BA

$$= \overline{O_1A} \, \frac{\phi^2}{2}$$

$$= \frac{\overline{OB}^2}{\overline{O_1A}} \, . \, \frac{\theta^2}{2} \text{ approx.}$$

Also, since P' is in BA, the ratio of these two distances is

$$\frac{\overline{P'B'}}{\overline{P'A'}} = \frac{\overline{PB}}{\overline{PA}}$$

Hence
$$\frac{\overline{PB}}{\overline{PA}} = \frac{\overline{OB}\,\frac{\theta^2}{2}}{\frac{\overline{OB}^2}{\overline{O_1A}}\,\frac{\theta^2}{2}} = \frac{\overline{O_1A}}{\overline{OB}} \times \frac{\theta^2}{\theta^2}$$

i.e.
$$\frac{\overline{PB}}{\overline{PA}} = \frac{\overline{O_1A}}{\overline{OB}} \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(VI.3)}$$

Thus, the point P divides the coupler AB into two parts whose lengths are inversely proportional to the lengths of the adjacent cranks. The dotted line shows the complete locus of P. A portion of this locus is indistinguishable from the straight line BA. In practice P is used to guide a point to which it is desired to give rectilinear reciprocatory motion over a distance not greater than the length of the approximately straight part of the locus of P. If O is on the same side of BA as O', we have the mechanism of Fig. 105. In any displacement from the mean position shown, B and A will both be displaced to the right of the line AB. The horizontal displacement of B will be greater than that of A, and as a consequence some point P on AB produced will remain on the original line. This point will be the correct tracing point. The reader should prove that $\frac{\overline{PB}}{\overline{PA}} = \frac{\overline{O_1A}}{\overline{OB}}$ as before. In some forms of the mechanism, the coupler AB is not perpendicular to the cranks OB, O_1A, but is inclined to both at an angle a little less than a right-angle, as shown in Fig. 106.

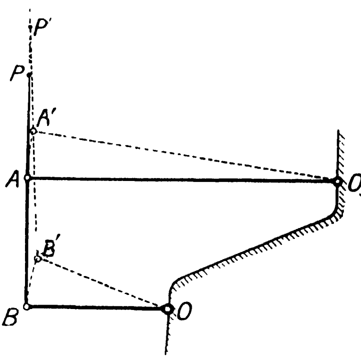

FIG. 105. WATT'S PARALLEL MOTION WITH TRACING POINT ON EXTENSION OF THE COUPLING BAR

On examination of Fig. 104, it will be seen that for all possible configurations of the coupling link AB, except that in which it is vertical, it is rotated in the clockwise sense from its initial position. In order that the obliquity of the coupler when in the extreme positions shall be as small as possible, it may be given a small initial angular displacement in the opposite sense, as shown in Fig. 106. The initial displacement should be such that in its extreme and mean positions the coupler is equally inclined to the vertical. This will be the case if the horizontal components of the total displacements of A and B have their middle points on the same vertical line. If, further, the tracing-point P is on this vertical line when the mechanism is in its mean position, the extreme positions of P will also lie on the line. The proportions are the same as those in Fig. 104,

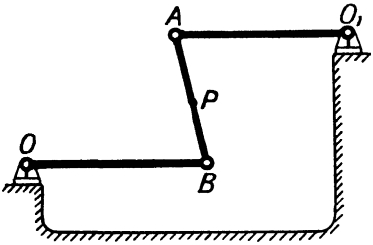

FIG. 106. WATT'S PARALLEL MOTION WITH INCLINED COUPLING LINK

i.e. $\frac{\overline{PB}}{\overline{PA}} = \frac{\overline{O_1A}}{\overline{OB}}$.

112. Other Approximate Straight Line Mechanisms derived from the Four-bar Mechanism. The arrangement of Fig. 107 is a four-bar mechanism, which, in its mean position, has the form of a trapezium $OABO_1$, with the two non-parallel sides OA and O_1B of equal lengths. OO_1 is fixed, and the bar PQ is rigidly attached to the bar AB at its middle point P. The dotted lines show the mechanism in a displaced position. Q is the tracing point, and traces out the approximately straight line QQ'. A', B', P', and Q' are the positions of the points A, B, P, and Q respectively when the mechanism is displaced

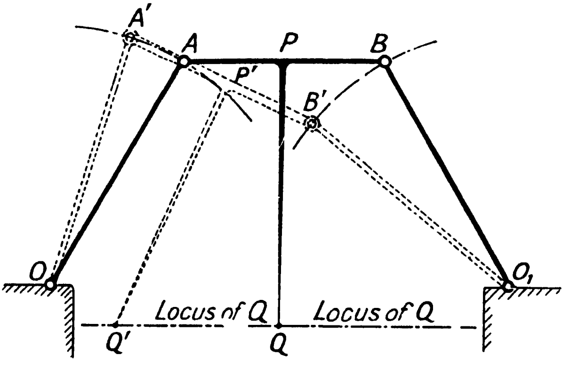

FIG. 107. FOUR-BAR STRAIGHT LINE MECHANISM

to the dotted position. If Q' lies on the horizontal line through Q, the distance of P' below the line AB must be equal to the difference between $\overline{PQ}$ and the vertical projection of $P'Q'$. Thus, if l is the length of PQ, and a is the angle between $P'Q'$ and PQ we have

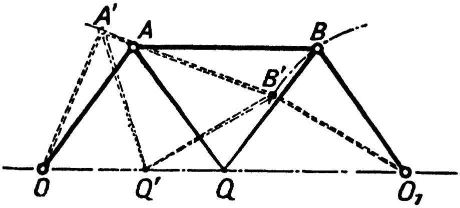

FIG. 108. ROBERTS' PARALLEL MOTION

d = depth of P' below AB

$$= l - l \cos \alpha$$

$$d = l(1 - \cos \alpha) \quad \text{(VI.4)}$$

If the dotted position is an extreme position of the mechanism, the point Q will trace out a line $Q'Q$, which is very nearly straight. The end points and the mid-point of the locus lie on an horizontal straight line. If the dotted position is not an extreme position, the point Q' will deviate but slightly from the line in a further small displacement to the left. The dimensions of the mechanism limit the travel of Q, the extreme left-hand position occurring when the links AB and BO_1 are in a straight line. In order that the deviation from the straight line should be as small as possible, the point Q', at which the tracing point is exactly on the line, should be chosen, so that there are equal maximum amounts of deviation on the left of Q' and between Q' and Q. For any mechanism the values of d and a may be determined by drawing, and the length of l may then be calculated from (VI.4).

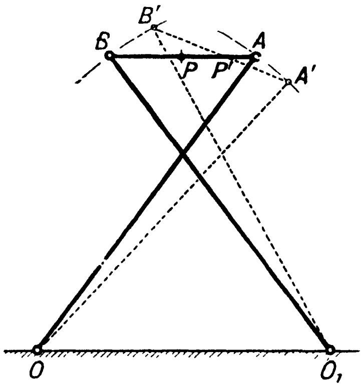

FIG. 109. TCHEBICHEFF'S PARALLEL MOTION

One form of this mechanism is known as *Roberts' Parallel Motion*. In this, $\overline{AB} = \frac{1}{2}\,\overline{OO_1}$ and $\overline{PQ}$ is equal to the distance between AB and OO_1, so that Q traces out part of the straight line OO_1. In the actual mechanism (Fig. 108) there is no bar

PQ, but Q is joined by equal bars AQ and BQ to the points A and B. If the point Q can move to O or O_1, it must coincide exactly with these points, because of the equality of the four inclined links.

If we use the mechanism of Fig. 107 with the piece PQ removed, and the links OA, OB crossed (Fig. 109), P being now the tracing point, we obtain the mechanism known as *Tchebicheff's Parallel Motion.*

In the position shown in full P is moving horizontally, because the instantaneous centre of the link AB is at the intersection of OA and O_1B, which is directly below P. P traces out an approximate straight line parallel to OO_1. The mechanism is usually made so that P is exactly on the line at points directly above O and O_1, one of the long links and the short link being vertical for those positions of P. The point P is on the straight line, at the middle and ends, and at each of these positions its velocity is in a horizontal direction. The ratios of the lengths of the links for this arrangement are easily determined, either by geometry or by algebra (Ex. 13, page 178).

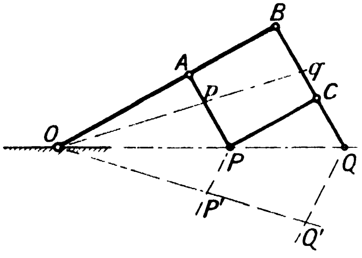

FIG. 110. THE PANTOGRAPH

113. **The Pantograph.** This instrument consists of a jointed parallelogram $ABCP$ (Fig. 110), made up of bars connected by turning pairs. The bars BA, BC are extended to O and Q respectively, so that $\frac{\overline{OA}}{\overline{OB}} = \frac{\overline{AP}}{\overline{BQ}}$. Thus, for all relative positions of the bars, the triangles OAP, OBQ are similar and so O, P, and Q are in one straight line. Also $\frac{\overline{P}}{\overline{OQ}} = \frac{\overline{AP}}{\overline{BQ}}$ = constant. Let O be fixed, and let P be moved to some new position P'. Let Q' be the new position of Q. Then $\frac{\overline{OP}}{\overline{OQ}} = \frac{\overline{OP'}}{\overline{OQ'}}$, and therefore the straight line PP' is parallel to the straight line QQ'. Hence, if O is fixed to the frame of a machine by means of a turning pair, and P is attached to a point in the

machine which has rectilinear motion relative to the frame. Q will also trace out a straight line path. Owing to the constant ratio of $\overline{OQ}$ to $\overline{OP}$, the travel of Q will be $\frac{\overline{OQ}}{\overline{OP}}$ times the travel of P, and the velocities of the two points will also have the ratio $\frac{\overline{OQ}}{\overline{OP}}$. If Q is constrained to move in a straight line, then P will trace out a straight line parallel to the former. The pantograph used in this way along with the frame to which O is connected, and the constraint at either P or Q, is a mechanism. It is also used as a geometrical instrument for drawing similar figures; if, for instance, P traces out the sides of a triangle, Q will trace the sides of a similar triangle.

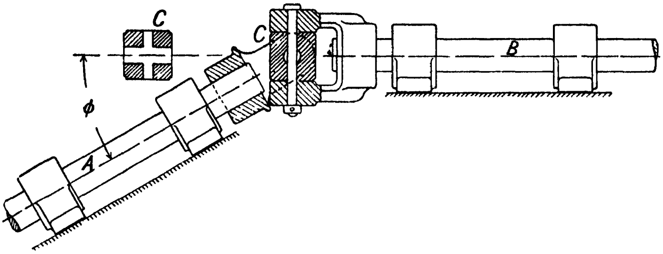

FIG. 111. PLAN OF HOOKE'S JOINT

In some applications the point P is attached to the tracing point of Watt's parallel motion, O being pivoted at a fixed point. In this case, Q will move in a path which is an enlarged copy of the motion of P. It is not essential that the point P should be at an angular point of the parallelogram. Thus, if Opq is a straight line, and one of the points p or q is guided in a straight line, the other point will trace out a parallel straight line. O, being fixed to the engine frame, and Q carried by the crosshead, P will, if properly connected by means of a cord to the drum of an indicator, give to the drum a motion proportional to that of the piston. The pantograph is also used for the reproduction of plane areas, and figures, such as maps, plans, etc., on enlarged or reduced scales.

114. **Hooke's Joint** is a means of connecting two rotating shafts whose axes lie in one plane, their directions making a small angle with each other. Two horizontal shafts A and B

are shown in Fig. 111, and it will be seen that they each form one element of a turning pair, the other elements of the two pairs being two cylinders rigidly connected together with their axes at right-angles. The two cylinders form a cross, as shown at *C*. The angle between the axes of the shafts is ϕ.

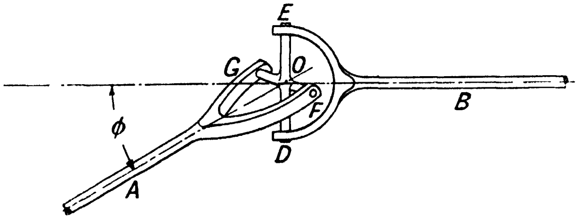

FIG. 112. PICTORIAL VIEW OF SIMPLE FORM OF HOOKE'S JOINT

The shafts *A* and *B* are carried in cylindrical bearings, and the axes of these shafts and those of the cylinders forming the cross *C* intersect in one point. Fig. 112 gives a conventional view of the joint in another position. *GF* and *DE* are the axes of the cylinders forming the cross, *O* is their point of intersection. We shall assume that $\overline{OE} = \overline{OD} = \overline{OF} = \overline{OG}$, though the arrangement will give the same variation of velocity ratio whatever the length of the two cylinders if *O* lies at the point of intersection of the horizontal shafts *A* and *B*. The four points *EFDG* will move on the surface of a sphere of centre *O*, *G* and *F* describing a great circle in the plane perpendicular to the shaft *A*, and *E* and *D* describing a great circle in a plane perpendicular to the shaft *B*.

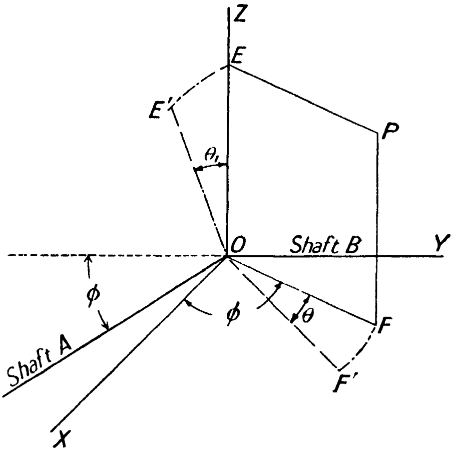

FIG. 113

115. Relation Between the Angular Velocities. In Fig. 113, let *OX*, *OY*, *OZ* be three axes of co-ordinates, each axis being perpendicular to the other two. Suppose the shafts A and B of Fig. 112 to be placed so that their intersection lies at the origin O, the shaft B along the axis of y, and the shaft A in the plane of XOY. Let the arm OE of the cross coincide with OZ. Then, since OF is perpendicular to OE, OF will lie in the horizontal plane and also in a plane $EPFO$ perpendicular to the shaft A. This latter plane makes an angle ϕ with the plane ZOX. If now the shaft B is rotated through an angle θ_1, E will move to E', so that $\widehat{EOE'} = \theta_1$, and F will move to F' in the plane $EPFO$. Let $\widehat{FOF'} = \theta$, then θ is the angle turned through by the shaft A. OE' and OF' are perpendicular in all positions, hence if l, m, and n are the direction cosines of OF' with OX, OY, and OZ respectively, and l', m', and n' are those of OE', we have by a well-known theorem in solid geometry,

$$ll' + mm' + nn' = 0$$

Since OE' is in the plane ZOX and inclined at θ_1 to OZ,

$$l' = \sin\theta_1,\ m' = o, \text{ and } n' = \cos\theta_1$$

Let x, y, z be the co-ordinates of the point F'. Then z is the projection of OF' on the axis $OZ = -\overline{OF'}\sin\theta$. The projection on the plane XOY is $\overline{OF'}\cos\theta$. Resolving this latter along the axes of x and y respectively, we have

$$x = \overline{OF'}\cos\theta\cos\phi$$

$$y = \overline{OF'}\cos\theta\sin\phi$$

Hence,
$$l = \frac{x}{\overline{OF'}} = \cos\theta\cos\phi$$

$$m = \frac{y}{\overline{OF'}} = \cos\theta\sin\phi$$

and
$$n = \frac{z}{\overline{OF'}} = -\sin\theta$$

Substituting in the equation $ll' + mm' + nn' = 0$ we have

$$\cos\theta\cos\phi\sin\theta_1 - \cos\theta_1\sin\theta = 0$$

or $$\tan\theta_1\cos\phi = \tan\theta \quad . \quad . \quad . \quad . \quad . \qquad \text{(VI.5)}$$

which gives the relation between the angles turned through by A and B.* Differentiating with respect to time t secs., both sides of (VI.5), in which ϕ is constant,

$$\cos\phi \sec.^2\theta_1 \frac{d\theta_1}{dt} = \sec.^2\theta \frac{d\theta}{dt}$$

or, writing $\omega = \dfrac{d\theta}{dt}$ = angular velocity of shaft A

$$\omega_1 = \frac{d\theta_1}{dt} = \text{angular velocity of shaft } B$$

$$\therefore \frac{\omega}{\omega_1} = \frac{\cos\phi \sec.^2\theta_1}{\sec.^2\theta}$$

$$= \frac{\cos\phi\,(1 + \tan^2\theta_1)}{1 + \tan^2\theta}$$

Substituting for $\tan\theta_1$ from (VI.5)

$$\frac{\omega}{\omega_1} = \frac{\cos\phi\left(1 + \dfrac{\tan^2\theta}{\cos^2\phi}\right)}{1 + \tan^2\theta}$$

$$\therefore \frac{\omega}{\omega_1} = \frac{\cos^2\phi + \tan^2\theta}{\cos\phi\,(1 + \tan^2\theta)}$$

$$= \frac{1}{\cos\phi}\left(\frac{\sec.^2\theta - \sin^2\phi}{\sec.^2\theta}\right)$$

or

$$\frac{\omega}{\omega_1} = \frac{1 - \sin^2\phi\cos^2\theta}{\cos\phi} \quad . \quad . \quad . \quad \text{(VI.6)}$$

For any given value of ϕ, the maximum velocity ratio is $\dfrac{1}{\cos\phi}$, and occurs when $\cos\theta = 0$, or $\theta = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$. The minimum velocity ratio occurs when $\theta = 0$ or π, in which cases $\dfrac{\omega}{\omega_1} = \cos\phi$. Thus, the velocity ratio fluctuates between $\cos\phi$ and $\dfrac{1}{\cos\phi}$. The total fluctuation of velocity ratio is $\dfrac{1}{\cos\phi} - \cos\phi$, or $\sin\phi\tan\phi$. This is small for small values of ϕ, but is relatively large for values of ϕ greater than 30°. If $\phi = 30°$, the fluctuation is $\sin 30° \tan 30° = {\cdot}289$ and, as the mean velocity ratio is one, this is the proportional fluctuation. This is a large fluctuation and will produce considerable stress in the material of the rotating shaft if its rotational inertia is at all considerable. Hooke's joint is only suitable for connecting shafts the angle between which is small, except in cases where a large variation of velocity ratio is permissible, and the rotational inertia of the shaft and attached masses is small.

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* Example 1, page 176, indicates an alternative method of proof of (VI.5).

116. **Use of Double Hooke's Joint to produce Uniform Velocity Ratio.** Fig. 114 shows two horizontal shafts A and B, each connected by a Hooke's joint to a third horizontal shaft C. The shafts A and B are in the same plane, and are equally inclined to the shaft C. The forks at the ends of C are in the same plane. With this arrangement, owing to symmetry, the velocity ratio between A and C is equal to that between B and C, and so the velocity ratio between A and B is constant. Double Hooke's joints are also used for connecting two parallel shafts.

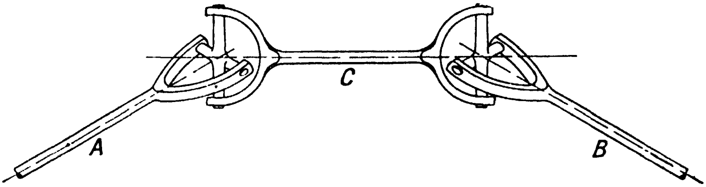


FIG. 114. DOUBLE HOOKE'S JOINT

117. **Graphical Method of Determining the Velocity Ratio Between Two Shafts Connected by a Single Hooke's Joint.** With centre O and radius unity, draw a circle DEF (Fig. 115), OD being a horizontal line and OE a vertical line. Draw a radius OK so that $\widehat{EOK} = \phi$ (Fig. 111). Draw KA perpendicular to OD. Describe a circle centre O and radius OA. Set off $\widehat{POA}$ to represent θ (Fig. 113), P being on the circumference of the smaller circle. Through P draw a line PM perpendicular to OD, and, when produced upwards, meeting the larger circle in G. Along MG mark off MH of length equal to MG^2. Determine the position of H for different values of θ, and draw its locus. If we take rectangular axes OX and OY along OD and OE respectively, the equation to the large circle is

$$y^2 = 1 - x^2$$

Hence, if $OM = x$,

$$\therefore \overline{MG}^2 = 1 - \overline{OM}^2$$

or $$\overline{MH} = 1 - \overline{OM}^2$$

Thus, the equation to the locus of H is

$$y = 1 - x^2$$

which represents a parabola with OE as axis and E as vertex; the parabola intersects the large circle at the ends of the horizontal diameter FD. The method of constructing this parabola is shown in the square $OETF$. From (VI.6),

$$\frac{\omega}{\omega_1} = \frac{1 - \sin^2\phi \cos^2\theta}{\cos\phi}$$

$$= \frac{1 - \dfrac{\overline{OA}^2}{\overline{OK}^2} \cdot \dfrac{\overline{OM}^2}{\overline{OP}^2}}{\dfrac{\overline{KA}}{\overline{OK}}}$$

and since $\overline{OA} = \overline{OP}$, and $\overline{OK} = 1$,

$$\frac{\omega}{\omega_1} = \frac{1 - \overline{OM}^2}{\overline{KA}}$$

$$= \frac{\overline{MG}^2}{\overline{KA}}$$

$$= \frac{\overline{MH}}{\overline{KA}}$$

Thus, MH represents the velocity ratio on the same scale as that on which $\overline{KA}$ represents unity. For any given value of ϕ, $\overline{AK}$, $\overline{AS}$, and $\overline{AL}$ represent respectively to the same scale the mean velocity ratio (unity), the minimum velocity ratio, and the maximum velocity ratio. $\dfrac{\overline{LS}}{\overline{AK}}$ is the proportional fluctuation. If for each value of θ a length equal to $\overline{MH}$ is marked off from O along OP produced, and if the points thus obtained are joined by a smooth curve, we obtain a polar diagram of angular velocity ratios. The diagram is shown dotted for values of θ between 0 and 90°.

118. **Applications of Hooke's Joint.** The joint is used in some motor vehicles, in scientific machines such as tensile testing machines, and in cases where a shaft drive has to be arranged on a flexible frame, as in tractors and agricultural machinery. It is equivalent to a spherical joint, and care must be taken, when mounting the joint, to ensure that the centre of the cross C lies on the axis of each shaft. In a testing machine, for determining the elastic properties of engineering materials, the load is usually applied by means of a lever. The force applied to the specimen under test is exerted by a point in the lever or beam, the point being near to the fulcrum. On the other side of the fulcrum the lever carries a travelling weight, and by moving this weight along the lever, the pull on the specimen is gradually increased. The weight forms a nut which fits a screw running the whole length of the lever. This screw is rotated by gearing, the driving wheel of which is mounted on a shaft carried by the lever. This shaft is connected by means of a Hooke's joint to a driving shaft carried by the framework of the testing machine. Thus the power is transmitted to the screw without the introduction of any resistance to the motion of the lever. The axis, or

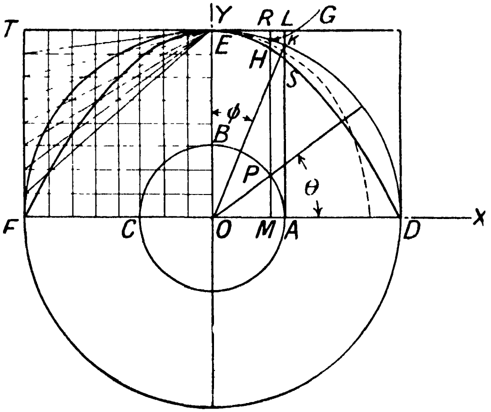


FIG. 115. VELOCITY RATIOS, HOOKE'S JOINT

fulcrum, of the lever, must pass through the centre of the cross of the joint. Hooke's joint is not so largely used as formerly and has been displaced, in many cases, by fabric couplings in which a sheet of fabric is used for coupling two shafts with intersecting axes.

119. **Note.** The relation (VI.5) may also be obtained by considering the projection of the path of F on the plane ZOX. The projection is shown in Fig. 116. F'' is the position taken up by F' (Fig. 113) when the plane $EPFO$ is rotated about OZ until it coincides with the plane ZOX. F_1' is the projection of F', $F''F_1'$ being perpendicular to ZO produced. Since OF' is perpendicular to OE' (Fig. 113), its projection OF_1' will also be perpendicular to OE'. Hence,

$$\widehat{F_1'OX} = E'OZ = \theta_1$$

In the projection all lengths perpendicular to OZ will be shortened in the ratio $\cos\phi : 1$. Then if $F''F_1'$ produced meets the axis of Z in N,

$$\tan\theta = \frac{ON}{F''N}$$

$$\begin{aligned}\tan\theta_1 &= \tan\widehat{F_1'OX}\\ &= \frac{ON}{F_1'N}\end{aligned}$$

$$\begin{aligned}\therefore \frac{\tan\theta}{\tan\theta_1} &= \frac{F_1'N}{F''N}\\ &= \cos\phi, \text{ which is the same as (VI.5)}\end{aligned}$$

Examples 6.

1. Find an expression for the distance between the points $E'F'$ in Fig. 113, in terms of the co-ordinates of the points. This is equal to the diagonal of a square with two equal sides $OE' = OF' = r$ where r is the length of each arm of the cross. By equating these two quantities, obtain the relation (VI 5), page 171.

2. Show that for small values of the angle between two shafts connected by a Hooke's joint, the total fluctuation of the velocity ratio varies as the square of the angle between the shafts.

3. Find the greatest and least values of the velocity ratio of two shafts connected by a Hooke's joint, the acute angle between their axes being 45°, and draw a polar diagram showing the velocity ratio for different values of the angle turned through by one of the shafts.

4. Draw two lines OA and OB, of the same length and containing an angle equal to the acute angle between two shafts connected by a Hooke's joint. Draw BC perpendicular to OA and meeting it in C. At O erect a perpendicular

to OA. Let P be any point in this perpendicular. Join PC and PA. Show that as P moves from O to infinity along OP the ratio of the angular velocities of PC and PA passes through the same series of values as does the velocity ratio between the two shafts as they rotate through 90° from the positions shown in Fig. 116.

5. The mechanism (Fig. 117) is a four-bar kinematic chain of which the centres A and B are fixed. The dimensions are $AB = 2$ ft., $AC = CD = DB = 1$ ft. Find a point G in the centre line of the cross-arm EF of which the locus is an approximate straight line even for considerable displacements from the position shown. (U.L.)

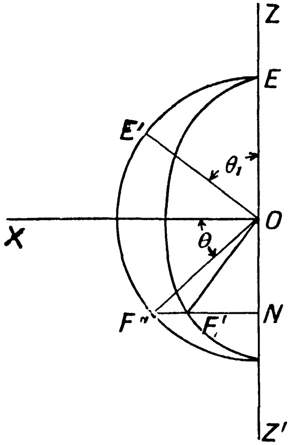


FIG. 116. VELOCITY RATIOS, HOOKE'S JOINT

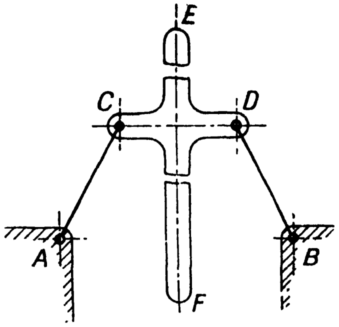


FIG. 117

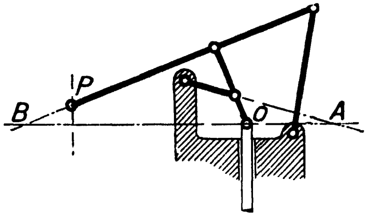


FIG. 118

6. A sketch is given of the Crosby indicator mechanism (Fig. 118), the motion of O being vertical and AB passing horizontally through O. What condition must be satisfied if the motion of P is to that of O as AB is to AO.

7. The link GAB (Fig. 119) oscillates on a fixed centre at A and the link FD on a fixed centre at F. The link AB is equal to AC, and BD, BE, EC, and CD are equal in length. (1) Find the length of FD and the position of the centre F so that the point E may move in a straight line. (2) If the point E is required to move in a circle passing through A, what will be the path of the point D? (U.L.A.)

8. Taking the dimensions of the Tchebicheff parallel motion from Fig. 109, page 167, draw the complete locus of the tracing point P.

9. A square $ABCD$ moves so that the points A and C slide along two fixed perpendicular lines. Show that each of the points B and D traces out a line which makes an angle of 45° with each of the two fixed lines; hence show that any point in BD traces out an ellipse whose major and minor axes are inclined at 45° to the fixed lines. (Compare this with the elliptic trammels, Art. 68.)

10. Show that if a triangle ABC moves so that B and A slide along fixed

rectangular axes, OX and OY respectively, in the plane of the triangle, the point C has a path whose equation is $\overline{CB}^2x^2 - 2\,\overline{CM}\,.\,\overline{AB}\,xy + \overline{CA}^2y^2 = (\overline{AM}\,.\,\overline{AB} - \overline{AC}^2)^2$, where M is the foot of the perpendicular from C to AB, and the order ACB is clockwise. Show also that if the angle ACB is a right-angle, the equation represents a straight line through the origin. (See Art. 56 and Example 9, above.)

11. Find an expression for the ratio of the speeds of the points D and E in the mechanism of Ex. 7.

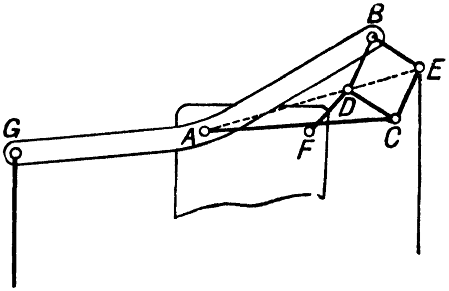


FIG. 119

12. Find the proportions of the links AB in Roberts' parallel motion (Fig. 108) if the inclined bars are all equal and the greatest possible travel of Q is from O_1 to O.

13. If in Tchebicheff's parallel motion (Fig. 109) the path of P is equal in length to OO_1, and if in the extreme positions the coupler AB and one of the levels OA and O_1B are vertical, find the ratio $\dfrac{\overline{AB}}{\overline{OA}}$

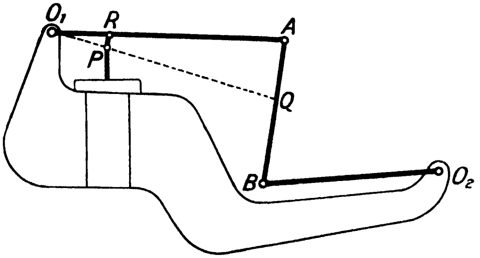


FIG 120

14. Fig. 120 shows part of the mechanism of Richards' steam engine indicator, and Fig. 121 shows the corresponding mechanism of the Simplex indicator. The dots indicate turning pairs connecting the links. P is the upper end of the indicator piston-rod and oscillates along a vertical straight line. Q is the point to which the pencil is attached, and in each case O_1PQ is a straight line in the mean position of the mechanism. Show that Q traces out an approximate straight line in the mechanism of Fig. 120, and an exact straight line in that of Fig. 121, and in each case give the name of the straight line motion from which the mechanism is derived.

15. Two parallel shafts are connected by an intermediate shaft inclined at θ to both, the end connections being Hooke's joints. Find the ratios of the angular velocities of the two parallel shafts when the forked ends of the connecting shaft are (1) in the same axial plane, (2) in axial planes at right-angles. (U.L.A.)

16. State the requirements as to the motion of (*a*) the pencil, (*b*) the drum of a steam-engine indicator, and give sketches of suitable mechanisms. In each case give the proportions, and state whether the resulting motion fulfils the requirements exactly or only approximately. (L.U.)

17. A crank OC rotates about a fixed centre O with uniform angular velocity ω clockwise. A second crank CD rotates about C with uniform angular velocity $r\omega$ counter-clockwise relative to OC. $\overline{CD} = \overline{OC}$. Show that D traces out a fixed straight line with simple harmonic motion.

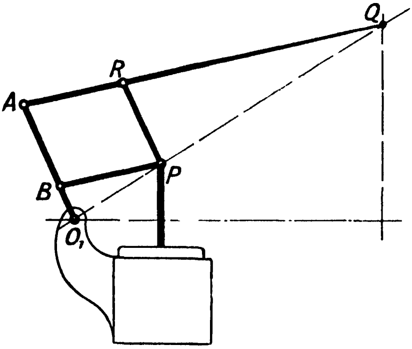


FIG. 121

18. Two shafts whose axes intersect at an angle of 150° are connected by a Hooke's joint. If one rotates uniformly, find a general expression for the velocity of the other. (L.U.)

19. In the " Grasshopper " parallel motion, shown in Fig. 103, page 162, prove that Q will describe a path which is approximately a straight line if $\overline{PR}^2 = \overline{SR}\,.\,\overline{QR}$.

Determine the vertical force at Q necessary to resist a torque T applied to the link SR.

20. In a " Grasshopper " parallel motion, the tracing point P is at one end of a rod PQ 1 ft. long. A point 3 in. from the other end Q of the rod moves on a small circular arc; find the radius of this arc. Find, by drawing, the maximum deviation of P from a vertical straight line in a travel of 3 in. on each side of its mean position. Assume $\overline{O_1P} = 1$ ft.

21. A Watt's parallel motion has two bars OA and O_1B joined by a coupling bar AB, in the form of a crossed four-bar mechanism. When the mechanism is in its mean position the links OA and O_1B are perpendicular to AB. If $\overline{OA} = 3$ in., $\overline{O_1B} = 5$ in., and $\overline{AB} = 4$ in., find the position of the tracing point P, and find how far P is from the straight line which is the mean position of AB (1) when OA and AB are in one straight line, and (2) when O_1B and BA are in one straight line.

22. Sketch the Peauceller and the Hart straight line mechanisms. Give dimensions to the links and find in each case what force acting at P (Figs. 96 and 97) perpendicular to OP will be required to balance a force of

W lb. acting at Q along the locus of P. Prove that the locus of Q is a straight line.

23. Describe the Scott Russell parallel motion, proving that it describes an exact straight line; sketch and describe the mechanisms which are derived from it.

24. The pantograph of Fig. 110, page 168, has only one point fixed. Which other point would you fix so as to convert it into a mechanism? What is the name of this mechanism? Show that if we require to make use of its motion-copying properties we must constrain the motion of some point not on the link OB.

25. The point P of the pantograph in the last example is moving at a given instant with a velocity of 10 ft. per sec., and an acceleration of 30 ft. per sec. per sec., both parallel to CP and in the sense C to P. Given $\overline{OB} = 1$ ft. $\overline{CP} = 4$ in., $\overline{BQ} = 6$ in., draw the velocity and acceleration images, if at the instant the angle OBQ is 100°. Find from your images the velocity and acceleration of the point Q. Also calculate these two quantities by a more direct method.

CHAPTER VII

FRICTION—THEORY OF LUBRICATION—MICHELL BEARINGS —BALL AND ROLLER BEARINGS

120. THE elements of lower pairs are in contact over a common surface, and the relative motion between the elements at any point of the common surface is a sliding of one surface over the other. It is impossible to produce perfectly smooth surfaces and, owing to the roughness of the surfaces, tangential forces, or resistances, are set up between them tending to prevent relative motion. These resistances are known as *frictional forces.* When there is relative motion between the elements the friction is termed *limiting friction.* When there is no relative motion, the force of friction is less than that of limiting friction, and the resultant frictional force acting on the element is equal and opposite to the resultant of all the other forces acting on it. The relative motion between the elements of a higher pair is in many cases a mixture of sliding and rolling. The sliding will set up frictional resistances, just as in the case of lower pairs. It is usually assumed that there is only line contact in higher pairs, but there is elastic deformation of the elements along the line of contact which produces a small area of contact. The laws concerning sliding contact in lower pairs will in general be applicable also in the case of higher pairs.

121. **Laws of Solid Friction.** The following laws have been found by experiment to be approximately true. They relate to cases of limiting friction, that is, cases in which relative motion is just occurring, or just about to occur.

1. The force of friction is independent of the area of contact.

2. For low speeds the force of friction is independent of the speed.

3. The force of friction on a body is proportional to the normal pressure between the surfaces of contact, and acts in such a direction as to directly oppose the relative motion.

If P lb. is the normal pressure between the surfaces, and

F lb. is the force of friction acting on either of the bodies, we have from the third law,

$$\frac{F}{P} = \text{constant}$$

$$= \mu$$

$$\therefore F = \mu P \qquad \text{(VII.1)}$$

The constant μ is called the *coefficient of friction*, and is constant only for a given pair of surfaces. The actual value of μ depends upon the material of which the bodies are composed, and upon the degree of smoothness of the surfaces. Friction between bodies in relative motion always leads to a waste of energy, and with lower pairs and some higher pairs it is necessary to reduce this waste by the use of lubricants and special types of bearing surfaces. In some higher pairs, such as friction clutches, belt and rope drives, driving wheels, and the tracks on which they run, it is desirable to prevent or reduce slipping. In these cases, friction is increased as much as possible by the use of surfaces having a high coefficient of friction, and by increasing the normal pressure between the surfaces. In some load-raising machines friction fulfils a useful function in preventing overhauling (Chapter II). The nature of the frictional resistance between two surfaces is imperfectly understood. As a body starts from rest, the force of friction is usually greatest when the body is at rest, and decreases slightly as the velocity increases up to a small value, afterwards remaining approximately constant. In some cases the friction increases slightly from the static condition to the kinetic condition. Law 2 above is only closely true for low speeds. Experiments by Galton in 1878 and more recent experiments by others, including one of the authors, show that the coefficient of friction between unlubricated metallic surfaces diminishes very appreciably and progressively at high speeds. Prof. Perry suggested that the reduction might be due to a film of air being drawn between the surfaces, the thickness of this film increasing with the speed. Some confirmation of this theory is given by the fact that at a given speed the coefficient increases with increase of pressure between the surfaces. Similar experiments on Ferodo brake linings show an increase of the coefficient of friction with increased speed of rubbing.

122. **Limiting Angle of Friction.** LM (Fig. 122) is a fixed surface of a solid B, and A is a second solid in contact with the former over a small area in the vicinity of O. ON is the normal to the surface of contact. Suppose the body A to be moving in the direction of the arrow, and let $F = \overrightarrow{OT}$ be the resultant pressure of B on A. Let the angle $NOT = \phi$. The component of F perpendicular to ON is the force of limiting friction, and acts so as to oppose the motion. Resolving along and perpendicular to ON, we have

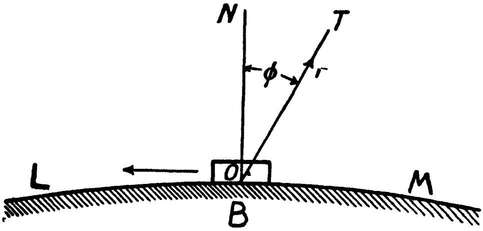


FIG. 122. LIMITING ANGLE OF FRICTION

Normal component of pressure $= F \cos \phi$

Tangential component of pressure $= F \sin \phi$

Hence, from (VII.1)

$$\mu = \frac{\text{force of limiting friction}}{\text{normal pressure}}$$

$$= \frac{F \sin \phi}{F \cos \phi}$$

$$\therefore \ \mu = \tan \phi \qquad \text{(VII.2)}$$

or

$$\phi = \tan^{-1} \mu \qquad \text{(VII.3)}$$

$\text{Tan}^{-1}\mu$ is called the limiting angle of friction.

If ϕ has a value less than that given by (VII.3), the body A will not move, however large the value of F. If ϕ has the value given by (VII.3), the body A will move over the surface of B, however small the force F may be. The value of ϕ cannot exceed $\tan^{-1}\mu$. In general one body will slide over the surface of another when, and only when, the forces exerted across the common surface are inclined at an angle ϕ to the common normal, where ϕ has the value $\tan^{-1}\mu$. For all possible directions of motion along the common surface F will lie on the surface of a cone, called the cone of limiting friction, having ON as axis and $\tan^{-1}\mu$ as semivertical angle.

Angle of Repose. If a weight rests on a plane and the plane is gradually tilted through an increasing angle to the horizontal, it is found that, at a certain value of the angle of tilt, the weight, once started, will run down the plane with uniform velocity. The angle at which this occurs is called the angle of repose. With the plane inclined at the angle of repose, there are two forces acting on the body, its weight and the reaction of the plane. Since the weight moves with uniform velocity the two forces are in equilibrium, hence the reaction of the plane is vertically upwards, and is consequently inclined to the common normal at an angle equal to the angle of repose. But the reaction when motion takes place is inclined to the normal at an angle which is the limiting angle of friction. Thus, the angle of repose is equal to the limiting angle of friction.

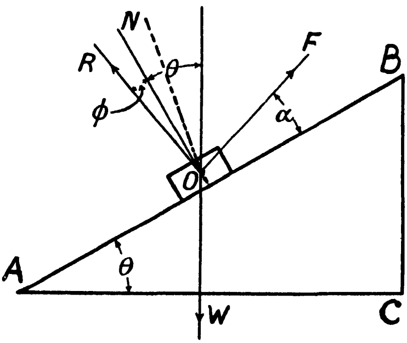


FIG. 123. THE INCLINED PLANE

123. **Equilibrium of a Body on a Rough, Inclined Plane.** Consider a weight W lb. on an inclined plane AB (Fig. 123), the inclination of AB to the horizontal AC being θ. Suppose the forces acting on W to be its weight of W lb., R the reaction of the plane, and a force F inclined at an angle α to AB. If the weight is moving along the plane with uniform velocity, the three forces are in equilibrium. If the motion is up the plane, the reaction R will make an angle ϕ with the common normal ON on the left of the normal. If the motion is down the plane, the reaction will act along a line making an angle ϕ on the other side of the normal as shown dotted. By the use of Lami's Theorem, we may write down the relation between the three forces.

For *motion up the plane*,

$$\frac{F}{\sin (180^\circ - \overline{\theta + \phi})} = \frac{R}{\sin (90^\circ + \alpha + \theta)} = \frac{W}{\sin (90^\circ + \phi - \alpha)}$$

or

$$\frac{F}{\sin (\theta + \phi)} = \frac{R}{\cos (\alpha + \theta)} = \frac{W}{\cos (\phi - \alpha)} \quad . \qquad \text{(VII.4)}$$

from which F and R may be found in terms of W and the three angles.

For *motion down the plane,*

$$\frac{F}{\sin (180^\circ - \theta - \phi)} = \frac{R}{\sin (90^\circ + \alpha + \theta)} = \frac{W}{\sin (90^\circ - \alpha - \phi)}$$

or

$$\frac{F}{\sin (\theta - \phi)} = \frac{R}{\cos (\alpha + \theta)} = \frac{W}{\cos (\alpha + \phi)} \quad . \quad \text{(VII.5)}$$

which could be obtained by changing the sign of ϕ in (VII.4).

If F acts along the plane, α is zero. If F acts in a horizontal direction $\alpha = -\theta$. If in the case of motion down the plane, $\alpha + \phi$ is less than 90°, and θ is less than ϕ, the value of F found from (VII.5) will be negative. In such cases the direction of F would need to be reversed in order to produce motion down the plane.

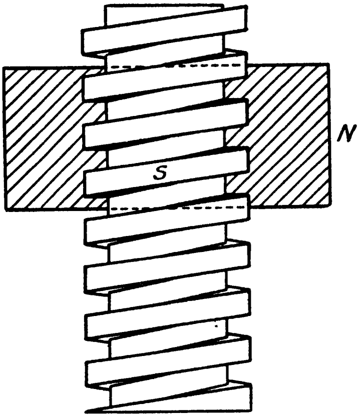


FIG. 124. FRICTION OF A SCREW

124. Friction Between a Screw and Nut. A screw is a cylinder with a uniform helical ridge projecting from its surface. A nut is a body with an internal surface made to fit closely the outer surface of a screw. The two form a screw pair and are such that the relative axial displacement is proportional to the relative angular rotation about the common axis. The section of the helical ridge by a plane containing the axis is usually either rectangular or triangular. The ridge is termed the *thread* of the screw, the two shapes mentioned being known as square threads or triangular threads respectively.

S in Fig. 124 is a square-threaded screw, fitted into a nut N. The nut may be of any convenient shape on its outer surface, and is usually of the shape of a square, or hexagonal, prism. In screw pairs used in machines the nut is sometimes part of the fixed framework, in which case the screw, on rotating, advances in the direction of its axis. In other cases the screw forms a turning pair, and the nut a sliding pair with the frame of the machine, and in such cases the nut moves along the axis of the screw. The screw-jack for load lifting is an example of the former arrangement, and the screw and nut arrangement for moving the slide-rest of a lathe is an example of the latter. Screws are right-handed or left-handed, according as they move away from, or towards, an observer looking at one end, when they are rotated in the clockwise sense relative to a fixed nut. The distance moved by a screw along its axis, when rotated once in a fixed nut, is called the *pitch*. This is the distance between the intersections of an axial plane and corresponding edges of two adjacent turns of the thread. If we neglect the dimensions of the thread, a nut and bolt are equivalent to two cylinders, the outer surface of the screw being in contact with the inner surface of the nut, and the relative motion is such that equal helices, one on each cylinder, remain in contact. If we develop the surfaces of the cylinders by cutting them along a line parallel to the common axis, and laying them flat on the vertical plane with this line vertical, each turn of the helix will become a straight line inclined to the horizontal at an angle whose tangent is $\frac{p}{2\pi r}$ where p is the pitch, and r the common radius of the cylinders. The relative motion is thus the same as that of a weight moving along an inclined plane. If W is the axial thrust between the two surfaces and F the force needed to produce relative motion when F acts tangential to the cylinders in the plane of a cross section, the conditions are analogous to those of Art. 123, in which F is assumed to act in a horizontal direction. Putting $\alpha = -\theta$, and $\tan\theta = \frac{p}{2\pi r}$, in (VII.4) we have,

$$F = W\frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} \quad . \quad . \quad . \quad \text{(VII.6)}$$

$$F = W\tan(\theta+\phi)$$

$$= W\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

$$= W\frac{\frac{p}{2\pi r}+\mu}{1-\mu\frac{p}{2\pi r}} \text{ since } \mu = \tan\phi$$

Hence,
$$F = W\frac{p+2\pi\mu r}{2\pi r-\mu p} \quad . \quad . \quad . \quad \text{(VII.7)}$$

This result applies to the case in which the two elements move outwards against the end thrusts. In an actual square-threaded screw, r will be the mean radius of the thread. If the elements move inwards, the case will correspond to that of motion down an inclined plane. Hence, changing the sign of μ and also that of F, since if the arrangement does not overhaul F would be reversed in order to reverse the relative motion,

$$F = W\frac{2\pi\mu r - p}{2\pi r + \mu p} \quad . \quad \text{(VII.8)}$$

(VII.7) and (VII.8) give the forces necessary when acting tangential to the cylinder of radius r. If the force F is applied at right-angles to an arm of length l perpendicular to the axis of the screw, its value will be $\frac{r}{l}$ of the above. For a triangular thread the above results will need correction, because of the increased inclination of the working faces of the threads to the horizontal. If we assume that the angle between the two faces of the thread is 2β (Fig. 125), the normal pressure P between the threads is approximately $W \sec \beta$. The effect of this may be allowed for by writing $\mu \sec \beta$ for μ, in (VII.7) and (VII.8).

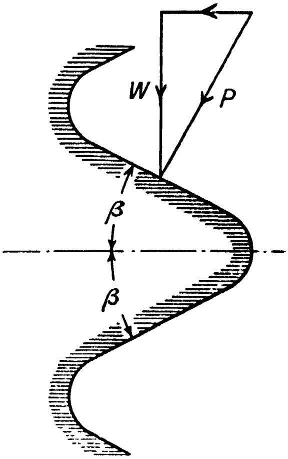


FIG. 125. TRIANGULAR THREAD

From Art. 34

$$\text{Mechanical advantage} = \frac{W}{F}$$

$$\text{Mechanical efficiency} = \frac{\text{mechanical advantage}}{\text{velocity ratio}}$$

In a square-threaded screw-jack for load lifting, the force F is applied at the end of an arm of length l measured from the screw axis. Hence,

$$\text{Mechanical advantage} = \frac{W}{F} = \frac{2\pi r - \mu p}{2\pi\mu r + p} \times \frac{l}{r} \quad . \quad \text{(VII.9)}$$

$$\text{Velocity ratio} = \frac{l}{r} \times \frac{2\pi r}{p} = \frac{2\pi l}{p} \quad . \quad . \quad \text{(VII.10)}$$

$$\therefore \text{ Mechanical efficiency} = \frac{2\pi r - \mu p}{2\pi\mu r + p} \times \frac{l}{r} \times \frac{p}{2\pi l}$$

$$= \frac{p}{2\pi r} \cdot \frac{2\pi r - \mu p}{2\pi\mu r + p} \quad . \quad . \quad \text{(VII.11)}$$

The mechanical efficiency and the mechanical advantage are constant for all values of *W*. If the weight of the screw and the parts attached to it is appreciable, and *W* represents only the weight of the load lifted (not including the screw), there will be an additional force required to raise the screw, in which case the mechanical efficiency and the mechanical advantage will increase with the load as in other load-lifting machines, and will approach asymptotically to the values given in (VII.9) and (VII.11) above.

Substituting $\tan\alpha$ for $\frac{p}{2\pi r}$ and $\tan\phi$ for μ in (VII.11),

$$\eta = \text{mechanical efficiency}$$

$$= \tan\alpha \frac{1 - \tan\phi \tan\alpha}{\tan\phi + \tan\alpha}$$

$$= \frac{\tan\alpha}{\tan(\alpha + \phi)}$$

To find the value of α, which gives maximum efficiency, we have

$$\frac{d\eta}{d\alpha} = 0$$

i.e.
$$\frac{\sec^2\alpha \tan(\alpha + \phi) - \sec^2(\alpha + \phi)\tan\alpha}{\tan^2(\alpha + \phi)} = 0$$

therefore
$$\sec^2\alpha \tan(\alpha + \phi) = \sec^2(\alpha + \phi)\tan\alpha$$

or
$$\sin(\alpha + \phi)\cos(\alpha + \phi) = \sin\alpha\cos\alpha$$

Hence
$$\sin 2(\alpha + \phi) = \sin 2\alpha$$

and therefore
$$2\alpha + 2\phi = 180° - 2\alpha$$

or
$$\alpha = 45° - \frac{\phi}{2}$$

The maximum efficiency is therefore given by

$$\eta_{max} = \frac{\tan\left(45^\circ - \frac{\phi}{2}\right)}{\tan\left(45^\circ + \frac{\phi}{2}\right)}$$

$$= \frac{\left(1 - \tan\frac{\phi}{2}\right)^2}{\left(1 + \tan\frac{\phi}{2}\right)^2}$$

$$= \frac{1 - \sin\phi}{1 + \sin\phi}$$

125. **Friction in a Journal Bearing.** The outer stationary element of a turning pair is called a *bearing*, and that portion

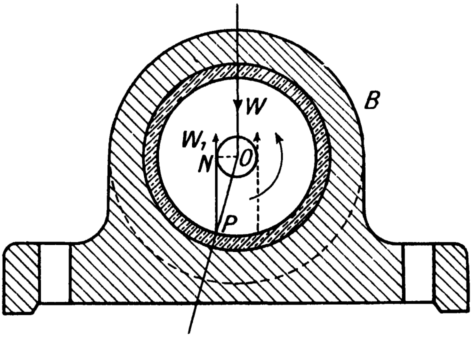


Fig. 126. Friction in a Journal Bearing

of the inner element which fits the bearing is called a *journal.* The journal is a working fit in the bearing, and is therefore slightly less in diameter than the latter. If the bearing is not lubricated, there will be line contact between the elements. When relative rotation occurs, there is limiting friction at the line of contact, and the resultant pressure of one element on the other will make an angle ϕ, equal to the limiting angle of friction, with a radial plane through the line of contact.

In Fig. 126 we have a shaft inside a bearing B. O is the end view of the axis of the shaft, and P that of the line of contact. The shaft is rotating in the counter-clockwise sense. W is the load on the shaft, and W_1 is the reaction of the bearing on the shaft. In order that rotation may be maintained, there must be a couple C rotating the shaft. The angle between OP and the force W_1 is ϕ where $\tan \phi = \mu$, the coefficient of friction. For uniform motion, the resultant force acting on the shaft must be zero, and the resultant turning moment on the shaft must be zero.

Thus, $$W_1 = W$$

and $$C = W \,.\, \overline{OP} \sin \phi$$

or, if r ft. is the radius of the shaft,

$$\begin{aligned} C &= \text{friction couple} \\ &= Wr \sin \phi \qquad \text{(VII.12)} \end{aligned}$$

As ϕ is usually small, $\sin \phi$ and $\tan \phi$ are very nearly equal, and we may write

$$C = W\mu r \text{ ft. lb.} \qquad \text{(VII.13)}$$

Also,

$$\begin{aligned} \text{Work done per min.} &= C \times \text{angle turned through in radians per min.} \\ &= C \times 2\pi N = 2\pi\mu r N W \end{aligned}$$

where N = no. of revs. of the shaft per min.

If P = horse-power wasted in friction

$$P = \frac{2\pi\mu r N W}{33000} \qquad \text{(VII.14)}$$

If ON is a perpendicular to the line of action of W_1, this latter line is a tangent to a circle centre O, and radius $ON = r \sin \phi$. This circle is called the *friction circle.* The force exerted by one element of a turning pair on the other element acts along a tangent to the friction circle. If the sense of rotation of the shaft is changed, W_1 will move to the position of the other vertical tangent to the friction circle. This position is shown dotted, and it should be noticed that the force always resists relative motion. When the bearing is efficiently lubricated, the two elements of the pair will be separated by a film of oil. It will still be necessary to apply a couple C to overcome friction, and the resultant pressure of the oil film on the shaft will be a vertical force equal and opposite in sense to W. The conditions are the same as in Fig. 126, except that since lubrication reduces friction, ϕ will have a smaller value.

If, in this case, we define μ as the value of sin ϕ, the above results will be applicable to either lubricated or unlubricated bearings.

126. **Effects of Friction on the Force transmitted by a Rod joining Two Turning Pairs.** *ABCD* (Fig. 127) represents a four-bar mechanism with *B* fixed; the cylindrical bearing surfaces between *D* and *A* and *C* respectively are indicated by section lines. Inside the bearing surfaces are drawn the friction circles. The radius of the friction circle is in each case *r* sin ϕ, where *r* is the radius of the bearing surface, and ϕ the angle of friction for the particular bearing. The line along which the forces are exerted by *D* on *A* and *C* respectively, will be a common tangent to the two circles. There are four such tangents, but the required tangent is the one which will cause the forces to oppose the relative rotation. Thus, if *C* is the driver and *A* the follower, and they are rotating in a counter-clockwise direction, *D* will exert thrusts on *A* and *C*. The line of thrust on *A* will touch the friction circle on its lower portion, so as to oppose the relative rotation. The line of the thrust on *C* will act on the upper side of the friction circle. *LM* is the line of thrust.

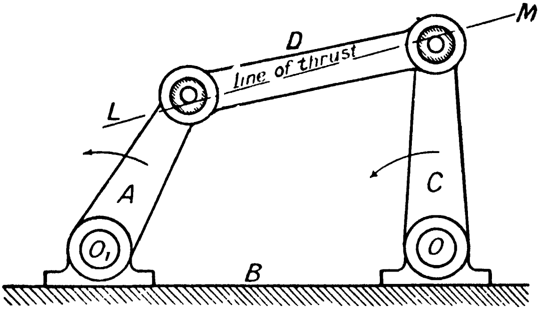


FIG. 127. FRICTION ON COUPLING LINK

127. **Effect of Friction in Simple Engine Mechanism.** The forces in a simple engine mechanism will act at angles $90° - \phi$ with surfaces of contact of sliding and turning pairs. Fig. 128 is a diagram of the mechanism, the small circles being friction circles. With the crank rotating in the clockwise sense, the lines of action of the reaction of the guide bar and the force in the connecting-rod are as shown. *P* is the thrust of the piston.

R is the pressure of the guide bar on the crosshead, and is inclined at an angle ϕ to the normal to the sliding surface. *T* is the thrust of the connecting-rod on the crosshead and on the crank-pin. *T* may be found by means of the triangle of forces. *NK*, which is tangential to the two friction circles, is the line of thrust in the connecting-rod. If *O* is the centre of the crank-shaft $T\,.\,\overline{OK}$ is the turning moment on the shaft (Art. 27), and if now *H* is the point in which *OK* cuts the friction circle, the turning moment needed to overcome the friction of the crank-shaft is $T\,.\,\overline{OH}$, and we have

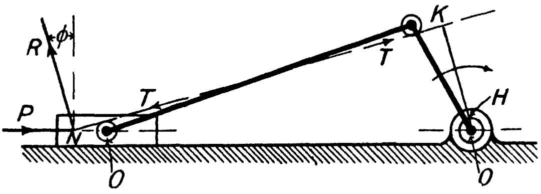


FIG. 128. FRICTION IN SIMPLE ENGINE MECHANISM

$$\text{Effective turning moment} = T\,.\,\overline{OK} - T\,.\,\overline{OH}$$
$$= T\,.\,\overline{HK}$$

In Figs. 129, 130, and 131, the line of thrust or pull in the connecting-rod, the line of action of the reaction *R*, and the line *HK*, are shown for different positions of the crank. In practice with lubricated pins the radii of the friction circles are very small compared with the lengths of the links.

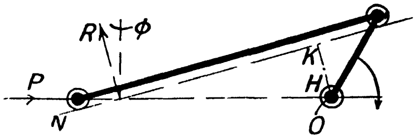


FIG. 129

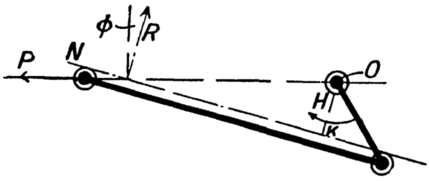


FIG. 130

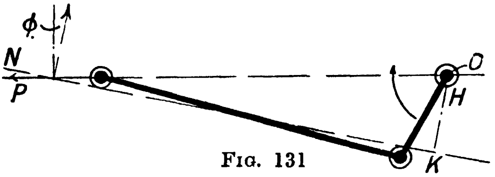
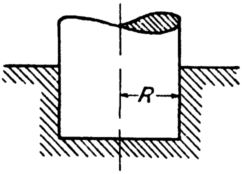


FIG. 131

128. Thrust Bearings. If there is end thrust on a shaft, it is necessary, in order to prevent lateral motion of the shaft, to provide one or more bearing surfaces in the form of surfaces of revolution other than cylinders, having the same axis as the shaft. If the surface of revolution is a plane, the bearing is a *flat pivot bearing*, as in Fig. 132. If the surface is conical, we have the conical pivot bearing of Fig. 133. If the surface is a flat annular ring, we have the bearing shown in Fig. 134.

FIG. 132



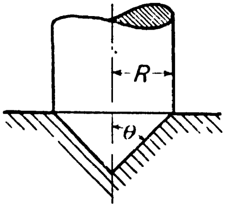


FIG. 133

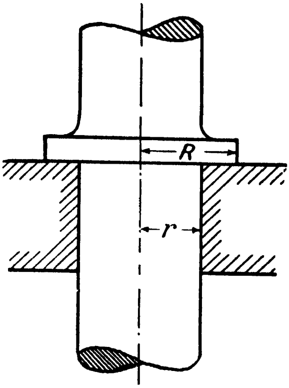


FIG. 134

When the bearing is at the foot of a vertical shaft, it is known as a footstep bearing.

Consider the bearing shown in Fig. 135, under a thrust of P lb. along the axis. Let $AOPB$ be the profile of the solid of revolution forming the bearing surface, the surface lying between an inner radius r and an outer radius R, both in inches. Let p be the normal pressure per square inch between the bearing surfaces at any point. If OX and OY are taken as rectangular axes of co-ordinates, OY being the axis of the shaft,

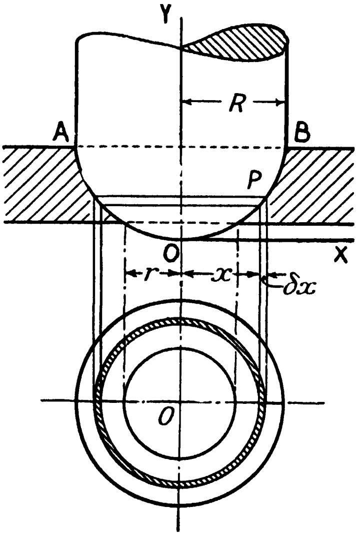


FIG. 135

then, from symmetry, p is a function of x. Let θ be the angle between the tangent at P to $AOPB$ and the axis of the shaft. Consider the strip of surface between the radii x and $x + \delta x$, and let P lie in this strip.

$$\text{Area of strip} = \frac{2\pi x\delta x}{\sin\theta}$$

$$\text{Total pressure on strip} = \frac{2\pi xp\delta x}{\sin\theta}$$

$$\begin{array}{l}\text{Component pressure along}\\ \quad\text{axis of shaft}\end{array} = \frac{2\pi xp\delta x}{\sin\theta} \times \sin\theta$$

and P = total end thrust

or
$$P = \int_r^R 2\pi p\, xdx \quad . \quad . \quad \text{(VII.15)}$$

Now, the moment about the axis of the frictional force on the ring is $\dfrac{2\pi xp\delta x}{\sin\theta} \times \mu \times x = \dfrac{2\pi\mu px^2\delta x}{\sin\theta}$, hence

C = total friction moment in lb. ins.

$$= \int_r^R \frac{2\pi\mu px^2dx}{\sin\theta} \quad . \quad . \quad \text{(VII.16)}$$

From (VII.15) and (VII.16),

$$\frac{C}{P} = \frac{\mu\displaystyle\int_r^R \frac{px^2dx}{\sin\theta}}{\displaystyle\int_r^R pxdx}$$

or
$$C = \mu P \frac{\displaystyle\int_r^R \frac{px^2dx}{\sin\theta}}{\displaystyle\int_r^R pxdx} \quad . \quad \text{(VII.17)}$$

129. Friction Couples.

(*a*) *Constant Pressure.* If we assume that p is constant, (VII.17) becomes

$$C = \mu P \frac{\displaystyle\int_r^R \frac{x^2dx}{\sin\theta}}{\displaystyle\int_r^R xdx} \quad . \quad \text{(VII.18)}$$

If θ is constant also,

$$C = \frac{\mu P}{\sin\theta} \frac{\int_r^R x^2 dx}{\int_r^R x dx}$$

$$\therefore C = \tfrac{2}{3}\mu \frac{P}{\sin\theta} \cdot \frac{R^3 - r^3}{R^2 - r^2} \quad . \quad \text{(VII.19)}$$

This formula also gives the couple to cause slipping between the elements of a friction clutch. (See Ex. 5, page 209.)

Flat Pivot (Fig. 132). In this, $\theta = 90°$ and $\sin\theta = 1$, also $r = 0$. Hence, from (VII.19),

$$C = \tfrac{2}{3}\mu PR \quad . \quad . \quad . \quad \text{(VII.20)}$$

Conical Pivot (Fig. 133). In this, $r = 0$. Hence,

$$C = \tfrac{2}{3}\mu P \frac{R}{\sin\theta} \quad . \quad . \quad \text{(VII.21)}$$

Collar Bearing (Fig. 134). Here, $\sin\theta = 1$, and

$$C = \tfrac{2}{3}\mu P \frac{R^3 - r^3}{R^2 - r^2} \quad . \quad . \quad \text{(VII.22)}$$

(*b*) *Pressure Varying Inversely as the Radius.* In an unlubricated bearing the bearing surface will wear more or less uniformly, as, if one part tends to wear less than the remainder, the increase of pressure on that part will then cause increased wear. The rate of wearing is approximately jointly proportional to the pressure and the velocity of rubbing. For equal wear, then, the product of the pressure and the velocity is constant, or, since the velocity is proportional to the radius,

$$px = \text{constant} = c$$

$$\therefore p = \frac{c}{x} \quad . \quad . \quad . \quad . \quad \text{(VII.23)}$$

Substituting in (VII.17),

$$C = \mu P \frac{\int_r^R \frac{cx dx}{\sin\theta}}{\int_r^R c dx}$$

or

$$C = \mu P \frac{\int_r^R \frac{x dx}{\sin\theta}}{\int_r^R dx} \quad . \quad . \quad \text{(VII.24)}$$

If θ is constant,

$$C = \frac{\mu P}{\sin\theta}\frac{\int_r^R x dx}{\int_r^R dx}$$

i.e.

$$C = \frac{\mu P}{2\sin\theta}\frac{R^2 - r^2}{R - r}$$

$$\therefore C = \frac{\mu P}{2\sin\theta}\cdot(R + r) \qquad . \quad \text{(VII.25)}$$

As before, we have for a flat pivot $\theta = 90°$, $\sin\theta = 1$, and $r = 0$,

$$\therefore C = \tfrac{1}{2}\mu PR \qquad . \quad . \quad . \quad \text{(VII.26)}$$

For a conical pivot $r = 0$, and

$$C = \frac{\mu PR}{2\sin\theta} \qquad . \quad . \quad . \quad \text{(VII.27)}$$

For a collar bearing, $\sin\theta = 1$, and

$$C = \tfrac{1}{2}\mu P(R + r) \qquad . \quad . \quad \text{(VII.28)}$$

We must remember that the law $p = \frac{c}{x}$ is obtained on the assumption that the friction is that between two solids. The relation leads to the absurd conclusion that the pressure is infinite at the centre where $x = o$. We shall see later that in lubricated bearings there is usually a film of oil between the two surfaces, and that the friction is, consequently, fluid friction. Thus, the assumption is not justifiable. The correct value of the turning moment due to friction is probably intermediate between that given by (VII.18), and that given by the relation (VII.24). The reader should use the former relation, unless it be expressly stated that the assumption of equal wear is to be made.

130. **Lubrication.** During the period in which machinery has been in use, it has been the continual endeavour of machine makers to reduce the friction between parts of machines. The first and simplest device was the introduction of *lubricant* between the rubbing surfaces. A lubricant is usually a fluid having the two essential properties of viscosity

and oiliness. Viscosity means the ability to sustain shear stress, and is a necessary property of a lubricant between two surfaces under pressure, but oiliness is a property not yet clearly understood. Animal and vegetable oils possess the latter property to a much greater degree than do mineral oils. This difference between the two classes of oils is thought to be due to the fact that animal and vegetable oils spread over any surface with which they come into contact, whilst mineral oils have not so high a power of spreading. The spreading power appears also to depend upon the material of the surface (*see below*).

It was thought until comparatively recent times that the lubricant simply greased the rubbing surfaces, and, in this way, by making the surfaces smoother, reduced the coefficient of friction. It has, however, been shown that in an efficiently lubricated bearing there is no actual contact between the two surfaces, and that the surfaces are separated by a film of fluid lubricant, there being a pressure in this film which increases from zero at the boundaries of the area of contact to a maximum at some point in the area.

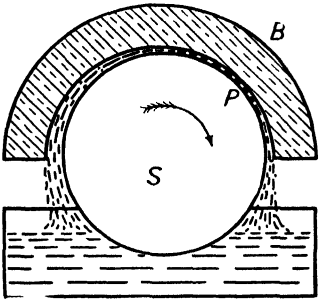


Fig. 136. Railway Axle Bearing

This knowledge was the result of experiments carried out in 1883 by Mr. Beauchamp Tower on behalf of the Institution of Mechanical Engineers. He used a railway carriage bearing (Fig. 136) in which a shaft or journal S supported a load W applied to brass step B. The journal was 4 in. in diameter, and its lower surface dipped into a bath of oil. Tower drilled radial holes in the bearing B at different points of its section and, plugging all but one at a time, and connecting a manometer to this one, he measured the pressure at each hole. He found that there was a fluid pressure at all parts of a well-lubricated bearing, and thus demonstrated the existence of a pressure oil film between the bearing surfaces. In addition to this discovery, he found that the frictional resistance was practically independent of the load on the bearing, and was very nearly proportional to the square root of the velocity of rubbing. With the oil-bath system of lubrication, the friction was considerably less than with less perfect systems. Tower also found that the frictional resistance fell considerably as the temperature of the lubricant increased. The frictional resistance to motion of a shaft in a lubricated bearing increases as the viscosity of the lubricant increases, and decreases as the thickness of the film of lubricant increases. In practice, provided there is an ample supply of oil and sufficient clearance between the shaft and the bearing, increased viscosity is largely compensated for by increased thickness of film. The circulating film carries heat away from the bearing. If there is not an ample supply of lubricant the temperature of the oil film will increase and will produce a reduction in the viscosity of the lubricant, thus reducing the coefficient of friction. Increase of temperature, however, also diminishes the power of the film to withstand the load, and this explains why a bearing seizes when the rise of temperature is excessive.

The temperature of a bearing will rise until the heat generated per second by friction is equal to the heat dissipated per second by radiation and conduction, and a bearing for high speeds should therefore be longer than a similar bearing for low speeds, in order to provide increased radiating surface. Such a bearing should also be flooded with lubricant in order to assist in the removal of the heat generated.

Professor Osborne Reynolds investigated the phenomena discovered by Tower, and propounded a theory of lubrication (see the Philosophical Transactions of the Royal Society, 1886). He applied the hydrodynamical equations of motion to the problem and explained how, and under what conditions, the pressure oil film was maintained. He showed that in Tower's experiments there was a small difference between the diameters of the journal S and of the step B (Fig. 136), and that the point P at which the surfaces approached most closely together was not directly under the load, but was situated, as shown, nearer to that edge at which the shaft surface leaves the bearing. Reynolds worked out the case of a journal in a cylindrical bearing which he assumed to be of infinite length in the direction of the axis of the shaft.

As a result of this theory it was realized that the most suitable point in a bearing at which to introduce lubricant was the point of lowest pressure. Other investigators have continued the study of lubrication, and the mathematical analysis has been much simplified. The greatest development of the theory since its inception was that made by Mr. A. G. M. Michell, in 1905, and this development has led to a radical improvement in the design of bearings.*

It is now generally recognized that in addition to *film lubrication* as described above, in which the two metallic surfaces are separated by a continuous film of liquid oil of measurable thickness which forces itself between them, there is a system of lubrication known as *boundary lubrication,* or greasy lubrication, in which the two surfaces have between them a more or less complete layer of oil which is only, at the most, a few molecules thick. We have seen that film lubrication occurs in a journal bearing when there is a plentiful supply of oil and relative motion takes place in one direction only. Boundary lubrication is likely to occur in heavily loaded slow running bearings, in bearings where the relative motion is slow and reciprocating, and in all cases where the supply of oil is not sufficient for the maintenance of a film. When a bearing is at rest oil is squeezed out and so boundary lubrication is likely to occur at starting up even in a bearing in which film lubrication occurs under normal running conditions. Consequently, if a bearing is designed so as to provide adequate bearing surface for film lubrication it is possible that the bearing surfaces will be damaged when boundary lubrication occurs, as when starting or under shock pressure.

Boundary lubrication is very imperfectly understood and an authoritative statement of what it means is still difficult to find. The above definition is anything but precise. Several different explanations are given of how the lubrication is brought about, all of which, however, agree on the importance of the part played by intermolecular forces. These forces are said by some to be gravitational, by others electrical, and again by others to be both gravitational and electrical. The following is one possible explanation. All substances are made up of molecules each of which is powerfully attracted by neighbouring molecules. Inside the material, where a molecule is completely surrounded by other molecules, the attractive forces balance. A molecule near the surface, or on it, is pulled more towards the inside of the material than towards the outside, and there is a resultant attraction inwards along the normal to the surface. When a fluid of lower molecular attraction than that of the metal is poured on to a metallic surface, the attractions near the surface are only partially balanced and the attractive forces in the fluid due to the metal are greater than the internal forces due to the fluid. Consequently the molecules adhere to, or wet, the surface, forming a layer of fluid of molecular thickness. This is known as *adsorption.* A fluid of greater molecular attraction such as mercury does not adhere to the surface. A lubricant must therefore be a fluid whose inter-molecular attractions are less than the mutual attractions between its molecules and those of the material of which the bearing and journal are made. Attractions between molecules on opposite sides of the common surface between two elements of an unlubricated bearing cause the phenomenon known as "seizing," in which the metal surfaces adhere and then tear owing to relative motion.

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* Readers should consult the following treatises on the subject: Stanton, *Friction*; Michell and others, *The Mechanical Properties of Fluids.*

From the above it seems probable that friction under boundary conditions will follow similar laws to those of solid friction. It will be dependent on the load, as this affects the distance between the surfaces, and therefore the magnitudes of the attractions of the molecules on each other. It will be dependent on the nature of the materials in contact, and also on the lubricant used, as these also determine the magnitude of the attractions. It will also be largely independent of the speed unless there is an abnormal rise of temperature.

The characteristic of the lubricant referred to above as *oiliness* is necessary for boundary lubrication and is usually possessed by lubricants containing a proportion of fatty acids, e.g. vegetable lubricants such as rape seed oil, castor oil, etc. Viscosity is of little importance in this connection. Two oils having the same viscosity and density at the same temperature may differ considerably in oiliness, and therefore in their capacity for serving as boundary lubricants. "Anti-friction" metals are those whose surface attractions are reduced to the greatest extent by the neutralizing action of lubricants.

Power Loss and Heat Generated in a Bearing with Boundary Lubrication. If W is the load in pounds then

$$W = p \, . \, l \, . \, d$$

where p = pressure in lb. per sq. in. of projected area;
l = length of bearing in inches;
d = diameter of bearing in inches.

Then from (VII.14)

$$P = \frac{\mu p d^2 l N}{126000} \quad . \quad . \quad . \quad . \qquad \text{(VII.29)}$$

If $l = nd$

$$P = \frac{\mu p n d^3 N}{126000} \quad . \quad . \quad . \quad . \qquad \text{(VII.30)}$$

Now 1 h.p. is equivalent to 0·707 B.Th.U. per second.

$\therefore$ heat generated per second

$$= \frac{\mu p d^2 l N}{178000} \text{ B.Th.U.} \quad . \quad . \quad . \qquad \text{(VII.31)}$$

and heat generated per second per sq. in. of projected area

$$= \frac{\mu p d N}{178000} \text{ B.Th.U.} \quad . \quad . \quad . \qquad \text{(VII.32)}$$

It will be seen from (VII.32) that the heat generated per second per square inch is proportional to pV, where V is the surface speed of the shaft. The temperature of the bearing will rise until the heat generated per second is equal to the sum of the heat radiated per second from the external surface of the bearing and the heat conducted away along the shaft or carried away by the oil. With boundary lubrication μ is practically independent of both p and V, whilst with film lubrication, as we shall see below, μ is a function of both p and V.

Now let us consider a bearing in which there is film lubrication. For the formation and maintenance of this film there must be a finite difference between the radius of the journal and that of the bearing, and the bearing must be slightly eccentric to the journal so that the film formed is wedge-shaped. Under these circumstances a film is capable of withstanding very high pressures at very high speeds. Messrs. Boswall and Brierley (*Proc. I.Mech.E.*, 1932, Part 2) experimenting with a bearing $2\frac{1}{2}$ in. diameter and 4 in. long, showed that the coefficient of friction μ in a film-lubricated bearing is a function of ZN/p, where Z (defined below) is the coefficient of viscosity of the lubricant at entry to the bearing, N is the speed in r.p.m., and p is the load per square inch of projected area. In these experiments N ranged from 2500 r.p.m. to 6000 r.p.m., p varied from 40 lb. to 120 lb. per square inch, and Z varied from 0·22 to 0·56 poises.

For a centrally loaded bearing with the arc of contact varying from 45° to 90°, the formula deduced from these results is

$$\mu = 0{\cdot}0028\left(\frac{ZN}{2p}\right)^{0{\cdot}58} \quad . \quad . \quad . \quad (\text{VII.33})$$

With central loading and 90° of arc of contact the film thickness varied from 0·004 in. with $ZN/p = 4$ to 0·006 in. with $ZN/p = 22$.

From (VII.32) and (VII.33) it follows that if Z, p and d are constants the heat generated per second per sq. in. is proportional to $N^{1{\cdot}58}$. The value of Z, however, decreases rapidly as the film temperature increases. This decreases the value of μ, but at the same time diminishes the sustaining power of the film, so that at high speeds the lubricant should be supplied to the bearing at as low a temperature as possible and should be cooled before being recirculated.

Since the viscosity of a lubricant diminishes rapidly with increase of temperature, it is necessary to employ a lubricant of high viscosity at normal temperature if the running temperature of the bearing is likely to be high. In such cases greases are sometimes used.

Z, the coefficient of viscosity, is defined as follows. Consider two parallel flat surfaces each of area A sq. cm. separated by a film of oil t cm. thick. Let F dynes be the magnitude of each of the tangential forces which must be applied to the surfaces in order to maintain a constant relative velocity of v cm. per sec. in a tangential direction. Then

$$F = Z\frac{Av}{t}$$

Using the methods of Appendix C we see that the dimensions of Z are $ML^{-1}T^{-1}$. The unit value of Z is called the *poise*. Now with the ordinary meaning of μ, the coefficient of friction, we have $F = \mu pA$ and therefore $\mu pA = \frac{ZAv}{t}$

i.e.
$$\mu = \frac{Zv}{pt} \quad . \quad . \quad . \quad . \quad (\text{VII.34})$$

This would seem to show that μ varies as Z, but as increase of viscosity is usually accompanied by an increase in film thickness t, we see that this is not so and that this relation is not necessarily in conflict with (VII.33).

[For a fuller treatment of viscosity and descriptions of experiments to determine the coefficient of viscosity, readers should consult *The General Properties of Matter*, by Newman and Searle, or any other textbook of physics which treats of viscosity.]

131. Michell Bearings. Mr. Michell considered the case of a bearing with surfaces of finite length in the direction perpendicular to that of the motion. He assumed a square-faced block *EGDC* (Fig. 137), (*CD* being an edge view of the square face) to be supported by an oil film, the latter being carried along by a plate *AB* moving in the direction of the arrow. He showed that if the block is loaded over the centre of pressure of the face *DC*, it will set itself at a small inclination to the face *AB* so that the space between the two surfaces tapers, being wider at the entering than at the leaving end. The angle of tilt varies with the pressure and with the velocity. The distribution of pressure over the area of contact is shown in Figs. 137 and 138. The curve *EFG* shows the distribution of pressure across the middle section *LM* of the surface *CD*, vertical heights above *EG* representing pressures. The closed curves in Fig. 138 show the lines of equal pressure, the pressure increasing from zero at the edges of the square to a maximum at *K*. *P* is the point at which the resultant pressure acts.

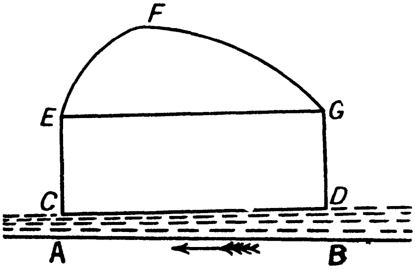

Fig. 137. Film of Fluid

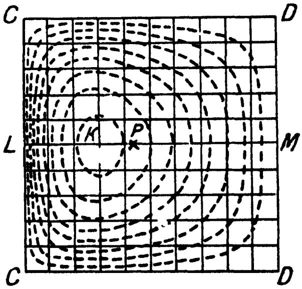

Fig. 138. Curves of Constant Pressure

In the half-cylindrical bearing of Fig. 136, we saw that the shaft sets itself in a position of equilibrium in a similar manner to that described above, but, as the shaft moves bodily to that position, the taper of the film at any particular place is not the correct one for the conditions at that place. In Fig. 136 it will also be noticed that the oil moves through a channel which diminishes in width up to the point *P*, and then begins to increase in width, whereas in Fig. 137 there is a continual diminution in the width of the channel. The former is, of course, the only possible form of tapered film for two curved surfaces. If, however, the bearing *B* is split up into a number of portions by planes passing through the axis, and each portion is pivoted about a point on the radius through the centre of pressure of its bearing surface, then any portion can take up its own correct position and produce a more nearly correct form of tapered film. By enabling the blocks to take up the proper position for the flow of the fluid film, we ensure the maintenance of the film and thus produce efficient lubrication; also at the same time we reduce the maximum pressure in the film by bringing into action all the bearing surface on the loaded side of the bearing.

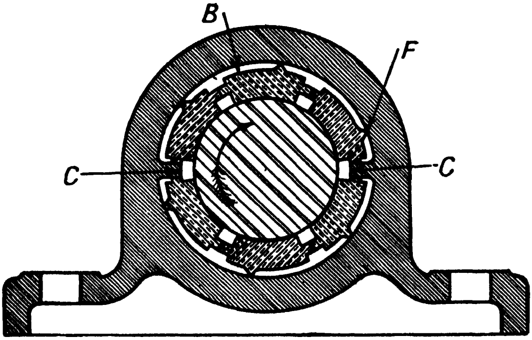

FIG. 139. SCHEME OF MICHELL JOURNAL BEARING

Fig. 139 shows a section through a journal bearing constructed on this principle. The portions *B* are blocks pivoted along lines of contact *F* with the housing. *CC* are lugs, to prevent rotation of the blocks.

Fig. 140 shows two views of a Michell thrust bearing. The tilting blocks are in this case pivoted segments *B*; the lines of the pivot are shown dotted in the end view. The blocks rest with the pivots on the plane surface of *D*, and are prevented from rotating by lugs *C* projecting from *D*. The radial edges of adjacent blocks either touch, or are separated by small distance pieces. The angle of tilt of the blocks is never more than $\frac{1}{50}$ of a degree, and the distance from the pivot to the entering edge is about ·62 to ·66 of the distance from the entering to the leaving edge. Sometimes the pivot is a point opposite the centre of pressure instead of a radius through that point, in which case the block can adjust itself laterally. It is claimed that these bearings will permit of a pressure of 5,000 lb. per sq. in. of bearing surface, as against pressures of 600 lb. per sq. in. in Tower's experiments, and 50 to 60 lb. per sq. in. the working pressure in ordinary thrust bearings. Michell bearings are the practical outcome of the hydrodynamical theory and are considerably superior to ordinary bearings in their ability to withstand end thrust.

132. **Ball and Roller Bearings.** When two surfaces touch each other there is a resistance to their relative motion. If the relative motion is that of rolling, the resistance is usually less than when it is that of sliding. On this account, rolling motion is often substituted for sliding motion, so as to reduce the resistance to relative motion. In his *Scientific Papers*, Vol. I, Professor Reynolds developed a theory of rolling friction, and showed that it is caused by the distortion under pressure of the surfaces of contact of the two bodies, which produces relative sliding motion between the surfaces. The frictional forces thus set up give rise to a resistance to rolling which is known as rolling friction. In order to replace sliding motion by rolling motion, balls or rollers are interposed between the two surfaces of a sliding or turning pair.

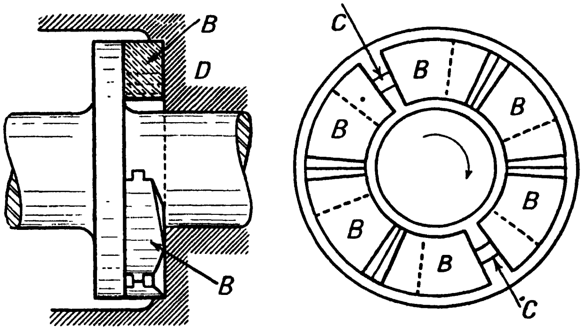

FIG. 140. SCHEME OF MICHELL THRUST BEARING

Fig. 141 shows a ball bearing. *R* is the outer race, or stationary element of the turning pair, *r* is the inner race, or moving element of the turning pair. *B*, *B*, *B*, etc., are balls arranged between the surfaces of the two races. Usually the inner race is fitted on to a shaft, being a tight press fit, and is held endwise between the surfaces of a lock-nut and an annular projection on the shaft. If the race is not firmly located on the shaft, relative motion, or creep, occurs between the shaft and the race. This leads to wear and consequently to slackness of fit. The outer race *R* is a press fit into a fixed housing by which it is supported. A small amount of creep between the outer race and the housing is beneficial, as it brings all parts of the outer race in turn into the loaded position, and thus leads to uniform wear.

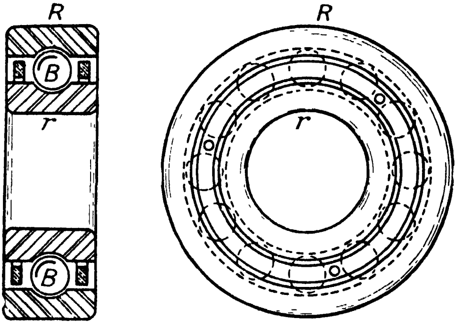

FIG. 141. BALL BEARING

The balls, rollers, and races are made of the best quality chromium steel containing also a relatively high percentage of carbon. They are best hardened throughout, as case-hardening does not give consistent results. The balls in a bearing are usually separated by a metal cage, not shown in the figure. This cage keeps the balls at a fixed distance apart, and prevents crowding of the balls, which would lead to unequal loading. The cage also prevents any two adjacent balls from rubbing together. The relative speed between two balls rubbing together is twice that between a ball and a cage, and at high speeds any rubbing would cause excessive temperature in the bearing, and would ultimately lead to failure of the bearing. For high speeds, the cage is usually made of bronze, and is in the form of a solid ring with cavities for the balls. For low speeds, a cage of pressed steel is used. It will be seen from Fig. 141 that the balls run in grooved tracks on the surfaces of the cylinders. The bearings are thus able to sustain end thrust up to about 30 per cent of the difference between the radial load for which they are rated and the actual radial load. Roller bearings are not capable of sustaining end thrust, as the paths of the rollers are cylindrical. We illustrate on opposite page two other types of anti-friction bearing.

Fig. 142 shows the Skefco double-row ball bearing, in which the outer race surface is spherical, so that the bearing permits self-alignment of the shaft, and is capable of withstanding a considerable end thrust.

Fig. 143 shows a typical method of mounting by the Hoffmann Co.; the roller bearings carry the radial load, but do not locate the shaft endwise. The outer race of the ball bearing is of slightly less diameter than its housing, and so the ball bearing, while not supporting radial load, locates the shaft endwise and carries the whole end thrust.

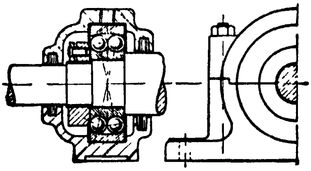

FIG. 142. SKEFCO TRANSMISSION BEARING

There is a decided advantage in the short lengths of ball and roller bearings, as compared with those of plain bearings. In the former, the length is seldom greater than the diameter of the bore, and is often only one-third of this; in the latter the length is often two or three times the diameter of the bore. The starting effort necessary to set in rotation a machine fitted with ball and roller bearings is considerably less than that necessary when the shaft is fitted with plain bearings.

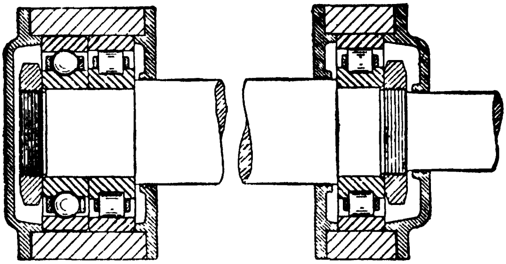

FIG. 143. HOFFMANN TRANSMISSION BEARING

In the report of a test conducted by Mr. W. E. Baker, B.Eng., M.C., it was claimed that the starting effort for a spinning machine was reduced in the ratio 1 : 1000, when ball and roller bearings were substituted for plain bearings, and that the saving in power was about 42 per cent of that previously wasted.*

EXAMPLE.

The screw *S* of Fig. 124, page 185, is raised by rotating the nut *N*, the screw being prevented from rotating. The mean diameter of the thread is 4 in.,

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* See *The Journal of the Textile Institute*, Vol. XVI, No. 9.

and the pitch $\frac{5}{8}$ in. The load raised by the screw is 1 ton, and the coefficient of friction is ·25. The under side of the nut is supported by a collar bearing the inner and outer diameters of which are 4·5 in. and 5·5 in. respectively. Find the turning moment or torque on the nut necessary to raise the load, and find the mechanical efficiency when raising this load

Let us first consider the friction on the collar bearing. From (VII.22) the friction couple C_f is given by

$$C_f = \tfrac{2}{3} \times \cdot 25 \times 1 \frac{\left(\dfrac{5\cdot 5}{2}\right)^3 - \left(\dfrac{4\cdot 5}{2}\right)^3}{\left(\dfrac{5\cdot 5}{2}\right)^2 - \left(\dfrac{4\cdot 5}{2}\right)^2} \text{ ton inch units}$$

$$\therefore C_f = \cdot 627 \text{ ton inch units}$$

Next, consider the raising of the load, taking account of the friction between the screw and the nut. If C_l is this couple, then from (VII.7),

$$\text{Tangential force} = F = 1 \frac{\pi + \frac{5}{8}}{4\pi - \frac{5}{32}} \text{ tons}$$

or

$$C_l = 2F$$

$$= \frac{2\pi + \frac{5}{4}}{4\pi - \frac{5}{32}}$$

$$= \cdot 606 \text{ ton inch units}$$

Thus, the total turning moment C is given by

$$C = C_f + C_l$$

$$= 1\cdot 233 \text{ ton inch units}$$

To find the mechanical efficiency, we have

$$\eta = \text{mechanical efficiency}$$

$$= \frac{\text{work done in raising load}}{\text{total work done}}$$

and considering one revolution of the nut,

$$\eta = \frac{\text{load} \times \text{pitch}}{\text{Couple} \times 2\pi}$$

$$= \frac{\frac{5}{8}}{1\cdot 233 \times 2\pi}$$

$$= \cdot 0806 \text{ or } 8\cdot 06 \text{ per cent.}$$

This efficiency is very low owing to (1) the high coefficient of friction, and (2) the large amount of collar friction. The efficiency of a worm and worm-wheel when transmitting power may be found by the method of the above example.

EXAMPLES 7.

1. Show that when a is variable in (VII.4) and (VII.5), the least force F lb. needed to keep the weight in motion is $W \sin (\theta + \varphi)$ for motion up the plane, and $W \sin (\theta - \varphi)$ for motion down the plane, and that, in each case, the force acts along a line making an angle φ with the plane.

2. Prove that the work done in pulling a weight along the whole length of a rough inclined plane, by means of a force parallel to the plane, is equal to the sum of the amounts of work done by (1) a vertical force lifting the weight through the height of the plane, and (2) a horizontal force dragging the weight along an equally rough horizontal plane through a distance equal in length to the base of the plane.

3. Prove that if θ is the angle of inclination of the helix to the horizontal in a vertical screw jack with square threads, the efficiency is a maximum, and is equal to $\left(\dfrac{1 - \tan\frac{\varphi}{2}}{1 + \tan\frac{\varphi}{2}}\right)^2$ when $\theta = 45° - \frac{\varphi}{2}$, φ being the angle of friction.

4. Two circular plates are in contact over their circular faces, the centres of these faces being coincident. If there is a normal pressure of N lb. between the plates, and the pressure is uniformly distributed, find the friction couple exerted by one plate on the other when the plates are in relative rotation about the common centre. The radii of the two circles are each r, and μ is the coefficient of friction.

5. A friction cone is made up of two conical frustra, one of which fits inside the other, the smallest radius of the area of contact is r, and the largest radius is R. The vertical angle of the cone is $2a$. Let P be the force acting along the common axis, pushing the inner cone into the outer one. Assuming the pressure to be uniformly distributed over the area, find the greatest turning moment about the axis of the cone which can be transmitted without slip taking place.

(NOTE. In textbooks which treat of friction cones, the assumptions are usually made that there is limiting friction in planes containing the axis, as well as in circumferential planes. Dr. Prescott has pointed out that these assumptions are mutually contradictory. When slipping takes place, it does so in a circumferential direction, hence there is limiting friction in this direction. See Prescott's *Mechanics*, Longmans, Green & Co., Ltd. ; Ex. 24, page 124, and Preface.)

6. A friction clutch is to be designed to transmit 25 h.p. at 600 revs. per min. ; the taper of the cones is to be 1 in 15, and the coefficient of friction ·18. If the mean diameter of the area of contact is 8 in., what axial force is necessary to just prevent slipping ? Taper = diameter ÷ length.

7. Schieles' pivot is a footstep bearing, designed to produce equal wear at all parts of the surface. Show that, on the assumption of equal pressure, an axial section of such a pivot is such that if a tangent be drawn to the section of the surface at any point, the length of this tangent intercepted between the point and the axis is constant.

8. Find the efficiency, when raising a load, of a screw-jack with a square-threaded screw in which the mean radius of the thread is three times the pitch, and $\mu = \cdot 15$.

9. With the screw-jack of example 8, find the force acting at the end of a bar 18 in. long which will raise a load of 10 cwt. Pitch = $\frac{1}{2}$ in.

10. A screw-jack has a square-threaded screw of 3 in. mean diameter. The angle of inclination of the thread is 3°, and the coefficient of friction is 0·06. It is operated by a handle 18 in. long. What pull must be exerted at the end of the handle (*a*) to raise, (*b*) to lower a load of 5 tons. (U.L.)

11. Find the efficiency of the screw-jack in example 10 when raising the load.

12. A column, weighing 2 tons, slides vertically between guides, and can be raised or lowered by a horizontal force applied to a wedge on which it rests. If the angle of inclination of the wedge is $\tan^{-1} \frac{1}{8}$, and if the coefficient of friction between all moving parts is $\frac{1}{5}$, what force is necessary (*a*) to raise, (*b*) to lower, the column ? (U.L.)

13. State, in general terms, how the friction varies with (*a*) the speed of rubbing, (*b*) the intensity of pressure, (*c*) a rise in temperature, in the case of surfaces which are (1) dry, (2) profusely lubricated. (I.C.E.)

14. A collar thrust bearing has an outer diameter of 9 in. and an inner diameter of 6 in. It rotates at 120 revs. per min., and carries an end thrust of 4 tons. If the coefficient of friction is 0·02, what is the horse-power lost in friction ?

15. A thrust block collar is 10 in. in external diameter, and 6 in. in internal diameter, and carries a load of 10 tons. It rotates at 120 revs. per min, and the coefficient of friction is 0·02. Determine the horse-power lost in friction (*a*) assuming the pressure intensity to be constant, (*b*) assuming the pressure to be inversely proportional to the rubbing velocity. (U.L.)

16. A horizontal drawer is 36 in. wide and 9 in. long. On the front are two handles, symmetrically placed. The drawer fits between two vertical slides 27 in. apart. Neglecting the friction between the base of the drawer and its supports, find the condition that a force P on one handle should open the drawer where $2x$ is the distance between the handles. The coefficient of friction is ·40. Find for what values of x the drawer will jam so that no value of P, however large, will open the drawer.

17. Explain the use of friction circles in finding the thrust, or tension, in a link forming an element of one or more turning pairs. Give diagrams showing the forces in the connecting-rod of a steam-engine for crank angles of 45°, 135°, 225°, and 315°. How is the effective turning moment found ?

18. Write a brief account of Tower's experiments on lubricated bearings, and Reynolds' deductions therefrom. What effects have these had on the development of lubrication and of bearings ?

19. Sketch the essential features of a Michell thrust block, and describe the principles on which its operation depends. (U.L.)

20. Describe the Michell journal bearing, giving a sketch showing the pivoted blocks.

21. Compare the advantages and disadvantages of plain bearings and ball bearings. (I.C.E.)

22. Two pairs of driving wheels of a locomotive are coupled together ; one pair has a radius r_2 slightly less than the other r_1. The axle having the smaller wheels has the heavier load, so that this pair of wheels does not slip. Show that, due to the unequal size of the wheels, there is a retarding force on the locomotive, and determine the magnitude of this at the instant when the coupling-rod cranks are vertical. Take the load on the axle of the larger pair of wheels as 20 tons ; $r_2 = \cdot 99 r_1$; and the coefficient of friction between wheel and rails as 0·10. (U.L.A.)

23. A worm reduction gear (Art. 163) transmits 30 h.p., with speeds of worm and wheel at 1,600 and 40 revs. per min. respectively. The worm is $3\frac{1}{2}$ in. diameter, 1 in. pitch, single thread. The thrust is taken by a flat collar bearing, mean diameter 5 in. The coefficient of friction for worm

and thrust may be taken as 0·05. Determine the efficiency of the gear and show whether the gear is self-locking if the drive is attempted from the wheel side. (U.L.A.)

24. A locomotive with bogie wheels unbraked, is coupled to a train which has all its wheels braked. The weight of the engine and tender is 50 tons, and of the rest of the train 350 tons. The weight carried by the bogie wheels is 15 tons. Assuming that the coefficient of braking friction is 0·2, calculate the maximum tension in the draw-bar between tender and train during braking and also the maximum retardation of the complete train on the level. (I.C.E.)

25. The diameter of the screw in a screw-jack is 2 in., and the pitch of the screw is ¼ in. Find an expression giving the relation between the torque exerted on the screw and the load supported by the jack (1) neglecting friction, (2) including the friction of the screw and assuming φ for the limiting angle of resistance between the screw and the nut. (I.C.E.)

26. Sketch a type of double-row ball bearing which permits self-alignment of the shaft. Compare this type with a single-row bearing. How is error of alignment corrected in the latter type ?

27. In a double-row ball bearing, or a single-row conical ball bearing, as used in a bicycle bearing, the cone of contact of the balls makes an angle a with the axis of the bearing. Find the thrust in each ball due to an end thrust T lb. Assume that there are n balls under thrust.

28 What is "creep" in ball and roller bearings ? Explain where creep is to be avoided and where a small amount is advantageous.

29. Discuss the effect of side force on a motor tyre in causing skid. Would it be wise to shut off power just before rounding a corner at high speed ?

30. In some scientific recording instruments where a carriage is used to carry a pen drawing a curve on a rotating drum, the carriage is arranged to slide along two cylinders which form sliding pairs with the carriage. Explain how rotation of these cylinders reduces the friction in the direction of motion of the carriage. (The reason for this reduction in friction is the reason why we rotate a cork at the same time as we pull it in order to withdraw it from a bottle-neck.)

31. The centre of gravity of a motor-car is at a height h above road level, and is at a distance a behind the front axle, and b in front of the back axle. Calculate the greatest retardation which can be produced by the brakes without skidding on a level road (a) if the front wheels are braked, (b) if the rear wheels are braked. Show that rear wheel braking is the more effective if $a > b + \mu h$ where μ is the coefficient of friction between tyres and road. (U.L.A.)

32. The angle of a wedge (Fig. 37) is 7°, and the coefficient of friction between the pairs A–B and A–C is 0·2. The pair C–B may be assumed to be frictionless. Find the mechanical advantage and the mechanical efficiency. What do these quantities become if the angle of the wedge is reduced to 2° ? If, due to scoring, the coefficient of friction is raised to 0·6, what is now the value of the mechanical advantage ?

33. A disc 36 in. diameter and ½ in. thick requires 1·24 h.p. to keep it running in air at 2,000 r.p.m. Journal friction absorbs 0·8 h.p., and the resistance in lb. due to air friction is kAv^2, where A is the area in square feet of any portion of the surface whose velocity is v ft. per second. Find k.

34. In a cone friction coupling prove that the couple which can be transmitted is equal to $\dfrac{F\mu r}{\mu \cos a + \sin a}$ where F is the axial force pressing the cones together; r is the mean radius of the driving surface, and $2a$ is the cone angle. (U.L.)

[By putting a for θ in (vii, 4) and $-a$ for a, we obtain $R = F/\sin(a + \phi)$. The normal component of R is $R \cos \phi$. Hence the normal pressure is proved to be $\dfrac{F}{\mu \cos a + \sin a}$. By assuming limiting friction in a circumferential direction the above result is obtained. See note to example 5 above.]

35. Outline briefly the Reynolds theory of lubrication, and show how it is applied in the Michell thrust bearing. (U.L.)

36. A wedge used for splitting wood has an apex angle of 15° and is driven 1 in. vertically into a baulk by a force equivalent to the blow of a 10 lb. weight falling freely through 9 ft. The timber offers a horizontal resistance of 500 lb. to splitting. Find the coefficient of friction between the wedge and the wood and the efficiency of the wedge. What pull would be required to withdraw the wedge and what is the greatest angle the wedge could have if it is to remain in position after each blow? (U.L.)

37. The angle of a Whitworth V thread is 55° and the diameter in inches at the bottom of the thread is $d = D - 0{\cdot}64p$, where p in. is the pitch of the screw and D in. is the external diameter of the bolt. The frictional resistance at the base of the nut may be taken as acting at a radius $0{\cdot}7d$. If $\mu = 0{\cdot}15$, show that the torque on the nut necessary to produce a pull P lb. in the bolt is given by

$$T = P(0{\cdot}19D + {\cdot}06p) \text{ lb.-in.}$$

Show also that the efficiency is given by

$$\eta = \frac{1}{1{\cdot}2\frac{D}{p} + {\cdot}38}$$

The following are particulars of an actual test on an unlubricated bolt: $D = 1\frac{1}{4}$ in.; $p = \frac{1}{7}$ in.; effort applied at 30 in. radius—

Load in tons . .	1·5	2·0	2·5	3·0	3·5	6·0	6·5
Effort in lb. . .	27	37	45	54	64	112	116

Test whether the above formula applies to these results and find the efficiency of the arrangement.

38. The motion of a vessel drifting away from a dock-side is retarded by a rope secured to the vessel and given three complete turns round a bollard on the dock-side. A pull of 80 lb. is applied to the free end of the rope at an instant when the speed of the vessel, which weighs 4,000 tons, is ¼ ft. per sec. After 10 sec. the rope begins to slip. Assuming the rope stretches elastically, calculate the stretch in the rope between the bollard and the vessel and the speed of the vessel when the rope slips. ($\mu = 0{\cdot}25$.) (U.L.)

39. Distinguish between boundary and film lubrication. Discuss carefully the conditions under which lubrication takes place between a journal and its bearing. Indicate the distribution of pressure and the position of the resultant pressure between the surfaces. Describe how the principle involved is applied in the Michell type of thrust bearing. (U.L.)

40. A single disc clutch is required to transmit 20 h.p. at 1,000 r.p.m. under conditions in which the mean torque may be exceeded by 50 per cent. The outside diameter of the disc is limited to 12 in. and the coefficient of friction between the fabric of the disc and steel is 0·25. If the maximum pressure is 12 lb. per sq. in., determine the width of the disc rings. (U.L.)

[*Note*: Assume uniform pressure in such cases unless the law of variation is given.]

41. A screw has a vee-thread of spiral angle α, i.e., $\tan \alpha = p/2\pi r$, and the actual angle between the two faces of the thread is 2θ. An axial plane intersects the thread surfaces in two lines meeting at an angle 2β as in Fig. 125. Show that

$$\tan \theta = \tan \beta \,.\, \cos \alpha$$

If $\alpha = 45°$ and $\beta = 27°$ show that θ is approximately 20°.

42. A plate clutch has 10 operating faces with inner and outer diameters of 6 in. and 10 in. respectively. It runs at 1200 r.p.m. with a total end load of 450 lb., the coefficient of friction being 0·08.

Find the maximum torque and horse-power that can be transmitted, assuming that wear has taken place so that the intensity of pressure between the plates varies inversely with the radius. (U.L.)

43. If R is the external radius of a plate clutch and $n.R$ is the radial width of the clutch lining, show that the percentage error due to the assumption that the total frictional resistance acts at the mean radius of the clutch lining (compared with VII. 22) is given by

$$\% \text{ error} = \frac{100}{12 \cdot \frac{1-n}{n^2} + 3}$$

Hence show that if n is not greater than 0·5, the percentage error does not exceed 3·4.

44. An engine is to transmit 40 h.p. at 2000 r.p.m. through a single plate clutch. The external radius of the clutch lining is to be $5\frac{1}{2}$ in., the intensity of pressure = 15 lb. sq. in. and $\mu = 0{\cdot}25$. Find the necessary internal radius of the clutch lining and the necessary axial load on the clutch lining.

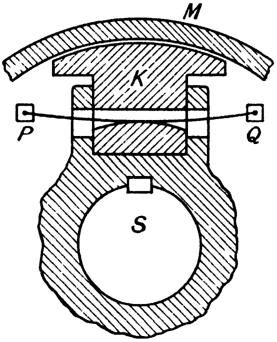

Fig. 143A

45. Fig. 143A shows a portion of a centrifugal clutch with four shoes K driven by a shaft S. When S is at rest K nearly touches the inner surface of the driver pulley M. The spring PQ exerts an inward radial pressure of 45 lb. on K, and this may be regarded as constant at all speeds. If the weight of each shoe is $3\frac{1}{2}$ lb., radius to centre of gravity of shoe = 4·8 in., radius of inner surface of pulley M = 7·5 in., calculate (a) the speed in r.p.m. at which the shoes begin to move outward, (b) the maximum torque which can be transmitted to pulley M at 800 r.p.m. $\mu = 0{\cdot}15$. (U.L C I.)

46. If the scale of the clutch shown in Fig. 143A is increased m times, show that the torque transmitted is increased m^5 times, provided that the spring load due to PQ is increased m^4 times.

CHAPTER VIII

HIGHER PAIRING—BELT, ROPE, AND CHAIN DRIVES

133. Belt Drives. In Chap. IV we saw that pairs of elements whose relative motions are sliding, turning, or screw motion, are known as lower pairs, and all other pairs are higher pairs. The latter term is not used to indicate that such pairs are more useful, but to show that the motion between them is of a more complicated nature. A belt and pulley form a higher pair. Here, if we neglect slip, the relative motion is that of rolling, the belt winding on to the pulley at one point and off at another. Let us consider two pulleys, A and B, mounted respectively on shafts S_1 and S_2 (Fig. 144), to which they are attached.

$CDEFGH$ represents an open endless belt connecting the two pulleys. If the straight portions of the belt are crossed as indicated by the dotted lines, the belt is then known as a crossed belt. With a crossed belt it is necessary to give the belt a half-twist in the straight portions shown, so that the same surface of the belt shall make contact with both pulleys. This also enables the portions of the belt to pass each other at the point L, as it brings the working faces of the belt together there. If the shaft S_1 is set in motion in the sense indicated, frictional forces will be produced between the pulley and the belt. If the belt is sufficiently tight, these forces will move the belt, and through the medium of a second set of frictional forces, the belt will cause the pulley B and the shaft S_2 to rotate. For motion of the shaft S_2 to take place, it is necessary that the frictional force between the belt and either pulley shall be not less than the tangential force necessary at the rim of the pulley B to rotate the shaft S_2. If the pulley B rotates, A is a driver, B a follower, and the belt is the connector. The belt is flexible and capable of stretching, and on this account, and because of the possibility of slip, it is *not a true connector.*

With this arrangement, the relative motion of the shafts is not a definite function of the dimensions and position of the mechanism as it would be, for instance, in the case of a four-bar mechanism connecting the two shafts. This indefiniteness of the relative speeds prevents the use of belts in cases where an exact speed ratio is essential, as in clockwork, or recording apparatus. Where a machine is subject to variations of torque, however, elasticity is in many cases an advantage, as it enables the belt to absorb shock, and thus to prevent damage to the more rigid parts of the machine.

Let r_1 and r_2 be the radii in feet, and n_1 and n_2 the speed in revolutions per minute of the pulleys *A* and *B* respectively. The surface speeds of the two pulleys are $2\pi n_1 r_1$ and $2\pi n_2 r_2$ ft. per min. respectively. If there is no slip between the belt and the pulleys, these are equal, i.e.

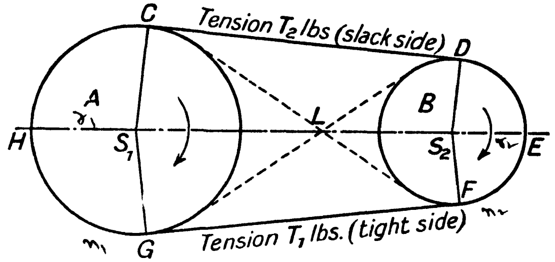
C
Tension T_2 lbs (slack side)
D
A
r_1
H
S_1
L
B
S_2
E
F
Tension T_1 lbs. (tight side)
G

FIG. 144. OPEN BELT CONNECTING PULLEYS

$$2\pi r_2 n_2 = 2\pi r_1 n_1$$

or

$$\frac{n_2}{n_1} = \frac{r_1}{r_2} \quad . \quad . \quad . \quad \text{(VIII.1)}$$

In this relation the radius of each pulley should be measured to the middle of the belt, that is, half the thickness of the belt should be added to the radius of each pulley. With an open belt, the senses of rotation of the two pulleys are both the same, with a crossed belt they are opposite. If T_1 and T_2 are the respective tensions in the tight and slack sides of the belt, then $T_1 - T_2$ is the driving force which acts at the circumference of the pulley *B*. The turning moment of this force is $(T_1 - T_2)r_2$.

In passing from the slack side to the tight side, any portion of the belt extends, and it contracts again in passing from the tight to the slack side ; owing to these changes of length there is relative motion between the belt and the pulley surfaces. This relative motion, called *creep*, leads to a loss of power, and also to a reduction of the velocity ratio. The difference between the actual speed of the driven pulley and that calculated by the above formula is called the *slip*, and is usually given as a percentage of the calculated speed. If S is the percentage slip, the expression (VIII.1) becomes

$$\frac{n_2}{n_1} = \frac{r_1}{r_2}\,\frac{100 - S}{100}$$

In arranging an open belt drive connecting two pulleys whose axes are not in the same vertical plane, it is usual to make the upper portion of the belt (CD in Fig. 144) the slack side. With this arrangement, the sag of the belt, being greater on the slack side, will cause the belt to envelop more of each pulley surface, and thus to obtain a better grip on the same. If the tight side were uppermost, sag would cause the belt to envelop less of the pulley surface, and thus lead to increased slip.

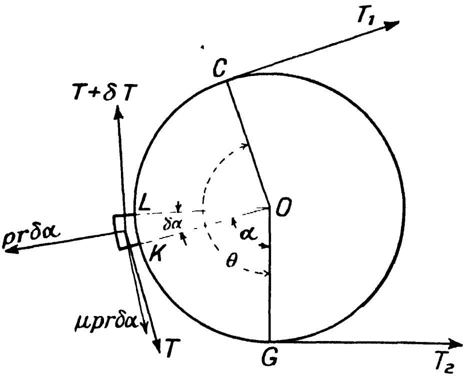

FIG. 145. LIMITING FRICTION BETWEEN BELT AND PULLEY

In drives from fluctuating sources of power, such as a gas engine, it is often better to have the tight side uppermost, as in such a case any sudden shock would cause the belt to vibrate and slip, and would thus reduce the shock on the driven pulley or machine. A belt drive is not so noisy as other types of drive. Belts are made of leather, camel hair, rubber, balata, steel, etc. The speed of belting should be arranged, where possible, at between 3,000 and 4,000 ft. per min.

134. **Tension in a Belt.** We shall now consider the variation of tension in a belt which is in contact with the surface of a pulley when the belt is just about to slip over the surface of the pulley.

Fig. 145 shows the pulley, whose centre is O and whose radius is r. The arc of contact is GKC Let $\widehat{GOC} = \theta$, and let μ be the coefficient of friction between the surfaces of the belt and the pulley. Consider an infinitesimal piece of the belt LK, where $\widehat{GOK} = \alpha$, and $\widehat{KOL} = \delta\alpha$. The piece is in equilibrium under the following forces, (1) the tension T acting along the tangent at K, (2) the tension $T + \delta T$ acting along the tangent at L, (3) the pressure $pr\delta\alpha$ acting along some radius between OK and OL, where p is the pressure per unit length of arc of contact exerted by the pulley on the belt, and (4) the force of friction $\mu pr\delta\alpha$ acting along a line perpendicular to the line of action of (3). The angle between the radius OK and the line of action of the force (3) is less than $\delta\alpha$. Let this angle be $k\delta\alpha$, where k is less than unity. .Resolving along the radius KO, we have

$$(T + \delta T) \sin \delta\alpha - pr\delta\alpha \cos (k\delta\alpha) - \mu pr\delta\alpha \sin (k\delta\alpha) = 0$$

If we neglect quantities of the second and higher orders in $\delta\alpha$, we may put $\sin \delta\alpha = \delta\alpha$, $\cos (k\delta\alpha) = 1$, and $\sin (k\delta\alpha) = k\delta\alpha$.

$$\therefore\ T\delta\alpha + \delta T\,.\,\delta\alpha - pr\delta\alpha - \mu prk(\delta\alpha)^2 = 0,$$

and, discarding terms of the second order,

$$T\delta\alpha - pr\delta\alpha = 0 \quad . \quad . \quad . \quad . \qquad (VIII.2)$$

or, dividing by $\delta\alpha$ and transposing,

$$T = pr \quad . \quad . \quad . \qquad (VIII.3)$$

Again, resolving along the tangent at K,

$$T + \mu pr\delta\alpha \cos (k\delta\alpha) - pr\delta\alpha \sin (k\delta\alpha) - (T + \delta T) \cos \delta\alpha = 0$$

whence $T + \mu pr\delta\alpha - prk(\delta\alpha)^2 - T - \delta T = 0$

or $\mu pr\delta\alpha = \delta T$

i.e. $$\delta T = \mu pr\delta\alpha \quad . \qquad (VIII.4)$$

Substituting in (VIII.4) from (VIII.3),

$$\delta T = \mu T\delta\alpha$$

Hence, $$\frac{\delta T}{T} = \mu\delta\alpha$$

or $$\int_{T_2}^{T_1} \frac{\delta T}{T} = \int_0^{\theta} \mu d\alpha$$

Integrating $$\log_e \frac{T_1}{T_2} = \mu\theta$$

and $$\frac{T_1}{T_2} = e^{\mu\theta} \quad . \quad . \quad . \qquad (VIII.5)$$

This relation shows us that the ratio of the tensions is independent of the radius of the pulley, and (VIII.5) is true for a non-circular pulley, θ in such a case being the angle between the radii of curvature at the extremities of the arc of contact.

It will be seen on examination of Fig. 144, that the angles of contact are equal in the case of a crossed belt, and unequal with an open belt, the smaller angle of lap being that on the smaller pulley. Thus, if μ is the same for the surfaces of the two pulleys connected by an open belt, slipping will occur first on the smaller pulley. For this reason, the value of θ in (VIII.5) should be that for the smaller pulley.

The reader should notice that (VIII.5) gives the limiting value of the ratio of the tensions.

In practice a value of μ which is somewhat smaller than the actual value is usually assumed.

The *effective* pull of the belt is $T_1 - T_2$, and if V is the speed of the belt in feet per second when transmitting P horse-power, then

$$T_1 - T_2 = \frac{550\,P}{V} \qquad . \quad . \quad . \qquad \text{(VIII.6)}$$

From (VIII.5),

$$T_2 = \frac{T_1}{e^{\mu\theta}}$$

$$\therefore\ T_1\left(1 - \frac{1}{e^{\mu\theta}}\right) = \frac{550P}{V}$$

$$\therefore\ T_1 = \frac{e^{\mu\theta}}{e^{\mu\theta} - 1} \cdot \frac{550P}{V} \qquad . \quad . \qquad \text{(VIII.7)}$$

and

$$T_2 = \frac{1}{e^{\mu\theta} - 1} \cdot \frac{550P}{V} \qquad . \quad . \qquad \text{(VIII.8)}$$

In order to transmit the required horse-power, the belt must be given an initial tension T_0. When the driving pulley begins to move, the "tight" side of the belt stretches until the pull is increased to $T_0 + \alpha = T_1$, and the "slack" side shortens until the pull is decreased to $T_0 - \alpha = T_2$ (assuming that the full power P is to be transmitted), so that

$$T_0 = \frac{T_1 + T_2}{2}$$

$$= \frac{1}{2} \cdot \frac{e^{\mu\theta} + 1}{e^{\mu\theta} - 1} \cdot \frac{550P}{V} \qquad . \qquad \text{(VIII.9)}$$

Since the length of the belt changes while it is moving from the " slack " to the " tight " side over the driven pulley, it is obvious that the relation given in (VIII.1) is only approximately satisfied when an appreciable torque is being transmitted to the driven shaft. The discrepancy increases with the distance between the centres of the driving and driven pulleys.

EXAMPLE.

A belt running on a pulley 3 ft. 6 in. in diameter, rotating at 230 revs. per min., is to transmit 12 horse-power. Taking $\mu = \cdot 27$ and the angle of lapping $= 165°$, find (*a*) the necessary width of the belt if the pull is not to exceed 90 lb. per in. of width, (*b*) the necessary initial tension in the belt.

In this case, $V = \dfrac{3\cdot 14 \times 3\cdot 5 \times 230}{60} = 42\cdot 13$ ft. per sec.

$$\log_{10}\frac{T_1}{T_2} = \frac{\cdot 27 \times 165}{57\cdot 3 \times 2\cdot 3} = \cdot 338$$

$$\therefore \frac{T_1}{T_2} = 2\cdot 18 = e^{\mu\theta}$$

$$\therefore T_2 = \cdot 46T_1 \text{ and } T_1 - T_2 = \cdot 54T_1$$

From (VIII.6) $T_1 - T_2 = \dfrac{550 \times 12}{42\cdot 13} = 157$ lb.

$$\therefore \cdot 54T_1 = 157$$

$$\therefore T_1 = 292 \text{ lb.}$$

and $T_2 = 135$ lb.

$\therefore$ necessary width of belt $= \dfrac{292}{90} = 3\cdot 25$ in.

Initial tension $= \dfrac{T_1 + T_2}{2} = 214$ lb.

(Note that doubling either μ or θ is equivalent to *squaring* the ratio of the tensions.)

From (VIII.9) it will be seen that with given values of μ, θ, and V, the power that can be transmitted without slipping is proportional to the initial tension given to the belt.

135. **Effect of Centripetal Force on the Tension in a Belt.** As a length $r\delta\alpha$ of belt passes round the pulley with velocity v ft. per sec., there is a centripetal force necessary of amount $\dfrac{wr\delta a \,.\, v^2}{gr}$ lb., where $\dfrac{w}{g}$ is the mass of unit length of the belt in

engineers' units. This force, which simplifies to $\frac{wv^2}{g}\,\delta a$, is a radial force, and should be added to the force $pr\delta\alpha$ in (VIII.2). This relation, when corrected and transposed, becomes

$$T\delta\alpha = pr\delta\alpha + \frac{wv^2}{g}\,\delta\alpha \quad . \quad . \quad \text{(VIII.10)}$$

from which
$$T = pr + \frac{wv^2}{g} \quad . \quad . \quad \text{(VIII.11)}$$

and, substituting for pr in (VIII.4),

$$\delta T = \mu\left(T - \frac{wv^2}{gr}\right)\delta\alpha \; . \quad . \quad \text{(VIII.12)}$$

or
$$\int_{T_2}^{T_1} \frac{dT}{T - \frac{wv^2}{g}} = \int_0^\theta \mu d\alpha \; . \quad . \quad . \quad \text{(VIII.13)}$$

from which
$$\frac{T_1 - \frac{wv^2}{g}}{T_2 - \frac{wv^2}{g}} = e^{\mu\theta} \; . \quad . \quad . \quad . \quad \text{(VIII.14)}$$

136. **Horse-power Transmitted by a Belt.** The horse-power is given by P = horse-power

$$= \frac{\text{force in lb.} \times \text{distance moved in ft. per min.}}{33{,}000}$$

$$= \frac{(T_1 - T_2) \times 60v}{33{,}000}$$

$$= \frac{1}{550}\,v(T_1 - T_2) \; . \quad . \quad . \quad . \quad \text{(VIII.15)}$$

To eliminate T_2, we have from (VIII.14)

$$e^{\mu\theta}\left(T_2 - \frac{wv^2}{g}\right) = T_1 - \frac{wv^2}{g}$$

$$\therefore T_2 - \frac{wv^2}{g} = e^{-\mu\theta}\left(T_1 - \frac{wv^2}{g}\right)$$

from which
$$T_2 = e^{-\mu\theta}\,T_1 + \frac{wv^2}{g}\,(1 - e^{-\mu\theta})$$

Hence,
$$T_1 - T_2 = T_1\,(1 - e^{-\mu\theta}) - \frac{wv^2}{g}\,(1 - e^{-\mu\theta})$$

$$= k\left(T_1 - \frac{wv^2}{g}\right) \text{ where } k = 1 - e^{-\mu\theta}$$

Substituting in (VIII.15)

$$P = \frac{k}{550}\left(T_1 v - \frac{wv^3}{g}\right) \quad . \quad . \quad \text{(VIII.16)}$$

If $P = 0$, the belt moves without touching the pulley surface, and

$$\frac{wv^3}{g} = T_1 v$$

or $$v = \sqrt{\frac{gT_1}{w}}$$

For maximum power

$$\frac{dP}{dv} = 0,$$

i.e. $$\frac{d}{dv}\left(T_1 v - \frac{wv^3}{g}\right) = 0$$

or $$T_1 = \frac{3wv^2}{g}$$

which gives $$v = \sqrt{\frac{gT_1}{3w}} \quad . \quad . \quad . \quad . \quad \text{(VIII.17)}$$

In this formula T_1 is the greatest permissible working tension of the belt, and v is the speed at which the maximum power is transmitted by the belt.

From (VIII.16) and (VIII.17) we have

$$P_{max} = \frac{k}{550}\left\{\sqrt{\frac{gT_1^3}{3w}} - \frac{w}{g}\sqrt{\frac{g^3T_1^3}{27w^3}}\right\}$$

$$= \frac{2}{3}\cdot\frac{k}{550}\cdot\sqrt{\frac{g}{3w}}\cdot T_1^{\frac{3}{2}} \quad . \quad \text{(VIII.18)}$$

Thus, given the safe maximum pull T_1 in a belt, the maximum power transmitted and the belt speed at which this power is transmitted are given by (VIII.18) and (VIII.17).

All the preceding formulae for belting problems are based upon the following assumptions—

(*a*) Limiting frictional resistance is proportional to the intensity of pressure between belt and pulley surfaces;

(*b*) Limiting frictional resistance is independent of speed of slip between belt and pulley surface;

(*c*) The tensile strain in a belt is proportional to the applied stress.

None of these assumptions agrees with the results of experimental determinations. Some experiments made by one of the authors (A. T. J. Kersey) in 1936, with specially designed apparatus, show that so far as flat surfaces are concerned the coefficient of friction between leather and cast iron, or steel, decreases as the load increases, and increases as the speed of slipping increases (load constant). It is suggested that the total frictional resistance consists of an adhesive force which is independent of the load and proportional to area of contact, and tends to increase with the speed of slipping, combined with an additional frictional resistance which is proportional to load, so that (for a small speed of slipping)

$$F = a \,.\, A + b \,.\, W$$

where a and b are constants and A is the area of contact.

The author was unable to continue the experiments in order to verify the above law.

If the above equation is assumed to be true, it leads to the following expression—

$$T_1 - T_2 = \frac{2(e^{b\theta} - 1)}{e^{b.\theta} + 1}\left(\frac{a \,.\, r}{b} + T_0\right) \quad . \quad . \quad . \text{(VIII.18A)}$$

where r is the pulley radius in inches, a is a constant whose dimensions are pounds per square inch, b is an absolute constant, $T_1 - T_2$ is in pounds per inch of width, and T_0 is the initial tension in pounds per inch of width.

Dr. Swift's experiments (*Proc. I. Mech. E.*, November, 1928) show that the coefficient of friction, calculated from the ordinary theory, increases considerably as the speed of slipping between belt and pulley increases, but laws connecting the true friction coefficient with speed of slipping still remain to be investigated.

With a short drive the angle of lapping on the smaller pulley is correspondingly small. If both pulleys are to be equally effective, the value of $\mu\theta$ should be the same for each, and the value of μ can be increased by facing the smaller pulley with cork or leather. The following table gives a selection from results of tests made by the author in collaboration with Dr. F. M. Schwabe in 1936. The increase in torque transmitted with a large percentage of slip and the effect of facing the smaller pulley with a leather ring are strikingly shown.

Material of Belt	Estimated initial tension (lb./in. width)	WITH SLIP CORRESPONDING TO NORMAL WORKING CONDITIONS					MAXIMUM VALUES					
			Without leather face		With leather face		Without leather face			With leather face		
		Slip %	Torque (lb.-ft.)	H.P.	Torque (lb.-ft.)	H.P.	Slip %	Torque (lb.-ft.)	H.P.	Slip %	Torque (lb.-ft.)	H.P.
Balata .	30	1	21·0	1·09	48·8	2·52	18·5	41·2	1·6	6·83	54·6	2·64
Rubber .	45	1	31·9	1·5	52	2·56	8·3	41·4	1·8	3·8	72·6	3·56
Hair . .	45	1	36·5	1·8	79	3·94	9·2	62·2	2·64	3·7	78·2	3·86
Leather .	50	1·5	33	1·65	62·5	3·30	12·1	70·7	2·93	6·68	89·4	4·19

The pulley diameters were 8 in. ($8\frac{5}{8}$ in. with leather ring) and $29\frac{3}{4}$ in. respectively, the centre distance was 10 ft. 6 in., the angle of the drive was about 40°, and the speed of driving pulley about 960 r.p.m.

Fig. 146 shows on the left graphs of one set of results with a leather belt.

Tests made in 1909 by the same author on three kinds of leather belting, using an extensometer with special grips, show that up to the usual maximum working stress (about 600 lb. per square inch) the rate of extension decreases with the load, the ratio of the unit strain at 50 lb. per square inch to that at 600 lb. per square inch being 2·5, a very substantial difference.

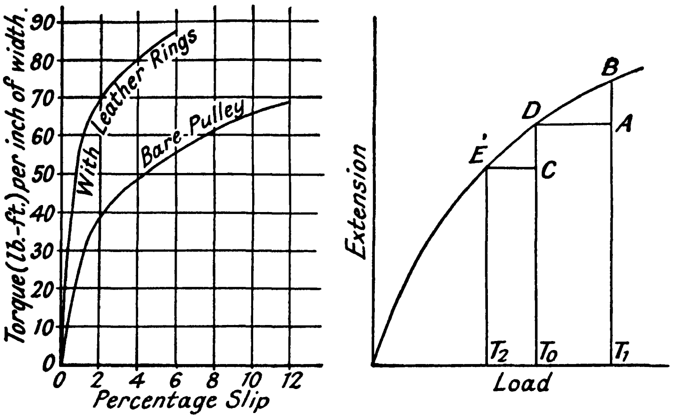

FIG. 146

Fig. 146 shows on the right the form of the load-extension curve for a leather belt. T_0 is the initial tension, T_1 and T_2 are the working tensions.

Increase of length from T_0 to $T_1 = AB$

Decrease of length from T_0 to $T_2 = CD$

If the total length remains constant, then $AB = CD$, and it will be evident that T_0 is less than $\frac{T_1 + T_2}{2}$.

In the case of a horizontal drive with a long belt, the sag on the "slack side" will be greater than that on the "tight" side, and this will accentuate the difference. Since, however, the tension in a belt is rarely measured when the belt is fitted, or checked after adjustment, and most belts are overtensioned, the simple assumption, (*c*) page 221, is probably sufficient for most practical cases.

137. **Lengths of Belts.** Suppose the two pulleys in Fig. 144 have radii r_1 and r_2 ft. respectively, and have their centres at a distance d apart. It is easy to prove, and we shall leave the proofs as exercises for our readers, that if l_1 is the length of the belt when crossed, and l_2 is the length when open,

$$l_1 = 2\left\{(r_1 + r_2)\left(\frac{\pi}{2} + a\right) + \sqrt{d^2 - (r_1 + r_2)^2}\right\} \quad . \quad \text{(VIII.19)}$$

where $\alpha = \sin^{-1}\dfrac{r_1 + r_2}{d}$ radians

and $l_2 = 2\sqrt{d^2 - (r_1 - r_2)^2} + \pi(r_1 + r_2) + 2(r_1 - r_2)\beta$ (VIII.20)

where $\beta = \sin^{-1}\dfrac{r_1 - r_2}{d}$ radians

In the expression for l_1, r_1 and r_2 only occur in the form of a sum, hence, if r_1 and r_2 are varied so that their sum is kept constant, the length of the belt will remain constant. This is a useful property in connection with speed cones, which are used for varying the velocity ratio between a pair of parallel shafts. Speed cones consist of two equal truncated cones mounted on parallel shafts, with the large end of each cone opposite to the small end of the other. The cones are connected by a crossed belt. By moving the belt in a direction parallel to the shafts, the velocity ratio may be varied, and the belt is equally tight in all positions. The expression for the length of an open belt may be simplified if $r_1 - r_2$ is small compared with d_1, for in such a case

$$\begin{aligned}\sqrt{d^2 - (r_1 - r_2)^2} &= d\left\{1 - \left(\frac{r_1 - r_2}{d}\right)^2\right\}^{\frac{1}{2}} \\ &= d\left(1 - \frac{(r_1 - r_2)^2}{2d^2}\right) \text{ approx.} \\ &= d - \frac{(r_1 - r_2)^2}{2d}\end{aligned}$$

and $\beta = \sin^{-1}\dfrac{r_1 - r_2}{d} = \dfrac{r_1 - r_2}{d}$ approx.

Substituting these in (VIII.20)

$$l_2 = 2d - \frac{(r_1 - r_2)^2}{d} + \pi (r_1 + r_2) + \frac{2(r_1 - r_2)^2}{d}$$

$$\therefore l_2 = 2d + \frac{(r_1 - r_2)^2}{d} + \pi(r_1 + r_2) \quad . \quad . \quad \text{(VIII.21)}$$

From this last relation it is clear that any variation in which $r_1 + r_2$ is kept constant will vary the length of the belt because of the term containing $(r_1 - r_2)$. If, then, speed cones are connected by an open belt, the belt will be slacker in some positions than in others.

138. **Relation Between the Speed of the Driven Shaft and the Lateral Displacement of the Belt.** If $\frac{r_1 - r_2}{d}$ is small, i.e. if the two straight portions of the belt are nearly parallel, the variation in tightness is negligible and, in practice, speed cones are often connected by open belts. Fig. 147 shows on the left a diagram of a pair of speed cones. r and R are respectively the least and greatest radii, l is the length of the cones, and y and z are the radii on which the belt is running when it has travelled a distance x from its extreme left-hand position. Then, if the upper cone is the driver, and it makes n revs. per min., where n is constant, we have

$$N = \text{revs. per min. of lower cone}$$

$$= \frac{y}{z} \times n \quad . \quad . \quad . \quad \text{(VIII.22)}$$

By simple geometry, since the profile is straight,

$$y = r + \frac{x}{l}(R - r)$$

or

$$y = \frac{(l - x)r + xR}{l} \quad . \quad . \quad \text{(VIII.23)}$$

Similarly,

$$z = \frac{xr + (l - x)R}{l} \quad . \quad . \quad \text{(VIII.24)}$$

Substituting in (VIII.22),

$$N = \frac{(l - x)r + xR}{xr + (l - x)R} \cdot n$$

$$= \frac{lr + x(R - r)}{lR - x(R - r)} \cdot n$$

$$= n\left(\frac{l(R + r)}{lR - x(R - r)} - 1\right) \quad . \quad \text{(VIII.25)}$$

Thus, N is not a linear function of x, and equal increases in the value of x do not produce equal changes in the speed of the driven shaft.

139. **Speed Cones with which equal Lateral Displacements of the Belt produce equal Changes in the Speed of the Driven Shaft.** Fig. 147 shows on the right a pair of modified cones designed to produce a change of speed which is proportional to the lateral displacement of the belt. In this case, we have from (VIII.22)

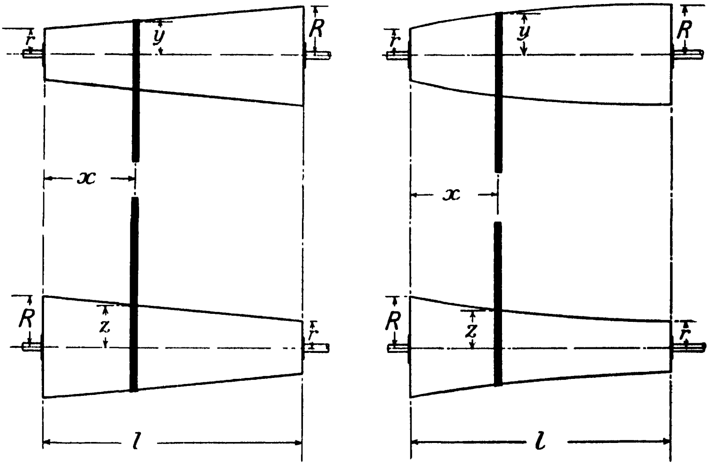

FIG. 147. CONE DRUMS CURVED SPEED "CONES"

$$N = \frac{y}{z} \times n$$

We shall assume that with an open belt, as with a crossed belt, the condition for equal tightness in all positions is

$$z + y = \text{constant}$$

i.e. $$z + y = R + r \qquad . \quad . \quad . \quad \text{(VIII.26)}$$

Substituting for y

$$z + \frac{N}{n} z = R + r$$

or $$z = \frac{R + r}{1 + \dfrac{N}{n}} \qquad . \quad . \quad . \quad \text{(VIII.27)}$$

As, by the conditions of our problem, equal increments in

x must produce equal increments in $\frac{N}{n}$, the latter must be a linear function of the former. That is,

$$\frac{N}{n} = \alpha + \beta\, x, \text{ where } \alpha \text{ and } \beta \text{ are constants}$$

Since, when $x = 0, \frac{N}{n} = \frac{r}{R}$, we have $\alpha = \frac{r}{R}$

also when $x = l, \frac{N}{n} = \frac{R}{r}$, therefore $\frac{R}{r} = \frac{r}{R} + \beta l$

or
$$\beta = \frac{R^2 - r^2}{rRl}$$

and, substituting for α and β

$$\frac{N}{n} = \frac{r}{R} + \frac{R^2 - r^2}{rR} \cdot \frac{x}{l}$$

Again, substituting this value of $\frac{N}{n}$ in (VIII.27), we have

$$z = \frac{R + r}{1 + \frac{r}{R} + \frac{R^2 - r^2}{rR} \cdot \frac{x}{l}}$$

or
$$z = \frac{R + r}{\frac{R + r}{R} + \frac{R^2 - r^2}{rR} \cdot \frac{x}{l}}$$

i.e.
$$z = \frac{R}{1 + \frac{R - r}{r} \cdot \frac{x}{l}} \quad . \quad . \quad . \quad \text{(VIII.28)}$$

This is the equation to a rectangular hyperbola with the axis of the shaft as one axis. Hence, the surface of the lower pulley is the surface of a hyperboloid of revolution. The profile of the upper pulley will fit that of the lower pulley because of the relation (VIII.26). Thus, the upper pulley has a convex profile and the lower one a concave profile. In spite of their curved profiles, the pulleys are called "cone drums," or "speed cones."

140. **Speed Cones in which equal Lateral Displacements of the Belt produce equal Changes in the Reciprocal of the Speed of the Driven Shaft.** In certain textile machines in which yarn or thread is wound on a spool, or bobbin, it is necessary to keep the surface speed of the bobbin constant whilst the diameter

is increasing owing to the winding on of the thread. "Cone" drums are used, and the belt is moved a fixed distance every time a layer of thread is wound on to the bobbin. Thus, for equal lateral displacements of the belt there are equal increases in the diameter of the spool. As the product of the diameter and the angular speed is required to be constant, the cones must be designed so that equal displacements of the belt produce equal increments in the reciprocal of the speed. We shall show that the speed cones on the right of Fig. 147 satisfy this condition if the lower cone is the driver and runs at a constant speed. Using the same symbols as before,

$$\begin{aligned} n &= \text{revs. per min. of upper cone} \\ &= \frac{z}{y} \times N \quad . \quad . \quad . \quad . \quad \text{(VIII.29)} \end{aligned}$$

where N is a constant. The value of z is given by (VIII.28).

Substituting this value in (VIII.26), we have

$$y = R + r - \frac{R}{1 + \frac{R - r}{r} \cdot \frac{x}{l}}$$

or

$$y = \frac{r + \frac{R^2 - r^2}{r} \frac{x}{l}}{1 + \frac{R - r}{r} \cdot \frac{x}{l}} \quad . \quad . \quad . \quad \text{(VIII.30)}$$

and substituting the values of z and y in (VIII.29),

$$n = \frac{\dfrac{R}{1 + \frac{R - r}{R} \cdot \frac{x}{l}}}{\dfrac{r + \frac{R^2 - r^2}{r} \cdot \frac{x}{l}}{1 + \frac{R - r}{R} \cdot \frac{x}{l}}} \times N$$

$$\therefore n = \frac{R}{r\left(1 + \frac{R^2 - r^2}{r^2} \cdot \frac{x}{l}\right)} N \quad . \quad \text{(VIII.31)}$$

The reciprocal of n is therefore a linear function of x, and the cone drums give the required motion.*

---

* (VIII.31) may be obtained by interchanging the values of N and n in Art. 139.

We saw in Art. 137 that, if open belts are used with cone drums designed so that $R + r$ is constant, the belt is not equally tight in all positions. Where $\frac{R}{r}$ is not greater than two, and the distance between the shafts is not less than five or six times R, the variation in tightness is usually negligible. Where it is desirable, a correction may be made to the values of y and z, so as to reduce or eliminate entirely the variation in tightness. In any modification the ratio $\frac{y}{z}$ must remain constant. Thus, having given the length of the belt and the ratio of the radii of the cones, the values of y and z may be found from (VIII.21) after writing y and z for r_1 and r_2 respectively.

In cases where a continuous variation is not essential, a variation in steps is made by having a series of cylindrical pulleys on each shaft, the diameter of the middle section of each pulley being that which the cone would have in the same plane section. This arrangement is used in such machines as lathes and drilling machines for varying the speed of the cutting tool.

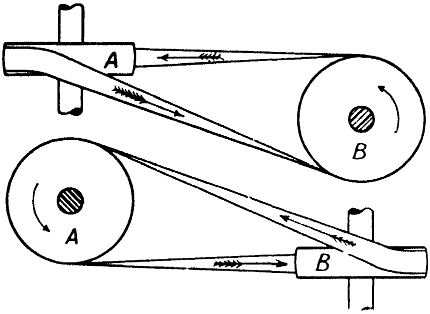

FIG. 148. BELT CONNECTING PERPENDICULAR SHAFTS

141. Belts Connecting Non-parallel Shafts. Non-parallel shafts may be divided into two classes, (*a*) those which intersect, and (*b*) those which do not intersect. The former class contains only cases in which the axes of the shafts lie in one plane, and the latter class contains only cases of shafts whose axes do not lie in the same plane. Two shafts in class (*a*) can only carry pulleys connected by a belt if guide pulleys, called jockey pulleys, are used, as it is clearly impossible to drive one pulley directly from the other. Any two shafts in class (*b*) may have their pulleys directly connected by a belt if the pulleys are properly arranged, and the shafts are not too close together. If shafts which are close together have to be connected by a belt drive, each portion of the belt, as it passes from one pulley to the other, is taken round part of the periphery of a jockey pulley, which is suitably placed.

In all cases of belt drives, the pulleys must be arranged so that the following condition is satisfied. *The point at which a belt leaves a pulley must be in the plane of the pulley to which it is moving.* As belts and pulleys have width, the point referred to is the middle point of the working surface of the belt, and the plane is the central plane of the pulley. If a belt were very elastic and the pulleys long cylinders, the belt would automatically take up when running the correct working position, in accordance with the above rule. If, however, the sense of rotation were reversed, the belt would move laterally with reference to one or both of the cylinders, so as to fulfil the above condition for the reversed rotation. As pulleys are usually only of slightly greater width than the belt, a reversal of the direction of rotation would, in general, cause the belt to run off the pulley. On this account, such drives can only be arranged to permit of motion in one sense.

Fig. 148 shows two pulleys, *A* and *B*, whose shafts are perpendicular. The arrows show the direction of motion of the belt. The middle plane of the pulley *A* is tangential to the cylindrical surface of the pulley *B*, and the middle plane of *B* is tangential to the surface of *A*.

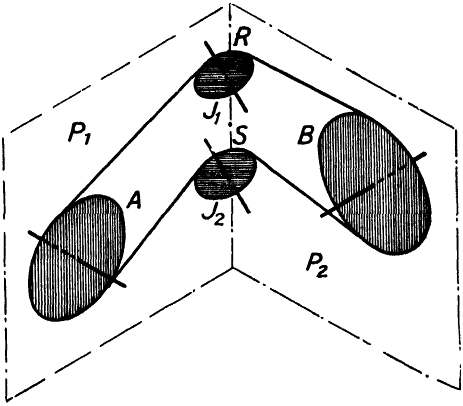

FIG. 149. USE OF JOCKEY PULLEYS

Fig. 149 shows the use of jockey pulleys. *A* and *B* are the centre planes of two pulleys lying respectively in planes P_1, P_2, and which are to be connected by an endless belt so as to be capable of rotation in either sense. The centre line of the portions of belt approaching or leaving either *A* or *B*, must lie in the plane of the pulley. Let *R* and *S* be any two points on the line of intersection of P_1 and P_2, and draw tangents from *R* to *A* and *B* respectively, and also tangents from *S* to *A* and *B* respectively. The jockey pulley J_1 should have its central plane in the plane of the tangents from R, and the central plane of the jockey pulley J_2 should contain the tangents from S. The condition for correct working will then be satisfied for either sense of rotation of A or B.

142. **Crowned Pulleys.** If a belt is required to run without any constraint other than that provided by the surfaces of the pulleys, cylindrical surfaces are not suitable. With such surfaces any lack of parallelism of the shafts or any small disturbances, such as those produced by swaying of the belt, would tend to make the belt move laterally and thus to slip off the edges of the pulleys. To prevent this, the edges of the pulleys are sometimes flanged. Another method of preventing lateral travel is to make the pulley surfaces convex, the diameter of the central section being greater than that of any other section.

With such pulleys, any slight displacement of the belt from its central position causes a reduction in the tension in the portion of the belt which is farthest removed from the central plane. The effect of the unequal tensions is to cause the belt to take a position on the surface of the pulley in which the portion of the belt approaching the pulley is nearer to its normal position. Thus the belt gradually moves back to its normal position. These pulleys are known as *crowned pulleys*, and the difference between the end diameters and the central diameter is usually from $\frac{1}{8}$ in. to $\frac{1}{4}$ in. per foot of width.

143. **Coil Friction due to a Rope on the Surface of a Cylinder.** The relation (VIII.5) holds in this case when slipping is about to occur. By coiling a rope a few times round a rotating cylinder or capstan, a large value of $\frac{T_1}{T_2}$ is produced, and a small force T_2 applied to the rope leaving the capstan will overcome a large resistance T_1 at the end of the rope approaching the capstan. If n is the number of coils of rope on the capstan,

$$\theta = 2\pi n \text{ radians}$$

and
$$\frac{T_1}{T_2} = e^{\mu(2\pi n)}$$

If $\mu = \cdot 25$,
$$\frac{T_1}{T_2} = e^{1\cdot 571 n}$$
$$= (2\cdot 718^{1\cdot 571})^n$$
$$= 4\cdot 811^n$$

The following table gives values of $\frac{T_1}{T_2}$ for different values of n, and shows how rapidly the ratio increases as n increases.

n = number of coils	0	1	2	3	4	5	6
$\frac{T_1}{T_2}$	1	4·811	23·14	111·4	535·6	2,576	12,400

These results are only true for a cylinder. The ordinary capstan is not a cylinder, having a concave profile, and the pressure of the capstan surface on the rope is not everywhere in the plane of the rope.

144. **Pulleys or Wheels Connected by Ropes.** Ropes of cotton, manilla, or hemp, are used for transmitting power, cotton ropes being usually considered the best on account of their greater driving power and flexibility. The ropes used are endless, being joined by a splice, and fit into circumferential grooves on the pulleys which they connect. There may be a single pair of grooves, one in each pulley, or there may be a number of pairs. By using a small jockey pulley to guide the rope on its return from the last groove of one pulley to the first groove of the other, a single endless rope may be used instead of a number of ropes as above. The single rope system is largely used in America, and has the advantages of only one splice in the rope, more uniform wear on the different parts of the rope as they all pass in succession round all the grooves, and the facility of tightening the rope by only one resplicing. The multiple rope system has the advantages (1) that a break in one rope does not destroy the drive, the repair being carried out at some time when the machinery is normally stopped, and (2) that tightening the ropes is not so frequently necessary.

In passing round a pulley the outer portion of the rope becomes more stressed than the portion which is nearer the shaft, and with large ropes a large proportion of the tension will fall on the outer portion. On this account, in transmitting large powers, it is better to use a number of ropes of small diameter rather than a single rope of large diameter. When the ropes of the drive are horizontal or nearly so, and the distance between the pulleys is fairly large, the weight of the ropes tends to wedge them in the grooves and so no initial tension is necessary. The upper side should be the slack side, so that any sag of the ropes increases the angle of lap. When the drive is vertical, the weight tends to reduce the pressure of the ropes in the grooves of the lower pulley, and initial tension is needed to increase this pressure. With initial tension, the life of the rope is very much shortened. The grooves on the pulleys are V-shaped (Fig. 150), the angle between the two faces being from 40° to 50°. The rope rests on the two sides as shown, and not on the bottom of the groove. In guide pulleys, the grooves are semi-circular, and the rope rests on the bottom of the groove.

145. **Ratio of the Tensions on the Two Sides of a Rope passing round a Grooved Pulley when just on the point of Slipping.** Consider the piece of rope, shown in section in Fig. 150, and of length δs along the circumference. The forces acting on it in the plane of the paper are—

$p\delta s$ along the radius, p being the radial force per unit of length, produced by the tensions T and $T + \delta T$ of Fig. 145.

$R\delta s$ perpendicular to each of the sides of the groove.

Let 2α be the angle between the sides of the groove. Then, neglecting the frictional forces in the plane of the paper along the faces of the groove we have

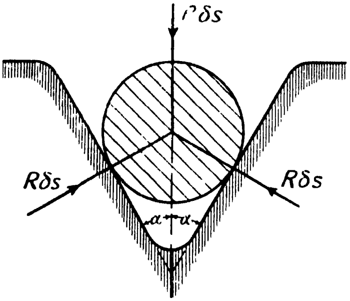

FIG. 150. ROPE IN GROOVE

$$p\delta s = 2R\delta s \, . \, \sin \alpha$$

$$\therefore 2R = \frac{p}{\sin \alpha} \qquad \text{(VIII.32)}$$

Thus, the total normal pressure per unit length between the rope and the faces of the groove is $2R$ or $\frac{p}{\sin \alpha}$. With this exception, the conditions are exactly the same as in the case of Art. 134. The friction per unit length is $\frac{\mu p}{\sin \alpha}$. Thus, if we substitute $\frac{\mu p}{\sin \alpha}$ for μp, or $\frac{\mu}{\sin \alpha}$ for μ in the equations of Art. 134 we obtain the correct solution for our present problem. (VIII.5) becomes

$$\frac{T_1}{T_2} = e^{\frac{\mu}{\sin \alpha}\theta} \quad . \quad . \quad . \quad \text{(VIII.33)}$$

or
$$\frac{T_1}{T_2} = e^{\mu\theta \operatorname{cosec} \alpha} \quad . \quad . \quad . \quad \text{(VIII.34)}$$

If
$$\begin{aligned} 2\alpha &= 40°, \\ \operatorname{cosec} \alpha &= \operatorname{cosec} 20° \\ &= 2{\cdot}924, \text{ and putting } \mu = {\cdot}25, \text{ and } \theta = 2\pi \\ \frac{T_1}{T_2} &= 2{\cdot}718^{4{\cdot}593} \\ &= 98{\cdot}8 \end{aligned}$$

Thus, the ratio of the tensions for a complete lap is $\frac{98{\cdot}8}{4{\cdot}81} = 20{\cdot}5$ times that required to produce slip between a rope and a cylindrical pulley (Art. 143).

When centrifugal force is taken into account, the results of Arts. 135 and 136 apply if μ cosec α be written for μ.

146. **Transmission by Steel Ropes.** Steel or wire ropes are used for transmission of power in cases where the parts to be connected are at a large distance apart, and where extra strength is needed. They are used in lifts, colliery winding and hauling arrangements, mill drives where a central engine drives shafts at a distance from the engine, etc. A steel rope is unsuitable for use with pulleys having V-shaped grooves, as the wedging acting between the rope and the pulley would cause serious injury to the rope. A steel rope should not touch the side of the groove, and should make contact only with a leather or wooden lining placed at the bottom of the groove. This lining has a fairly high coefficient of friction with the steel rope, the coefficient being about the same as that between the surfaces of a pulley and a leather belt.

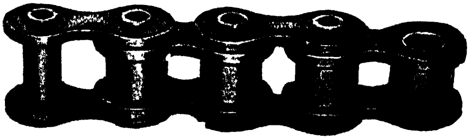

FIG. 151

147. **Steel Chains.** Steel chains (Figs. 151 and 152) are also used for the transmission of power. These chains are used in connection with chain-wheels or sprocket-wheels. Projections on the wheel fit into cavities in the chain, or vice versa, and the two are thus constrained to move together without slipping. The two principal forms of chain are shown; that in Fig. 151 is a roller chain, or *Renold's bush roller chain*, and that in Fig. 152 is the *Renold's inverted tooth chain.*

Bush Roller Chain. Each of the cylinders or rollers seen between the two sides of the chain in Fig. 151 is free to rotate on an inner cylinder or bush, the end of one of which may be seen showing through the side link on the right of the figure.

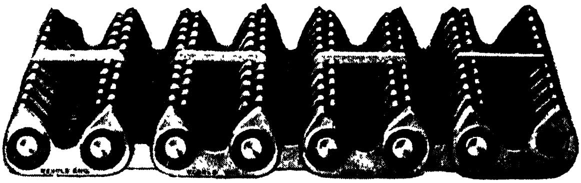

FIG. 152

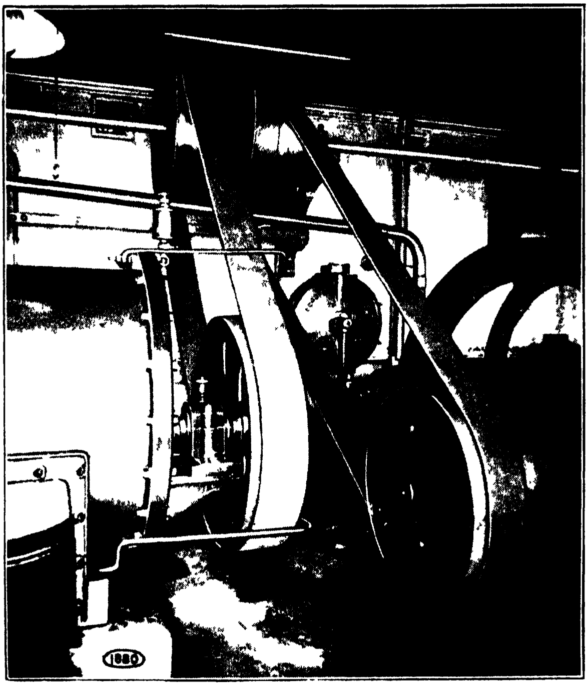

FIG. 153

The Bush Roller Chain is of extremely robust and simple construction, and is capable of giving good service under rather severe conditions. With this chain there is a little noise, or chatter, caused by the impact of the rollers on the wheel teeth, this being more pronounced when there is scanty lubrication. When one of these chains elongates slightly, owing to wear and stretching of the parts, the extended chain is then of greater pitch than are the teeth of the sprocket wheel. The rollers then fit unequally into the cavities of the wheel, the result being that the total load falls on one tooth, or on a few teeth, instead of being distributed over the whole of the teeth on the arc of contact. Stretching also leads to increased wear of the surfaces of the rollers and of the wheel teeth.

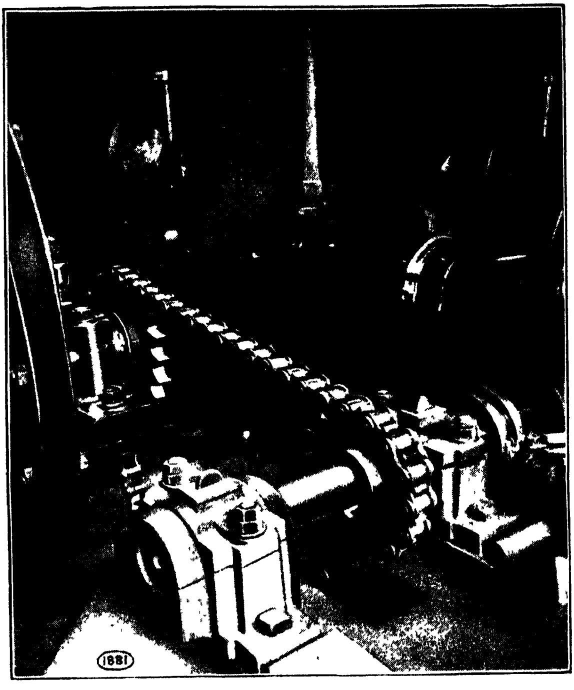

FIG. 154

The Silent Chain or Inverted Tooth Chain (Fig. 152) is designed to eliminate the evil effects caused by stretching, and also to produce quietness of running. As the chain stretches and the pitch of the chain increases, the links ride on the teeth of the sprocket wheel at a slightly increased radius, thus automatically correcting the small change in the pitch. There is no relative sliding between the teeth of the inverted tooth chain and the wheel teeth, and if properly lubricated this chain gives durable service and runs very smoothly and quietly.

Chain-gearing produces a positive drive, and may be used where an exact average velocity ratio is essential. A chain drive takes up less space than a belt drive. For belt or rope drives a certain minimum distance between the shafts is necessary, unless jockey pulleys are to be used. This prevents the use of belts or ropes for connecting directly two shafts which are close together. Chain drives may be used when the distance is much less than the minimum distance for a belt drive.

By the use of the highest grades of steel, and by accurate methods of production, manufacturers can now produce chains which give a transmission efficiency of 99 per cent when new, and 97 per cent when nearing the end of their useful lives. Chains were originally used only where greater strength was required, or where a positive drive was needed. They are now in general use for the transmission of power in cycles, motor vehicles, agricultural machinery, road rollers, etc., and for gearing in workshops and factories, and are continually being installed to displace belt or rope gearing and wheel gearing.

Fig. 153 and Fig. 154 show a gas exhauster before and after conversion from belt driving to chain driving. A Renold bush roller chain of 2 in. pitch replaced the two belt drives. The chain drive has the undoubted advantage of (1) economy of space, (2) absence of slip, and (3) the elimination of complex gearing.

EXAMPLES 8.

1. A pulley, 12 in. diameter, drives a pulley 7 in. diameter; on the same shaft as the latter is a third pulley, 15 in. diameter, driving a pulley of 4½ in. diameter, the second and third pulleys are both attached to the shaft. Find the ratio of the speeds of the first shaft to that of the last (1) assuming no slip, (2) assuming 6 per cent slip.

2. Prove that the ratio between the tensions on the tight and slack sides of a belt transmitting power between two pulleys is $e^{\mu\theta}$ where $\mu =$ coefficient of friction between belt and pulley, and $\theta =$ minimum arc of contact between belt and pulley. Explain clearly why the belt is re-tightened when it has stretched unduly after service. (I.C.E.)

3. Find the number of times a hauling rope must be wound round a rotating capstan in order to haul 10 trucks, each weighing 30 tons. Rolling resistance, 12 lb. per ton. Pull on free end of rope 20 lb.: coefficient of friction, 0·4. (I.C.E.)

4. Find the length of (1) a crossed belt, (2) an open belt, to connect two

pulleys 4 ft. diameter and 2 ft. diameter respectively, their axes being parallel and 6 ft. apart.

5. Write a note on the subject of flexibility in belt drives and discuss whether the tight side should be the upper or lower side of a horizontal drive.

6. Explain the effect of " centrifugal " force on the pressure between a belt and a pulley. If the weight of the belt is ¼ lb. per foot and the radius of the pulley is 3 ft., find the speed at which the pressure between the belt and the pulley will be entirely relieved (neglect gravity). Initial tension = 150 lb.

7. Find the ratio of the tensions on the two sides of a rope pulley when the rope is on the point of slipping, in terms of the wrap θ, the angle of the groove 2φ, and of μ. If this ratio is 6, the weight of the rope per foot being w, and the maximum permissive tension T_1, show that the maximum rate at which power can be transmitted is

$$\frac{5}{9}T^{\frac{3}{2}}\sqrt{\frac{g}{3w}}\text{ ft.-lb. per sec.} \qquad \text{(U.L.A.)}$$

8. The accompanying sketch (Fig. 155), shows a type of hand-brake. If the coefficient between brake and drum is 0·15, and if the angle of wrap is 270°, determine the ratio of x to y for the brake not to be self-locking. If x is zero and if y is 3 in., determine the magnitude of the force P, acting at a leverage of 3 ft. 6 in., necessary to hold a load of 1 ton, acting at a leverage of 15 in. The diameter of the brake drum is 36 in. (U.L.)

9. A flexible link belt 4 in. wide, weighing 1 lb. per ft. length, and having a coefficient of friction 0·32, is employed to drive a pulley 6 in. diameter, embracing an arc of contact of 180° on the pulley. Determine the torque which can be transmitted by the pulley (a) at starting, and (b) at a speed of 2,000 revs. per min., without the tension in the belt exceeding 300 lb. (U.L.A.)

10. A motor of 20 h.p. runs at 900 revs. per min., and drives a countershaft supported in bearings 6 ft apart at a speed of 250 revs. per min. From the countershaft a belt drives a machine which absorbs the full power of the motor when running at 80 revs. per min. Assuming a tension in the belt of 80 lb. per in. width of belt, determine suitable diameters for all the pulleys and the widths of the two belts, assuming the smallest pulley is 15 in. diameter and the largest 6 ft. diameter. Coefficient of friction μ = 0·32. Find also the lengths of the open belts. (U.L.A.)

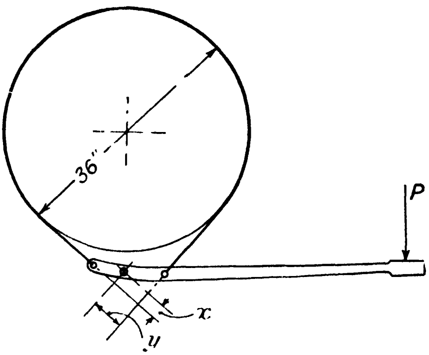

Fig. 155

11. Determine the width of belt necessary to transmit 7½ h.p. under the following conditions—

Revs. per min. of pulley = 150
Diameter of pulley = 18 in.
Angle of contact of belt = 160°
Coefficient of friction = 0·3
Maximum tension per inch of width = 80 lb. (I.C.E.)

12. Determine the length of a crossed belt to connect the pulleys on two shafts which are 18 ft. apart, when the diameters of the pulleys are 4½ ft. and 2½ ft. respectively. Prove any formula you use. (I.C.E.)

13. Find the shapes of the contours of a pair of speed cones, the largest and smallest diameters of each being 7 in., and 3½ in. respectively, and the length 30 in. The speed cones are to be such that while one runs at a constant speed the other must give (*a*) equal increments of speed, and (*b*) equal increments of the reciprocal of the speed, for every inch of travel of the belt.

14. Find the length of an open belt to connect the cones of example 13, when in an end position, and find how much it will be too tight or too slack when in its middle position. Axes of cones 40 in. apart.

15. What should be the diameters of the cones of example 13 at their middle section if the belt is to be equally tight there and in its end positions ?

16. State the general rules relating to the positions of pulleys connected by a single endless belt when the pulleys are not on parallel shafts. Explain the use and arrangement of jockey pulleys.

17. Explain why a rope fitting in a V-shaped groove on a pulley rim does not slip as easily as a belt on a cylindrical pulley. Prove the formula

$$\frac{T_1}{T_2} = e^{\mu\theta \operatorname{cosec} \alpha}$$

where 2α is the angle of the groove.

18. Write a short note on the transmission of power by steel ropes.

19. Describe any forms of chain which are used for the transmission of power. Describe their advantages or disadvantages as compared with belt drives.

20. Does a chain drive give a uniform velocity ratio between two shafts ? If not, explain how the variation arises. State the possible causes of loss of efficiency in a chain drive.

21. 12 h.p. is to be transmitted by a belt weighing 1½ lb. per sq. ft. running at a speed of 40 ft. per sec. The angle of lapping on the smaller pulley is 160°, and $\mu = 0{\cdot}27$. If the pull in the belt is not to exceed 85 lb. per in. of width, find (*a*) the necessary width of the belt, (*b*) the necessary initial tension.

22. 450 h.p. is to be transmitted by a rope pulley 14 ft. in diameter running at 80 revs. per min. There are 15 ropes and the angle of each groove is 45°. $\mu = 0{\cdot}28$, the working tension in each rope is to be $15C^2$, and the weight per foot is $0{\cdot}03C^2$, where C is the circumference of the rope. Estimate approximately the necessary diameter of each rope and the initial tension required. The smaller angle of lapping is 165°.

23. A length l ft. of rope hangs freely between two points in a horizontal line, The weight of 1 ft. length of the rope is W lb. Show that if the sag is small compared with the length, the relation between the tension T lb. at each end, and the sag S ft. at the centre, is given approximately by $T = \frac{ul^2}{8S}$

(*Note.* This result leads to the conclusion that the ratio of the sags on the tight and slack sides of a horizontal rope drive is approximately the reciprocal of the ratio of the tensions.)

24. In a rope drive the angle of lapping is 140°, angle of groove 40°, $\mu = 0{\cdot}16$, weight per foot of rope = 0·6 lb., maximum pull in each rope = 750 lb., speed 90 ft. per sec.

(*a*) Find the horse-power that can be transmitted per rope without slipping taking place.

(*b*) If the angle of lapping is increased 50 per cent, what will be the percentage increase in power transmitted, other conditions remaining the same ?

(*c*) If it is desired to increase the h.p. transmitted by 25 per cent, how much must the angle of lapping be increased, the other conditions being unaltered?

25. The following particulars are given of an overhead pulley for a colliery gear: Diameter of pulley = 20 ft.; weight of pulley = 10 tons; radius of gyration = 9·5 ft.; steady tension at loaded end (load + rope) = 38,000 lb.; angle of lapping = 125°. If the rope is suddenly accelerated by 10 ft. per sec. per sec., find the minimum value of the coefficient of friction between rope and pulley if slipping is not to occur, (*a*) load moving upward, (*b*) load moving downward.

26. Assuming that the initial tension T_0 in a belt is the mean of the running tensions, show that, in order to transmit P horse-power at v ft. per sec., the necessary initial tension is given by

$$T_0 = \frac{w \cdot v^2}{g} + \frac{275P}{v} \cdot \frac{r+1}{r-1} \quad \text{where } r = e^{\mu \cdot \theta} \cdot$$

Hence show that the speed for which T_0 is a minimum is given by

$$v = 16{\cdot}4 \sqrt[3]{\frac{r+1}{r-1} \cdot \frac{P}{w}}$$

27. A tug weighing 250 tons is to tow a ship weighing 1250 tons, the pull in the tow-rope being 2·4 tons when a steady speed is maintained. The rope is coiled round a bollard for 6 complete turns, the pull at the free end being 40 lb. and the coefficient of friction being 0·15. At the instant of tightening of the rope the speed of the tug is 5 ft. per sec. and the ship is stationary. Find (*a*) the time during which slipping of the rope occurs, (*b*) the common velocity at the end of this time, (*c*) the distance through which the rope slips during this period. The propulsive force exerted by the tug may be assumed to be constant.

CHAPTER IX

HIGHER PAIRING (*continued*)—WHEEL GEARING

148. **Train of Wheels.** When two shafts are to be connected so as to have a definite and invariable velocity ratio, trains of gear wheels are usually employed. The nature of the train used depends upon the velocity ratio required and the relative positions of the axes of the shafts.

Where the axes of the shafts are parallel and the distance *d* between them is small, the arrangements shown in Figs. 156, 157 and 158 may be used. The arrangement of Fig. 156 is used when the velocity ratio is not large and the shafts revolve in opposite directions; that of Fig. 157 is used when the velocity ratio is not large and the shafts are required to revolve in the same direction; the compound train (Fig. 158) is used when the velocity ratio is so large that one of the wheels of the simple train (Fig. 156) would be impracticably small. In certain cases an internal gear, such as shown in Fig. 159, might be used.

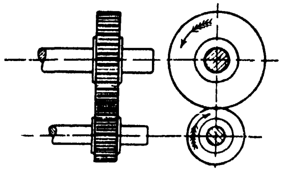

FIG. 156

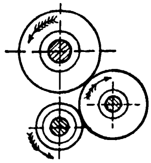

FIG. 157

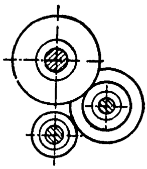

FIG. 158

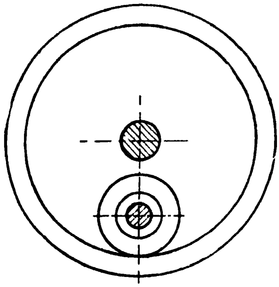

FIG. 159

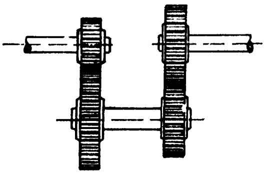

FIG. 160

Where the axes of the shafts are in the same straight line, the arrangement shown in Fig. 160 may be used, or, in cases where a high velocity ratio is required, an epicyclic train (Art. 151) may be employed.

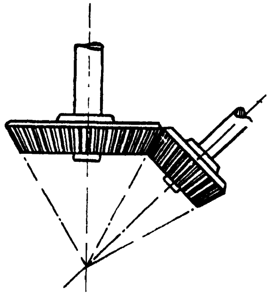

FIG. 161. BEVEL WHEELS

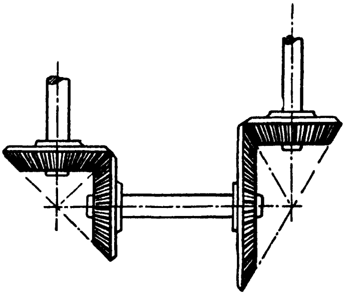

FIG. 162. BEVEL WHEEL TRAIN FOR PARALLEL SHAFTS

If the axes are in the same plane, but are inclined to one another, bevel wheels may be used, as shown in Fig. 161.

Fig. 162 shows an application of bevel wheels to the case of two parallel shafts where the distance between the centres is too large for the application of a train such as is shown in Fig. 156. A chain drive may be used in this case as an alternative, and is frequently necessary where it is not practicable to arrange for bearings for an intermediate shaft.

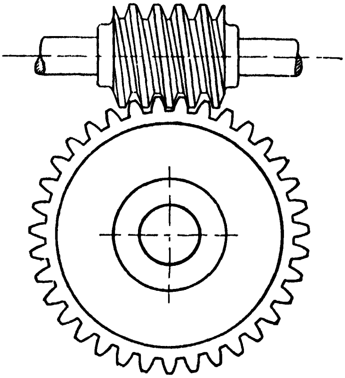

FIG. 163. WORM GEARING

Where the axes of two shafts are at right-angles, and a high velocity ratio is desired, worm gearing (Fig. 163) is necessary. On account of the smooth running and absence of vibration at high speeds of revolution, worm gearing is, however, frequently used, even when the velocity ratio is comparatively small. It will be obvious that in this case the axes of the two shafts cannot be in the same plane.

The object of using toothed wheels is to obtain the equivalent of rolling without slipping. The conditions to be satisfied in order to attain this object will be dealt with later, but for the present it will be assumed that the motion of two wheels in gear is the same as that of two cylinders in contact which roll without slipping, the respective diameters of the cylinders being the diameters of the *pitch circles* of the actual wheels.

The ratio of the speed of revolution of the last wheel to the speed of revolution of the first wheel of a train is known as the "velocity ratio" of the train. This is positive if the first and last wheels of the train revolve in the same direction, and negative otherwise. Thus, in Fig. 156, the velocity ratio is negative, and the velocity ratios in Figs. 157 and 158 are positive.

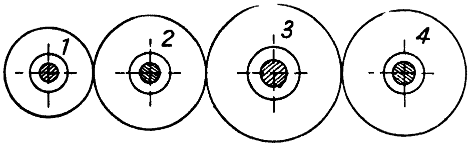

FIG. 164. SIMPLE TRAIN OF WHEELS

If the axes of the first and last wheels are co-incident, as in Fig. 160, the train is known as a "Reverted Train."

149. Simple Train (Fig. 164)

Let v = pitch line velocity

ω_1 = angular velocity of 1

ω_2 = angular velocity of 2

r_1 = radius of pitch circle of 1

r_2 = radius of pitch circle of 2

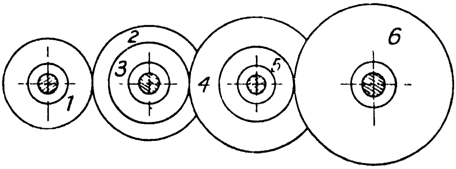

FIG. 165. COMPOUND WHEEL TRAIN

Then, $\omega_1 = \frac{v}{r_1}$; $\omega_2 = \frac{v}{r_2}$; $\frac{\omega_2}{\omega_1} = \frac{r_1}{r_2}$

By similar reasoning,

$$\frac{\omega_3}{\omega_2} = \frac{r_2}{r_3}; \quad \frac{\omega_4}{\omega_3} = \frac{r_3}{r_4}$$

$$\therefore \quad \frac{\omega_4}{\omega_1} = \frac{\omega_4}{\omega_3} \times \frac{\omega_3}{\omega_2} \times \frac{\omega_2}{\omega_1} = \frac{r_3}{r_4} \times \frac{r_2}{r_3} \times \frac{r_1}{r_2} = \frac{r_1}{r_4} \quad . \quad \text{(IX.1)}$$

The numerical value of the velocity ratio is thus independent of the number of intermediate wheels in the train. It will be positive if the number of spindles is odd and negative if the number of spindles is even. (The rule relating to sign also applies to compound trains.)

150. Compound Train (Fig. 165).

Reasoning as before,

$$\omega_2 = \frac{r_1}{r_2} \cdot \omega_1 = \omega_3$$

$$\omega_4 = \frac{r_3}{r_4} \cdot \omega_3 = \frac{r_1 \cdot r_3}{r_2 \cdot r_4} \cdot \omega_1 = \omega_5$$

$$\omega_6 = \frac{r_5}{r_6} \cdot \omega_5 = \frac{r_1 \cdot r_3 \cdot r_5}{r_2 \cdot r_4 \cdot r_6} \cdot \omega_1 \quad . \quad . \quad \text{(IX.2)}$$

The numerical value of the velocity ratio is thus given by

$$\text{Velocity ratio} = \frac{\text{product of radii of driving wheels}}{\text{product of radii of driven wheels}} . \quad \text{(IX.3)}$$

In the case of any pair of wheels in gear, the ratio of the radii of their pitch circles is the ratio of the numbers of teeth in the wheels, and the velocity ratio of the compound train may be expressed as

$$\text{Velocity ratio} = \frac{\text{product of teeth in driving wheels}}{\text{product of teeth in driven wheels}} \quad \text{(IX.4)}$$

151. **Epicyclic Trains.** An epicyclic train consists of a train of wheels in which there is relative motion between two or more of the axes of the wheels constituting the train.

A simple form of such a train is shown in Fig. 166. The *value* of the train (e) is given by

$$e = \frac{\text{speed of last wheel relative to the arm}}{\text{speed of first wheel relative to the arm}} \quad . \quad \text{(IX.5)}$$

The value of e, being the velocity ratio of the train with the arm D fixed, is easily calculated by the preceding rules, and is positive or negative according as the last wheel and the first wheel rotate in the same or in opposite directions when viewed in the same direction along the axis of either wheel.

Let f, l, and a be the number of revs. made in a given time by the first wheel, the last wheel, and the arm respectively, relative to a fixed framework.

then

$$\frac{l-a}{f-a} = e \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IX.6)}$$

The velocity ratio $\frac{l}{f}$ of any epicyclic train can be calculated by means of (IX.6), as the following examples show.

EXAMPLE 1.

In Fig. 166, let the number of teeth in wheel A be 30, and, in wheel C, 15.

Case 1. A fixed $(f = 0)$; arm makes 1 rev. clockwise $(a = +1)$. In this case $e = +2$.

$$\therefore \frac{l-1}{0-1} = +2$$

$$\therefore l = -1$$

i.e. C will make 1 rev. in space in an anti-clockwise direction.

Case 2. C fixed $(l = 0)$; arm makes 1 rev. clockwise $(a = +1)$

$$\therefore \frac{0-1}{f-1} = +2$$

$$\therefore f = \tfrac{1}{2}$$

i.e. A will make $\frac{1}{2}$ rev. in a clockwise direction.

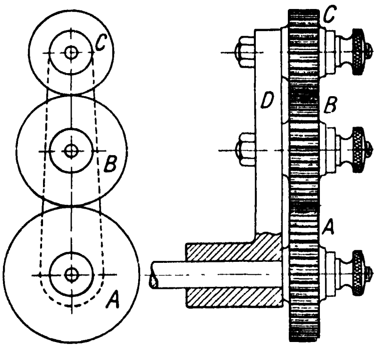

FIG. 166. EPICYCLIC TRAIN OF WHEELS

EXAMPLE 2.

In Fig. 166, B is removed, and A gears direct with C, so that $e = -2$. If A is fixed $(f = o)$ and $a = +1$,

$$\therefore \frac{l-1}{0-1} = -2$$

$$\therefore l = 3$$

i.e. C will make 3 revs. in a clockwise direction.

EXAMPLE 3.

Referring to Fig. 167, the numbers of teeth in the wheels are as follows: A, 45; B, 25; E, 50; C, 20. To find the revolutions per minute of wheel C, A being fixed and the arm driven in a clockwise direction by the wheel D at 75 revs. per min.

Regarding A as the first wheel and C as the last, then $f = o$. (The dotted wheel F is to be disregarded.)

Also, $$e = -\frac{45 \times 50}{25 \times 20} = -4{\cdot}5$$

$$\therefore \frac{l - 75}{0 - 75} = -4{\cdot}5$$

$$\therefore l = 412{\cdot}5 \text{ r.p.m.}$$

∴ Wheel *C* will rotate at 412·5 revs. per min. in a clockwise direction.

The speed of rotation of the intermediate wheel *B* may be found by treating the train *A*, *B*, and the arm *D*, as an isolated train. In this case, however, some convention is necessary regarding positive and negative directions of motion. Considering *A* and *B*, the directions of sight should be along arrows *X* and *Y*, so that *e* is negative and $= -\frac{45}{25} = -\frac{9}{5}$.

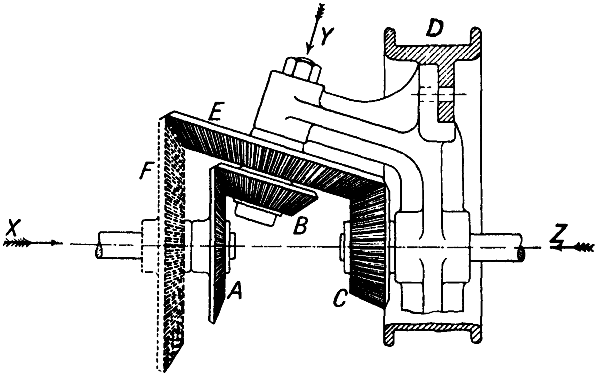

FIG. 167

Let x = r.p.m. of *B*

$$\therefore \frac{x - 75}{0 - 75} = -\frac{9}{5}$$

$$\therefore x = +210 \text{ (135 r.p.m. relative to } D\text{)}$$

Thus, *B* and *E* rotate on their common axis at 135 r.p.m. in a clockwise direction, looking along *Y*, when *D* rotates in a clockwise direction, looking along *X*.

Considering *E*, *C*, and the arm, directions of rotation would be taken looking along arrows *Y* and *Z*, so that r.p.m. of *D* = – 75 ; r.p.m. of *C* = – 412·5 ;

$$e = -\frac{50}{20} = -2{\cdot}5.$$

In this case, $e = \dfrac{\text{r.p.m. of } C \text{ relative to arm}}{\text{r.p.m. of } E \text{ relative to arm}}$

$$= \frac{-412{\cdot}5 + 75}{135}$$

$$= -2{\cdot}5$$

which confirms the method.

If now, the fixed wheel F (shown dotted in Fig. 166), gearing with E, be added, the procedure would be as follows—

Consider the train A, B, E, F, taking F as the last wheel ($70T$).

Then $$e = +\frac{45 \times 50}{25 \times 70} = +\frac{9}{7}$$

Let A make m r.p.m. clockwise.

$$\therefore \frac{0 - a}{m - a} = \frac{9}{7}$$

$$\therefore a = 4{\cdot}5m$$

$\therefore$ r.p.m. of arm $D = 4{\cdot}5m$ in a clockwise direction.

Now consider the train A, B, E, C ($e = -4{\cdot}5$).

Let C make n r.p.m. (taken as last wheel)

$$\therefore \frac{n - 4{\cdot}5m}{m - 4{\cdot}5m} = -4{\cdot}5$$

$$\therefore n = 20{\cdot}25m$$

Thus, if $m = 20$, $a = 90$, $n = 405$.

EXAMPLE 4.

Referring to the data of Example 3 (wheel F omitted), if A rotates at 110 r.p.m., and C rotates at 375 r.p.m., calculate the r.p.m. of the wheel D.

From (IX. 6), if $r = \dfrac{\text{r.p.m. of last wheel}}{\text{r.p.m. of first wheel}}$

$$\frac{\text{r.p.m. of arm}}{\text{r.p.m. of first wheel}} = \frac{a}{f} = \frac{e - r}{e - 1} \quad . \quad . \quad . \quad . \qquad \text{(IX.7)}$$

In this case $f = 110$; $r = \dfrac{375}{110} = 3{\cdot}41$; $e = -4{\cdot}5$,

$$\therefore \frac{a}{110} = \frac{-4{\cdot}5 - 3{\cdot}41}{-4{\cdot}5 - 1{\cdot}0} = 1{\cdot}44$$

$$\therefore a = 158{\cdot}2 \text{ r.p.m.}$$

Fig. 168 shows a type of reverted train which has many practical applications. *A* is a wheel with internal teeth. The wheel *B*, pinned to the rotating arm *D*, gears with wheels *A* and *C* (a compound wheel may be substituted for *B*).

EXAMPLE 5.

A (fixed wheel) has 60 teeth ; *C* has 24 teeth. Arm *D* rotates at 75 r.p.m. To find r.p.m. of *C*.

In this case $l = 0$; $a = 75$;

$$e = -\frac{24}{60} = -\frac{2}{5}$$

$$\therefore \frac{0-75}{f-75} = -\frac{2}{5}$$

$$f = +262{\cdot}5$$

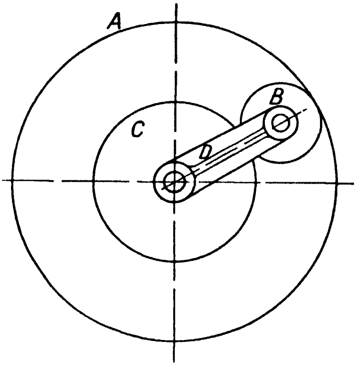

FIG. 168

∴ *C* rotates at 262·5 r.p.m. in the same direction as the arm.

To find the r.p.m. of *B*, consider a train consisting of *C*, *B*, and the arm. Regarding *B* as the last wheel (18 teeth),

$$e = -\frac{24}{18} = -\frac{4}{3}, f = 262{\cdot}5, a = 75.$$

$$\therefore \frac{l-75}{262{\cdot}5-75} = -\frac{4}{3}, \text{ i.e. } l = -175$$

∴ *B* makes - 175 r.p.m. in the opposite direction to the arm, relative to a fixed framework, or - 175 - 75 = - 250 r.p.m. relative to the arm, i.e. on its own axle.

Another method of investigating the motions of the wheels in an epicyclic train is to consider the motions in two parts—

(1) Motions in which all the wheels are relatively at rest but are carried round bodily by the arm ; and

(2) Motions in which the arm is fixed but the wheels are in motion relative to each other.

In the case of Example 1 above, we see that motion (1) consists of the rotation of the arms and wheels bodily through + 1 revolution, and motion (2) consists of the rotation of the wheel *A* through - 1 revolution with the arm fixed. The positive sign indicates that the rotation is clockwise and the negative sign indicates rotation in the counterclockwise sense. The total motion of A is, therefore, a rotation through $1 - 1 = 0$ revolutions, i.e. A is fixed as required by the conditions of the example. In motion (2), the arm being fixed, the rotations of the wheels of the train are calculated by the methods of Art. 149 and Art. 150. Thus during motion (2)

Number of revolutions of C

$$= \text{Number of revolutions of } A \times \frac{\text{Number of teeth in } A}{\text{Number of teeth in } C}$$

$$= -1 \times \frac{30}{15} = -2$$

The total number of revolutions of C is therefore $1 - 2 = -1$, i.e. C makes 1 revolution in the clockwise sense. These results are best worked out by tabulation as shown below. The first row of numbers gives the motion (1) and the second row gives motion (2). The third row, obtained by addition, gives the total motion of each of the wheels and of the arm—

	Arm	Wheel A	Wheel C
	1	1	1
	0	-1	$-1 \times \frac{30}{15} = -2$
Total	1	0	-1

The following table shows the application of the method to Example 3 above—

Arm D	Wheel C	Wheel A
$+75$	$+75$	75
0	$+75 \times \frac{45 \times 50}{25 \times 20} = 337\frac{1}{2}$	-75
$+75$	$+412\frac{1}{2}$	0

Thus the speed of the wheel C is $412\frac{1}{2}$ r.p.m. as before.

In these two examples there is no difficulty in deciding what numbers should be entered in the table; in Example 1 the number of revolutions in each of the spaces on the first row is clearly $+1$, being the revolutions of the arm. In the second part of the motion A is to be brought back to its original position, and so the number -1 is entered under A in the second row. The number under "arm" in this row is always zero and that under C is calculated.

In Example 3 the minute is taken as the interval of time. The number under D, C, and A in the first row is 75, being the total number of revolutions of the arm D. In motion (2) A must rotate through -75 revolutions back to its original position. Thus in the second row we enter -75 under A, 0 under D, and the number under C is calculated.

In some cases the total number of revolutions of the arm is not known, and so it is not obvious what number should be entered in the first row, though usually the number may be determined by the exercise of a little ingenuity. Any difficulty may be overcome by the following method. Assume motion (1) to be such that all the wheels make $+1$ revolution while in motion (2) the arm is at rest and a selected wheel (say the first wheel in the train) makes $+1$ revolution. The numbers in the first and second rows are then multiplied by α and β respectively and added to give the numbers in the third row. α and β are two unknown arbitrary constants; by a suitable choice of these constants we can obtain any possible motion of the epicyclic train under consideration. Below we apply this method to the case of Example 4 above—

	Arm D	Wheel A	Wheel C	Multiplier
	$+1$	$+1$	$+1$	α
	0	1	$-1 \times \frac{45 \times 50}{25 \times 20} = -4\frac{1}{2}$	β
Total	α	$\alpha + \beta$	$\alpha - 4{\cdot}5\beta$	

From the data,

$$\alpha + \beta = 110 \text{ and } \alpha - 4{\cdot}5\beta = 375$$

from which $$\alpha = 158\tfrac{2}{11}$$

Thus the arm *D* makes $158\frac{2}{11}$ revolutions per minute in the same sense as that of the rotation of the wheels *A* and *C*.*

152. **Forms of Wheel Teeth.** As a preliminary to the discussion of the correct shapes of the profiles for wheel teeth, the following investigation is necessary—

Two bodies *AP* and *BP* (Fig. 169), pivoted at *A* and *B*, respectively, are in contact at *P*. *YY* is the common tangent, and *PN* is the common normal to the surfaces at the point of contact. *PN* cuts the line joining *A* to *B* at *T*. To find the ratio of the instantaneous angular velocities in this position.

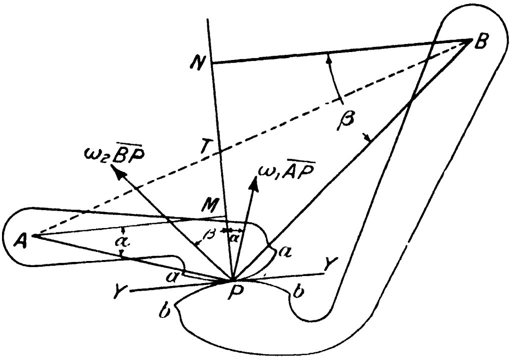

FIG. 169

Let ω_1 and ω_2 be the angular velocities of *AP* and *BP* respectively. Draw *AM* and *BN* each perpendicular to *PN*. Then, referring to the figure, since the two bodies remain in contact, the component of the velocity of the point *P* on *AP* in the direction *PN*, is the same as the component of the velocity of the point *P* on *BP* in the direction *PN*.

$$\therefore \omega_1 . \overline{AP} . \cos\alpha = \omega_2 . \overline{BP} . \cos\beta$$

$$\omega_1 . \overline{AP} . \frac{\overline{AM}}{\overline{AP}} = \omega_2 . \overline{BP} . \frac{\overline{BN}}{\overline{BP}}$$

$$\therefore \omega_1 . \overline{AM} = \omega_2 . \overline{BN}$$

$$\therefore \frac{\omega_1}{\omega_2} = \frac{\overline{BN}}{\overline{AM}} = \frac{\overline{BT}}{\overline{AT}} \quad . \quad . \quad . \quad . \quad (IX.8)$$

Compare with (IV.7). *T* is point of connection (Art. 73). Thus, the ratio of the angular velocities of *AP* and *BP* is the

---

* See also Appendix E.

inverse ratio of the segments in which the line of centres AB is divided by the common normal through the point of contact.

Therefore, if the ratio of the angular velocities of the bodies is to be constant for all positions, T must be a fixed point on AB. Thus the condition for constant velocity ratio is that *the common normal at the point of contact must always pass through a fixed point on the line of centres.*

In the case of two wheels in gear, T will be the point of contact of the pitch circles, and is known as the "pitch point."

The rate at which sliding occurs along YY is given by

$$\begin{aligned} v_s &= \omega_1 \cdot \overline{AP} \,.\, \sin\alpha + \omega_2 \,.\, \overline{BP} \,.\, \sin\beta \\ &= \omega_1 \,.\, \overline{AP} \,.\, \frac{\overline{PM}}{\overline{AP}} + \omega_2 \,.\, \overline{BP} \,.\, \frac{\overline{PN}}{\overline{BP}} \\ &= \omega_1 \,.\, \overline{PM} + \omega_2 \,.\, \overline{PN} \\ &= \omega_1 \,.\, (\overline{PT} - \overline{TM}) + \omega_2 \,.\, (\overline{PT} + \overline{TN}) \\ &= (\omega_1 + \omega_2) \,.\, \overline{PT} - (\omega_1 \,.\, \overline{TM} - \omega_2 \,.\, \overline{TN}) \end{aligned}$$

From (IX. 8) $\dfrac{\omega_1}{\omega_2} = \dfrac{\overline{BT}}{\overline{AT}} = \dfrac{\overline{TN}}{\overline{TM}}$

$$\therefore \omega_1 \,.\, \overline{TM} = \omega_2 \,.\, \overline{TN}$$

$$\therefore v_s = (\omega_1 + \omega_2) \,.\, \overline{PT} \quad . \quad . \quad . \quad . \qquad \text{(IX.9)}$$

which was proved in Art. 79.

Hence, *the velocity of rubbing between the teeth of a pair of gear wheels is proportional to the distance of the point of contact from the pitch point.*

The above investigation will serve to emphasize the fact that wheel teeth do not roll upon one another without sliding. As a matter of fact, in high-speed gearing, some sliding is desirable in order to preserve a film of lubricant between the teeth.

The importance of uniform velocity ratio when two wheels are in gear increases with the moments of inertia of the wheels and the speed of revolution.

Let N = number of teeth in a given wheel
I = its moment of inertia (Engineers' units)
n = speed of revolution (revs. per min.)

Suppose that, owing to incorrect tooth formation, *the speed is varied by* 0·1 *per cent* during the time of contact (i.e. AT increases to 1·001 AT).

Then, time of contact $= \dfrac{60}{N \,.\, n}$ sec.

Change of angular velocity in $\dfrac{60}{N \,.\, n}$ sec.

$$= \cdot 001 \times \frac{2\pi \,.\, n}{60} \text{ rad. sec.}$$

∴ Change of angular velocity per sec.

$$= \frac{\cdot 001 \times 2\pi \,.\, n}{60} \times \frac{N \,.\, n}{60} = \frac{\cdot 00628}{3600} \,.\, N \,.\, n^2$$

∴ Variation of torque from mean

$$= \frac{\cdot 00628}{3600} \,.\, N \,.\, n^2 I \text{ lb.-ft.} \qquad . \qquad . \qquad \text{(IX.10)}$$

EXAMPLE.

Gear wheel 8 in. diameter, 35 teeth, weight 12 lb., $I = \cdot 03$ running at 1500 r.p.m. Speed variation 0·1 per cent.

$$\text{Variation of torque from mean} = \frac{\cdot 00628 \times 35 \times (1500)^2 \times \cdot 03}{3600}$$

$$= 4 \cdot 12 \text{ lb.-ft.}$$

This is equivalent to a variation of tooth pressure of ± 12·4 lb., acting 875 times per sec. Such a variation in this case would not make an appreciable difference to the strength of the tooth, but might cause very perceptible noise in a gear-box.

EXAMPLE.

The following are the particulars of a turbine gear wheel: Weight 10 tons, diameter 6 ft., 240 teeth, $I = 2{,}800$, running at 250 r.p.m. Speed variation 0·1 per cent.

In this case, variation of torque from mean

$$= \frac{\cdot 00628 \times 240 \times (250)^2 \times 2800}{3600} = 73{,}300 \text{ lb.-ft.}$$

This is equivalent to a variation of tooth pressure of ± 24,400 lb. (11 tons approx.), acting 1000 times per sec. Obviously, in this case the degree of inaccuracy in the teeth would have to be very much less than that assumed above, and in any case some degree of flexibility in the drive would be necessary.

Given a profile *aa* (Fig. 169) of any shape (preferably convex and of single curvature), it is possible to construct a profile *bb* to gear with this and satisfy the condition for constant velocity ratio. (See Unwin's *Machine Design*, Part I, for suitable construction.) In practice, however, only two forms of profile for the teeth of wheel-gearing are in general use, i.e. "cycloidal" and "involute."

153. **Cycloidal Teeth.** Fig. 170 shows two teeth of a wheel with cycloidal teeth. The face eb of the tooth is a portion of an epicycloid generated by a circle rolling on the outside of the pitch circle pp. The *flank* bc is a portion of a hypocycloid generated by a circle rolling on the inside of the pitch circle. The circle aa, passing through the tip e of each tooth, is known as the "addendum circle"; the circle rr, passing through the root c of each tooth, is known as the "root circle." The "circular pitch" (C.P.) of the teeth, is the distance from b to d, measured along the pitch circle pp. In many cases the "diametral pitch" (D.P.) is used in preference to circular pitch. This is the number of teeth per inch of pitch circle diameter, the relation between the two being

$$CP = \frac{\pi}{DP} \quad . \qquad \text{(IX.11)}$$

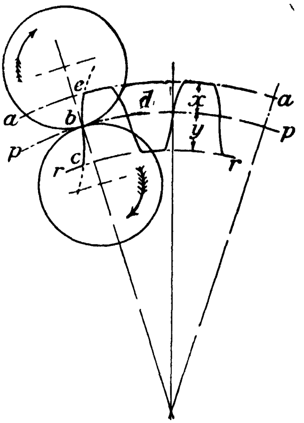

FIG. 170. CYCLOIDAL TEETH

The use of diametral pitch simplifies the calculations for distance between centres of two wheels in gear with each other. For instance, if two wheels with teeth of 4 D.P. have 25 and 55 teeth respectively, the distance between centres will be $\frac{25 + 55}{4 \times 2} = 10$ in. If teeth of $\frac{3}{4}$ in. C.P. were adopted, the distance between centres would be $\frac{\frac{3}{4}\,(25 + 55)}{3{\cdot}14 \times 2} = 9{\cdot}55$ in.

The height x of the tooth above the pitch-line pp is known as the "addendum," and the depth y (Fig. 170) is the "dedendum."

154. **To Show that Cycloidal Teeth Satisfy the Condition for Constant Velocity Ratio.** In Fig. 171, A is the centre of a wheel, radius R_1, and B is the centre of a wheel, radius R_2,

gearing with the first. *O* is the point of contact of the respective pitch circles, 11 is a portion of the face of a tooth of wheel 1 in contact with a portion 22 of the flank of a tooth of wheel 2, *P* being the point of contact. *POQ* is the rolling circle radius *r*, generating the curves 11 and 22. (The figure is drawn out of proportion for the sake of clearness.)

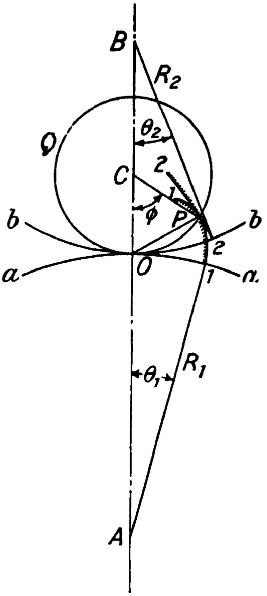

FIG. 171

Since the common normal at *P* must pass through the point of contact of the rolling circle generating 11 with *aa*, and also through the point of contact of the rolling circle generating 22 with *bb*, it must necessarily pass through *O*, which is common to the two circles. This satisfies the condition for constant velocity ratio.

Since the normal *OP* at *P* passes through *O*, *P* must be a point on an epicycloid 11 of which the rolling circle has moved backward through the angle ϕ.

$$\therefore \text{ arc } OP = 01 = R_1\theta_1 = r \,.\, \phi.$$

Similarly, *P* must be a point on the hypocycloid 22, the rolling circle of which has moved backward through the same angle.

$$\therefore \text{ arc } OP = 02 = R_2\theta_2 = r \,.\, \varphi$$

$$\therefore R_1\theta_1 = R_2\theta_2$$

$$\therefore \frac{\theta_2}{\theta_1} = \frac{R_1}{R_2} \quad . \quad . \quad . \quad . \quad \text{(IX.12)}$$

which again satisfies the condition for constant velocity ratio.

155. **Velocity of Sliding.** *OP* is the common normal to the teeth at their point of contact, and the point of contact moves along the circumference of the circle *OPQ* above *aa* and that of a similar circle below *aa*.

The faces of the teeth of wheel 1 and the flanks of the teeth of wheel 2 must be generated by the same rolling circle, but a rolling circle of different diameter may be adopted to generate both the flanks of wheel 1 and the faces of wheel 2. In practice, however, the same size of rolling circle is usually adopted for both.

From (IX.9) v_s = velocity of sliding

$= (\omega_1 + \omega_2)\overline{PT}$ (Fig. 169)

$= (\omega_1 + \omega_2)\overline{OP}$ (Fig. 171).

Thus, the velocity of sliding is proportional to the distance between the pitch point O and the point of contact P.

Hence, the greatest amount of wear will take place at the points and flanks of the teeth.

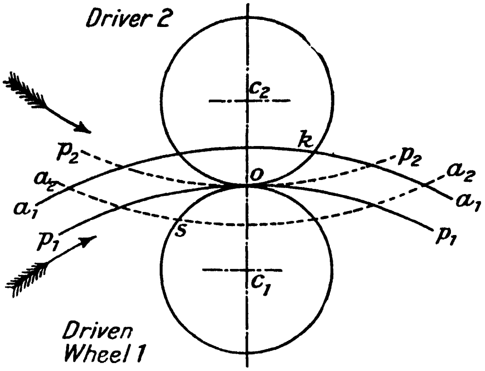

FIG. 172. ARCS OF APPROACH AND RECESS

In Fig. 172, a_1a_1 is the addendum circle of wheel 1, and a_2a_2 is the addendum circle of wheel 2. o is the pitch point,

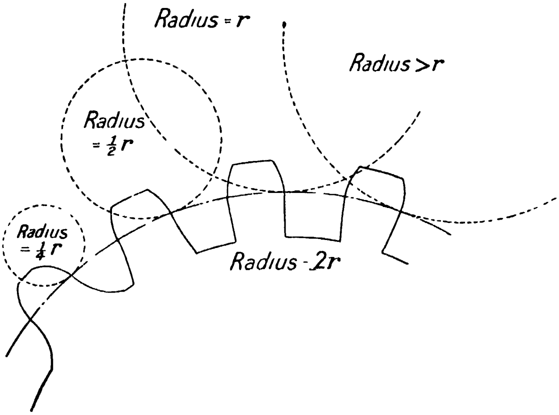

FIG. 173. EFFECT OF SIZE OF ROLLING CIRCLE ON SHAPE OF TOOTH

and c_1, c_2 are the centres of the rolling circles. The path of contact is then *sok*. An arc of the pitch circle from o towards p_1 equal to *os* is called *the arc of approach*, and one equal to *ok* on the other side of o is the *arc of recess*. The sum of these arcs is the *arc of action*, and this, divided by the C.P., gives the number of teeth in contact simultaneously.

The amount of sliding, and hence the amount of wear, can be reduced (*a*) by reducing the diameter of the rolling circle, (*b*) by reducing the depth of the tooth; (*a*) will involve an increase in the maximum obliquity of action, and (*b*) may be effected by a reduction of the pitch, which, however, involves a reduction in the strength of the teeth.

156. **Effect of Size of Rolling Circle on Shape of Tooth.** The shape of the tooth depends upon the *ratio* of the diameter of the rolling circle to the diameter of the pitch circle.

Fig. 173 shows the alteration of the shape of the tooth as the size of the rolling circle is increased, the ratios in this case being $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$ and more than $\frac{1}{2}$. A rolling circle whose diameter is half that of the pitch circle gives "radial flanks" (see Art. 71), and it is undesirable to increase the ratio beyond this, owing to the reduction in strength of the tooth. Fig. 174 shows the teeth of a rack generated by rolling circles of the same diameters as those in Fig. 173. It will be evident that the smaller rolling circle gives a stronger tooth, but with greater maximum obliquity of action, and hence larger radial thrust.

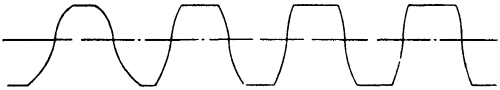

FIG. 174. RACK TEETH GENERATED BY VARIOUS ROLLING CIRCLES

In the case where a pinion of relatively small diameter drives a wheel of large diameter, the diameter of a rolling circle small enough to give the pinion teeth reasonably strong flanks would be so small a fraction of the pitch circle diameter of the driven wheel that pointed teeth and undesirable obliquity of action would result. In this case the rolling circle is usually chosen so that the pinion has radial flanks. Excessive obliquity of action can be avoided in the case of a large velocity ratio by the use of a compound train where practicable when a heavy torque has to be transmitted.

The conditions to be satisfied if a pair of wheels with cycloidal teeth are to gear together with uniform velocity ratio are

1. The pitch circles must touch each other.
2. The pitch of the teeth must be the same for both wheels.
3. The rolling circle generating the tooth profiles must be the same for the flank of one wheel as for the root of the other.

Unless the teeth are sufficiently hard and adequately lubricated, wear will occur at those parts of the tooth where the sliding velocity is greatest and the form will no longer be correct, so that vibration and noisy running will ensue.

157. **Involute Teeth.** In the case of slow speed wheels with cast teeth which are moulded from patterns, or machine moulded from segments, there is little difficulty in producing wheels with cycloidal teeth of reasonable accuracy, but even in this case care is necessary in assembling to see that the pitch circles are properly in contact, and that wear in bearings does not alter the centre distances appreciably. Where the teeth are to be machine-cut, the cutters are expensive to produce, and very careful setting is necessary. Involute teeth possess so many practical advantages in these and other respects, that, except in the case of large slow-speed wheels, with teeth of relatively large pitch, the use of involute teeth is almost universal.

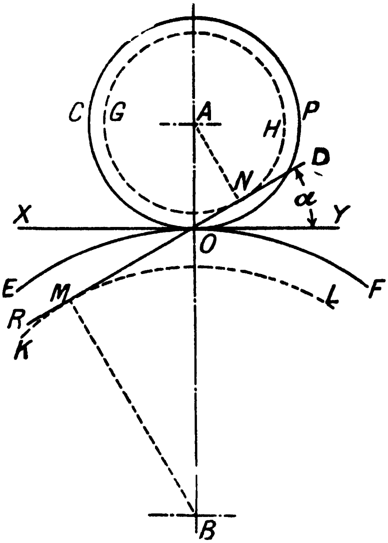

FIG 175. BASE CIRCLES AND PRESSURE ANGLE

An involute is generated by the rolling of a straight line on a circle, and is really a special case of an epicycloid with a rolling circle of infinite diameter. Since a straight line cannot roll on the inside of a circle, involute teeth are not generated on the pitch circles of the wheels, but on circles of smaller diameter, known as " base circles."

As will be shown later, the direction of the common normal, at the point of contact of involute teeth, is constant for the whole duration of contact, and the respective diameters of the base circles of two wheels in gear will depend upon what is known as the *pressure angle*, i.e. the angle between the common normal at the point of contact of two teeth and the common tangent to the pitch circles at their point of contact.

Let *COP* and *EOF* (Fig. 175) be the pitch circles of the two wheels, and let $DOY = a$ be the specified pressure angle. Draw AN and BM respectively perpendicular to RD. With radius AN, draw the circle GNH, and with radius BM draw the circle KML. GNH and KML will be the respective base circles, and the involute curves for the teeth will be described upon these circles.

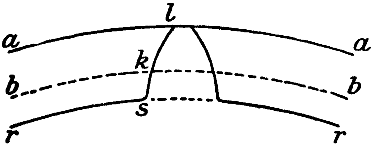

FIG. 176

In Fig. 176, *aa* is the addendum circle, *rr* is the root circle, and *bb* is the base circle. *kl* is the involute curve for the portion of the tooth above *bb*. Since no contact will take place below *bb*, the curve can be continued to *s* by a circular arc, or in any other convenient way.

158. **To Show that Involute Teeth Satisfy the Condition for Constant Velocity Ratio.** Let MN (Fig. 177) be the common tangent to the base circles RN, SM of the wheels, and let *aa*, *bb* be portions of two teeth which are in contact at P. Since the normal to an involute at any point is tangential to the base circle, lines PN, tangential to RN, and PM, tangential to SM, will represent the normals to *bb* and *aa* respectively at P. As the normal at P is common to both curves, PN and PM must lie in the same straight line, i.e. P must always lie on the common tangent MN, which passes through the pitch point O. This satisfies the condition for constant velocity ratio.

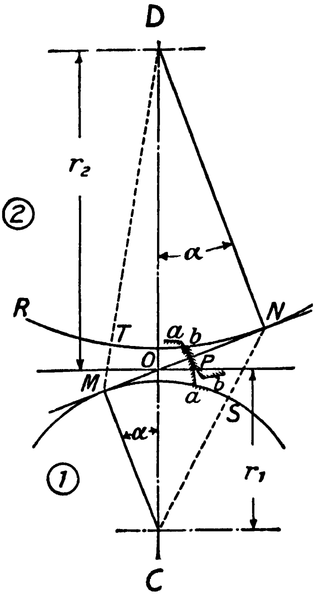

FIG. 177

The path of contact will thus lie along the line MN, and the direction of the thrust between the teeth will be constant.

To avoid "interference," and consequent "undercutting" of the flanks of the teeth of the pinion, the height of the teeth generated on the larger base circle NR, measured from NR, must not exceed the radial distance TM, since contact should not take place outside of MN.

Let r_1 and r_2 be the radii of the pitch circles, $r_2 > r_1$. Then $CN = r_1 + d_1$ where d_1 is the addendum on wheel 1. Then from the triangle CON

$$(r_1 + d_1)^2 = r_1^2 + r_2^2 \sin^2 \alpha - 2r_1r_2 \sin \alpha \cos (90° + \alpha).$$

Neglecting d_1^2 in comparison with $2\, r_1\, d_1$ we have on simplification

$$d_1 = r_2 \left(1 + \frac{r_2}{2r_1}\right) \sin^2 \alpha.$$

Similarly for wheel 2 the addendum is

$$d_2 = r_1 \left(1 + \frac{r_1}{2r_2}\right) \sin^2 \alpha.$$

If there is to be no interference these are the greatest possible values of the addenda.

Where interference is not likely to occur, the length of the addendum is usually taken as $\frac{1}{\text{D.P.}}$ This will not cause interference if the number of teeth in the smaller wheel is not less than $\frac{2}{1 + \frac{r_1}{2r_2}} . \operatorname{cosec}^2 \alpha$. (See Art. 167.)

EXAMPLE.

A wheel of 20 teeth, 4 D.P., is to gear with a wheel of 40 teeth. Taking the addendum as above, and a pressure angle of $14\frac{1}{2}°$, we have

Addendum of each tooth $= \frac{1}{4}$ in. $= \cdot 25$ in.

$$r_1 = \frac{20}{2 \times 4} = 2\tfrac{1}{2} \text{ in.}; \quad r_2 = \frac{40}{2 \times 4} = 5 \text{ in.}; \quad \sin \alpha = \cdot 25;$$

$$1 + \frac{r_1}{2r_2} = 1{\cdot}25; \ r_1 \left(1 + \frac{r_1}{2r_2}\right) . \sin^2 \alpha = \cdot 195 \text{ in.};$$

$$r_2 \left(1 + \frac{r_2}{2r_1}\right) . \sin^2 \alpha = \cdot 625 \text{ in.}$$

The addendum of the larger wheel is thus too great, and interference will occur.

(*Note.* The use of $14\frac{1}{2}°$ has no special significance other than convenience of setting-out an angle the sine of which is $\frac{1}{4}$. 15° is now frequently used.)

If "stub" teeth are used, the pressure angle will be increased to 20° and the addendum reduced to $\frac{1}{5}$ in. $= \cdot 2$ in.. the pitch remaining the same. In this case $r_1\left(1+\frac{r_1}{2r_1}\right).\sin^2\alpha = \cdot 382$ in., which is greater than the actual addendum, so that the wheels would gear without interference.

159. **Effect of Variation of Distance Between Centres of Wheels.** The effect of altering the distance AB (Fig. 175) will be to alter the pressure angle α, but as the base circle diameters are unaltered, the ratio $\frac{\overline{AO}}{\overline{OB}}$ remains unaltered, so that with involute teeth the velocity ratio remains as before, and the condition for uniform velocity ratio is still satisfied.

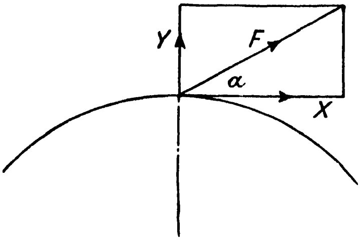

FIG. 178. RADIAL THRUST

Let F = pressure between the teeth (Fig. 178)

α = pressure angle

X = tangential component of F

Y = radial component of F (neglecting friction)

then, $X = F\,.\cos\alpha$ (IX.13)

$Y = F\,.\sin\alpha$ (IX.14)

$\therefore \frac{Y}{X} = \tan\alpha$

and $Y = X\,.\tan\alpha$ . . . . (IX.15)

(for $\alpha = 14\frac{1}{2}°$, $\tan\alpha = \cdot 258$; for $\alpha = 20°$, $\tan\alpha = \cdot 364$)

The tendency of the thrust Y is (a) to produce radial deflection of the wheel rims, and (b) to deflect the shafts carrying the wheels. In each case vibrations may be set up, even with teeth of correct shape, unless adequate stiffness is arranged for.

The magnitude of X depends upon the horse-power transmitted (P) and the linear speed (V) in feet per minute at the pitch line, and is given by

$$X\,.\,V = 33{,}000\,.\,P \qquad \text{(IX.16)}$$

Since the radial distance between the base circle and the pitch circle increases as α is increased for a given circular pitch, the thickness of the tooth at the root, and therefore the strength, will be increased by an increase of α.

160. **Helical Teeth.** Absolute accuracy of profile is impossible of attainment in gear cutting, although extremely high degrees of accuracy are possible with modern machines. If

a very small error in profile exists at some point, this will be repeated at the same point in each tooth, and some degree of shock will take place when this becomes a point of contact if the teeth are straight and parallel to the axis. If, however, each tooth is cut as a helix, the points of contact for any given cross-section of the wheels will vary according to the axial distance across the wheel, so that small positive and negative inaccuracies tend to balance each other.

The helix angle should be sufficiently large to ensure that the points of contact extend over the whole length of the profile. In practice, helix angles of from 25° to 45° are used, and end thrust is avoided by the use of right and left-hand helices on each wheel.

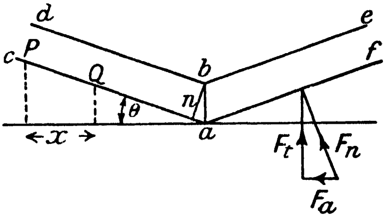

FIG 179. TEETH OF HELICAL WHEEL

Let *caf*, *dbe* (Fig. 179) be the centre lines of two consecutive teeth in a double helical wheel. The circular pitch (p_c) is *ab*, and the normal pitch (p_n) is *bn*, and these are connected by the equation

$$p_n = p_c \,.\, \cos\theta \qquad . \quad . \quad . \quad . \quad \text{(IX.17)}$$

where θ is the angle of the helix.

Let F_n = normal load on a length PQ of a tooth

x = axial distance of P from Q

$\therefore$ $F_t = F_n \,.\, \cos\theta$ = tangential load on PQ

$$\overline{PQ} = \frac{x}{\cos\theta} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IX.18)}$$

$\therefore$ Tangential load per inch width of wheel face

$$= \frac{F_t}{x} = \frac{F_n \,.\, \cos\theta}{x} \qquad . \quad . \quad . \quad \text{(IX.19)}$$

Normal load per inch length of tooth

$$= \frac{F_n}{\overline{PQ}} = \frac{F_n \,.\, \cos\theta}{x} \qquad . \quad . \quad . \quad \text{(IX.20)}$$

Thus, *the normal load per inch length of tooth is the same as the tangential load per inch width of face.* Since the normal thickness of the tooth for a given value of p_c decreases as θ

increases, a large spiral angle means an appreciable reduction in the resistance to bending of the tooth. In practice, however, the increased smoothness of engagement and reduction of shocks neutralize the effect of reduction of tooth thickness for spiral angles in ordinary use.

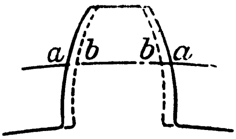

FIG. 180. SHAPE OF CUTTER

The shape of the cutter for the spaces between the teeth is shown by the dotted lines in Fig. 180, where the full lines show the section of the tooth in the plane of the wheel and $\frac{\overline{bb}}{\overline{aa}} = \frac{p_n}{p_c} = \cos\theta$.

161. **Methods of Cutting Teeth of Cylindrical Wheels.** The chief methods of cutting teeth may be classified under the following heads—

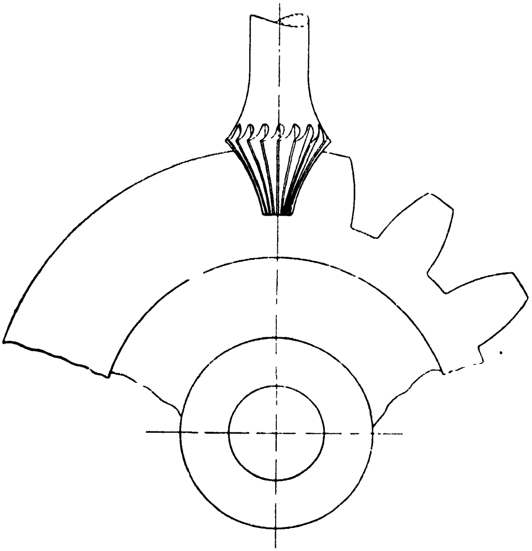

FIG. 181. MILLING TEETH WITH END MILL

1. Milling with circular cutter (axis of cutter at right-angles to wheel axis). (Fig. 182.)
2. Milling with circular cutter (axis of cutter intersecting the axis of the wheel at right-angles). (Fig. 181.)

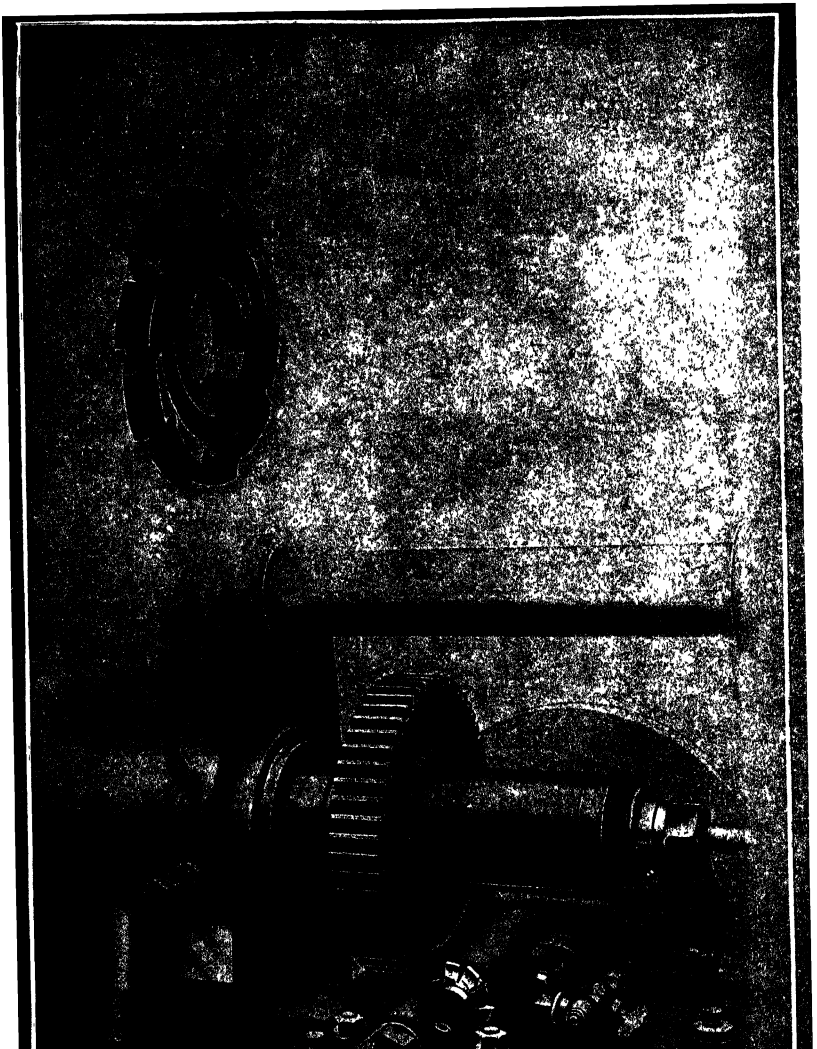

FIG. 182. MILLING WHEEL TEETH WITH SHAPED CIRCULAR CUTTER

3. "Hobbing" with worm-shaped cutter. (Fig. 184.)
4. Generation by planing with rack-shaped cutter. (Fig. 183.)
5. Generation by planing with pinion-shaped cutter.

(1) is the oldest method, but is not suitable for rapid production of very accurate gears, as the shape of the cutter must coincide exactly with the shape of the tooth space, and a different cutter is required for each size of wheel, even where the pitch and pressure angle are kept constant.

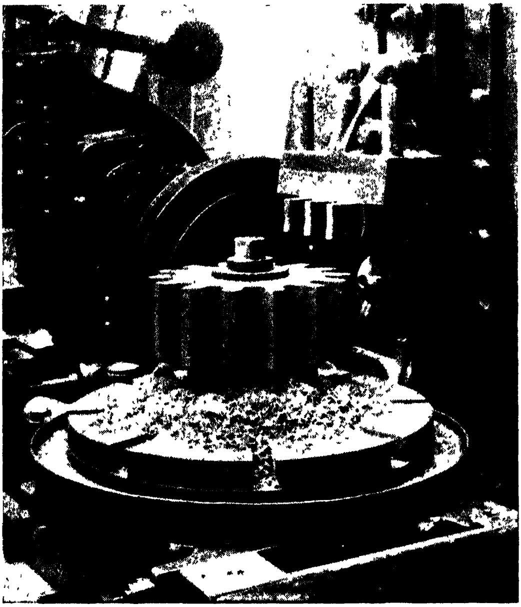

FIG. 183. GENERATING WHEEL TEETH BY MEANS OF RACK-SHAPED CUTTER

(2) is the method introduced by Messrs. Citroën for cutting double helical and similar wheels with continuous teeth. This has the same disadvantages as (1).

(3) This process is a continuous one, the table of the machine and the rotating hob being geared so that the relative motions shall be the same as if the hob were a worm driving the wheel. The normal pitch of the worm thread will be the same as the circular pitch of the wheel teeth, and the axis of the hob is inclined to a plane cross-section of the wheel blank at an angle

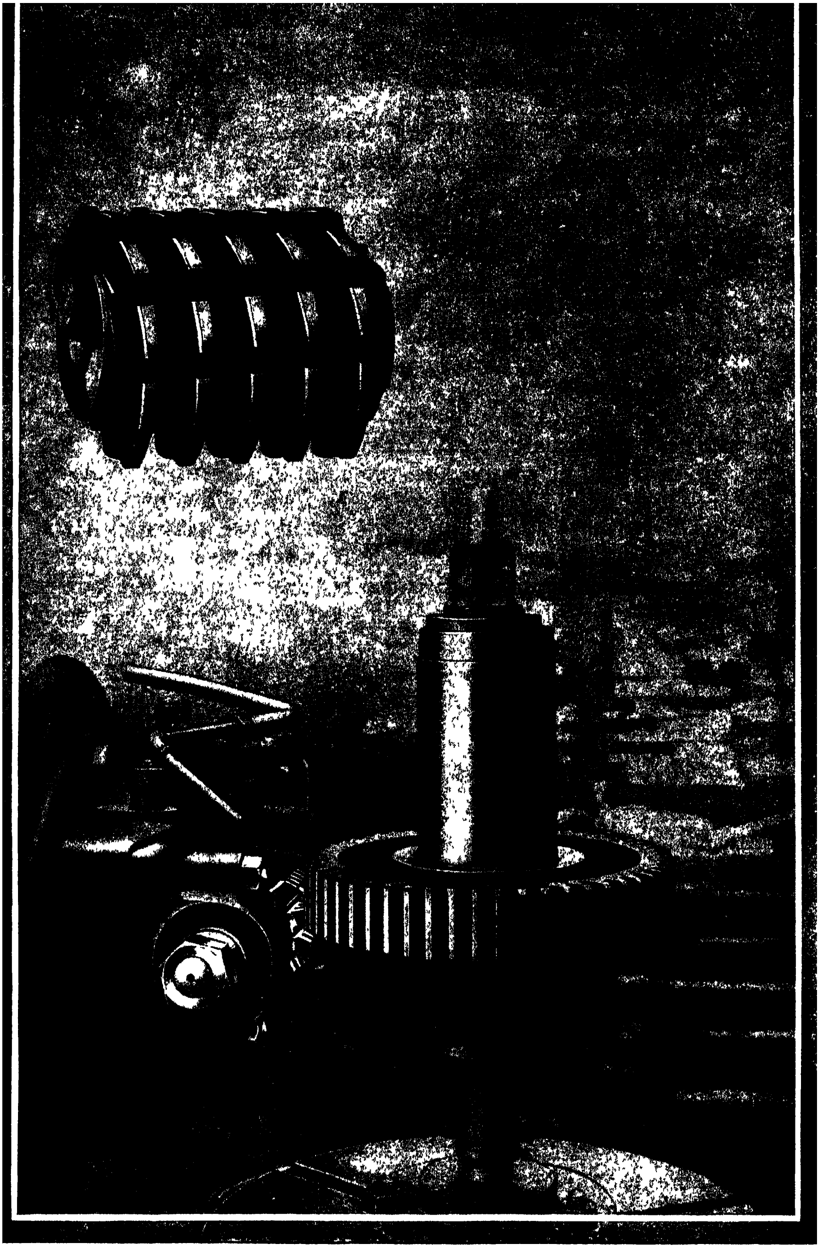

FIG. 184. HOBBING WHEEL TEETH WITH WORM-SHAPED CUTTER
(*B.S.A. Tools, Ltd., Birmingham*)

equal to the angle of the thread. The direction of feed of the hob is parallel to the axis of the wheel for ordinary spur wheels. Large helical wheels for turbine reduction gears, etc., are also cut by this process. In this case the setting of the worm axis must be such that the motion of the cutting edge *relative to the wheel* gives the desired spiral angle. As in (1) and (2), different cutters are necessary for different diameters of wheels.

(4) and (5) are true " generating " methods, as the one cutter in each case will produce teeth of the correct shape on wheels of all sizes, provided the pitch remains constant.

(4) makes use of the fact that the working surfaces of the teeth of a rack for gearing with involute pinions are planes, the angle between two adjacent faces being twice the pressure angle, and that such a rack will gear with pinions of any diameter, provided the pitch and pressure angle are the same as those of the rack. The cutter, in the form of a short rack, is geared with the blank to be cut, so that both move as if the rack geared with the wheel to be cut, the cutter making a series of planing strokes across the blank and being fed slowly in the direction of its length. As soon as the cutter has been fed beyond the blank, the wheel is indexed to a new position, and the cutter returned to its starting position.

The principle of (5) is similar, but in this case the cutter consists of a pinion instead of a rack, the cutter being rotated by a gear similar in principle to that used for feeding the rack in (4). Both methods (4) and (5) have been adapted for generating double helical gears. (5) is suitable for cutting internal gears.

162. **Bevel Gears.** In the case of bevel gears, the pitch cylinders will be replaced by pitch cones, as in Fig. 185, and the condition to be satisfied for pure rolling at all points of contact is that the apices of the pitch cones must coincide.

The pitch cones for two shafts AB, CD, which are to be geared together with a given velocity ratio, are determined by the following construction.

Let O be the point of intersection of the axes of the shafts. Set out on and om respectively proportional to the angular velocities of AB and CD, and complete the parallelogram $OnPm$. Then the angle $AOP = a$ is the semi-angle of the pitch cone for the shaft AB and $COP = \beta$ is the semi-angle of the pitch cone for the shaft CD.

The pitch of the teeth of a bevel wheel will be proportional to the distance from the apex O of the pitch cone, and it is necessary to specify where the pitch is to be measured. In practice this is measured at the largest diameter of the pitch cone of the wheel.

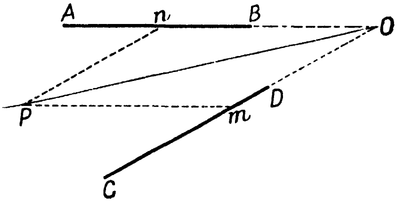

FIG. 185. SEMI-ANGLES OF PITCH CONES

The process of setting out the section of a bevel wheel with a given pitch cone angle 2α will then be as follows (Fig. 186).

The largest diameter of the pitch cone will be known from the number of teeth and pitch. Set out the angle $AOC = \alpha$. Draw OB at right-angles to OA, and equal to the maximum pitch cone radius, and draw BC parallel to OA, meeting OC in C. Draw DCA at right-angles to OC. CA is the radius of the pitch circle on which the teeth will be set out, DC is the addendum, and CE is the dedendum. CF is the length of tooth.

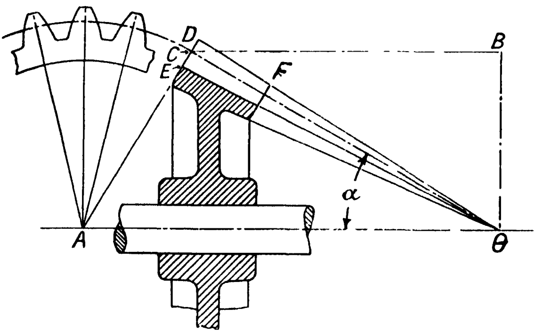

FIG. 186. SETTING OUT BEVEL WHEEL

The machining of the teeth of bevel wheels to a correct profile is obviously a more difficult problem than that of machining spur wheels, since the size of the tooth diminishes from C to F. All the machines which aim at producing correct outlines throughout operate by means of a planing cut directed toward the centre O. In the earliest type of bevel gear-planer a " former " was used, consisting of an enlarged tooth form situated on an extension of the line OC, which guided the tool round the required profile on the blank. This has the objection of requiring different formers for different sized wheels.

In one type of "generating" machine, the cutters *AA* (Fig. 187), which are straight-sided and inclined at the pressure angle of the involute, reciprocate towards *O* without lateral movement, and the blank moves about the centre *O* as if it were rolling on a circular rack *ODE*, of which the cutters *AA* represent one tooth. In this case the same cutters can be used for varying pitches and diameters, provided that the pressure angle is not altered.

Another machine, which has the advantage of somewhat greater simplicity in construction, using a variable four-bar mechanism and a plate with curved slots, produces teeth which approximate to involute form.

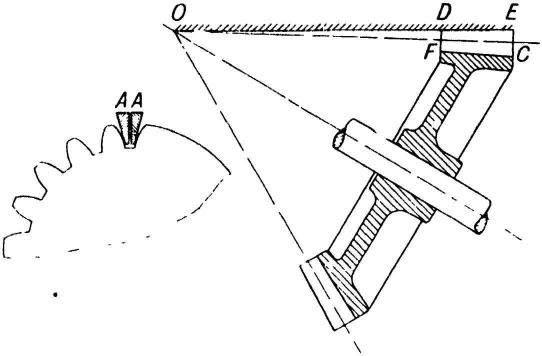

FIG 187 GENERATING BEVEL GEARS

Spiral bevels, which have the same advantages as spur gears with helical teeth, can be cut by any of the above processes if a device is added for rotating the blank while the cut is taking place.

163. **Worm Gearing***. The ordinary cylindrical worm is essentially a single or multiple-threaded screw, the section of the threads being that of a rack, and this would drive a spur wheel with straight teeth inclined to the wheel face at the spiral angle of the worm. Such a drive is used in practice for lightly-loaded transmissions such as feed motions, but would be unsuitable for heavy loads, as point contact only takes place between the worm and wheel. If, however, the wheel be cut by means of a hob which is a reproduction of the worm with

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* See Appendix D for more correct theory.

cutting edges, contact will then take place over the whole width of tooth which is cut by the hob (Fig. 188), i.e. line contact is substituted for point contact, so that much greater loads can be transmitted.

Since elastic deformation of the surfaces in contact must take place when a torque is being transmitted by the worm, the real advantages of the distribution of pressure are (*a*) less deformation of the surfaces in contact, and smaller wear, (*b*) a film of oil is more easily retained between the surfaces, thus reducing the friction coefficient. The following terms are used in connection with worm gearing—

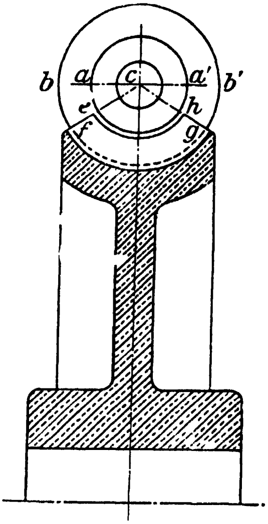


FIG. 188. WORM AND WORM WHEEL

Pitch = axial distance between centres of consecutive threads

Lead = axial distance between beginning and end of one turn of the same thread

= pitch × No. of threads on worm

Thread angle of worm (θ)

$$= \tan^{-1} \frac{\text{lead}}{\text{pitch circumference}} \quad . \quad . \quad (\text{IX.21})$$

No. of threads on worm = No. of spaces in end view of worm

$$= \frac{\text{lead} \times \cos \theta}{\text{normal pitch}} \quad . \quad . \quad . \quad . \quad . \quad (\text{IX.22})$$

The ratio of the angular velocities of the worm and wheel is given by

$$V.R = \frac{\text{No. of teeth in wheel}}{\text{No. of threads on worm}} \quad . \quad . \quad . \quad (\text{IX.23})$$

The mechanical efficiency is given by

$$\eta = \frac{\tan \theta}{\tan (\theta + \varphi)} \text{ where } \tan \varphi = \mu \quad . \quad . \quad (\text{IX.24})$$

and η is a maximum when $\theta = 45^\circ - \frac{\varphi}{2}$ (see Art. 124).

"Overhauling" will not occur if θ is not greater than φ, (see Art. 124) i.e. if

$$\frac{\text{lead}}{\text{pitch circumference}} \ngtr \mu$$

or if

$$\frac{\text{lead}}{\text{pitch diameter}} \ngtr 3{\cdot}14 \quad \mu \qquad . \quad . \qquad \text{(IX.25)}$$

If $\mu = {\cdot}05$, $\varphi = 3°$, $45° - \frac{\varphi}{2} = 43{\cdot}5°$, $\tan\left(45° - \frac{\varphi}{2}\right) = {\cdot}95$.

∴ For maximum efficiency,

$$\frac{\text{lead}}{\text{pitch diameter}} = {\cdot}95 \times 3{\cdot}14 = 2{\cdot}99$$

Thus, if the pitch diameter is 4 in., the lead will be approximately 12 in. If a normal pitch of about 1¼ in. is desired, the axial pitch will be $\frac{1{\cdot}25}{\cos 43{\cdot}5°} = 1{\cdot}73$ in.

∴ No. of threads per pitch $= \frac{12}{1{\cdot}73} = 6{\cdot}93$ (say 7).

Themaximum efficiency (neglecting bearing friction) in this case is $\frac{\tan 43{\cdot}5°}{\tan 46{\cdot}5°} = {\cdot}90$.

Actual tests of high efficiency worm gears with ball bearings and adequate lubrication have shown efficiencies of 95 to 97 per cent.

If, in the above case, a velocity ratio of 4 were required, the number of teeth in the wheel would be 4 × 7 = 28. Taking an axial pitch of worm $= \frac{12}{7} = 1\frac{5}{7}$ in., then

Pitch circumference of wheel $= \frac{12}{7} \times 28 = 48$ in.

∴ Pitch diameter of wheel $= \frac{48}{3{\cdot}14} = 15{\cdot}14$ in.

so that the distance between the axes of worm and wheel would be $\frac{15{\cdot}14 + 4}{2} = 9{\cdot}57$ in.

164. End-thrust in Worm Gears.

Let T = torque (lb.-in.) acting on worm (Fig. 189)
S = tangential thrust at circumference of pitch circle of worm (lb.)
= axial thrust on worm wheel

E = axial thrust on worm (neglecting friction)
= tangential thrust on wheel (neglecting friction)
θ = angle of thread, assumed to be square
d = pitch circle diameter of worm (in.)

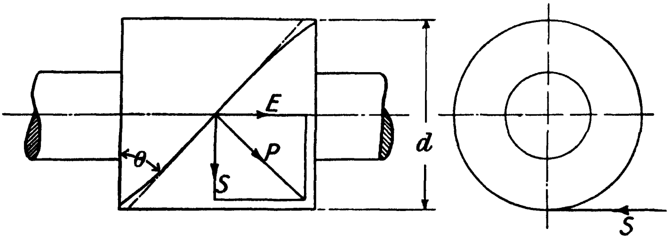


FIG 189. END THRUST IN WORM GEARS

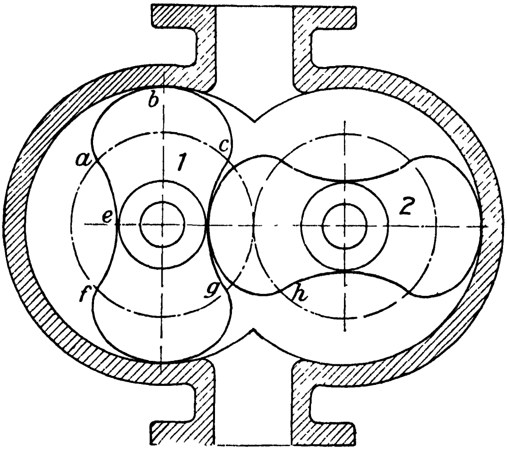


FIG. 190. ROOT'S BLOWER

Then $S = \frac{2T}{d}$, tending to bend the worm

$$E = S \,.\, \cot\theta = \frac{2T}{d} \,.\, \cot\theta$$

but $\cot\theta = \frac{\pi d}{\text{lead}}$

$$\therefore E = \frac{2\pi T}{\text{lead}} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IX.26)}$$

(IX.26) may also be obtained by consideration of the work done in one revolution of the worm. The actual end thrust on the wheel is $E \times$ efficiency.

An exhaustive analysis of the problem of worm gear contacts is contained in a paper by Dr. W. Abbott (*Proc. I. Mech. E.*, Vol. 133, 1936). This paper contains an analysis of theoretical lines of contact and surface attitudes of worm gears, of both the parallel and the enveloping types.

A paper by Dr. Merritt (*Proc. I. Mech. E.*, Vol. 192, 1935) should also be studied in this connection.

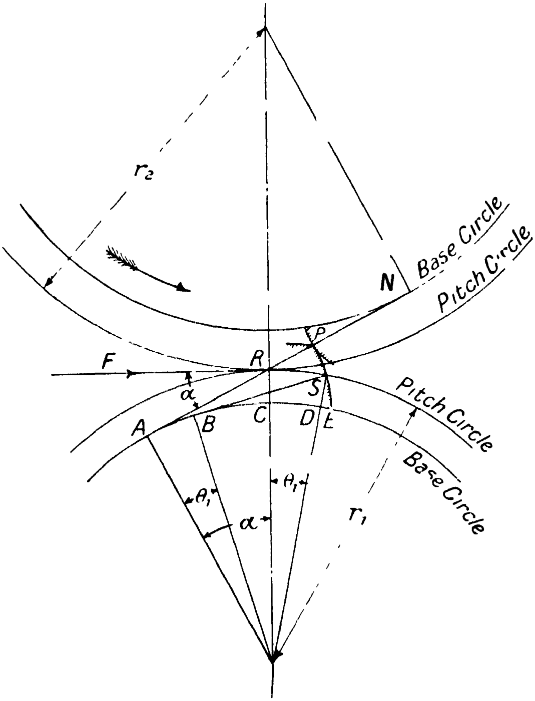


FIG. 191. FRICTION BETWEEN WHEEL TEETH

165. Root's Blower. The arrangement of Root's blower is shown in Fig. 190. Each of the rotors is essentially a wheel of two teeth only, the epicycloid *abc* and the hypocycloid *aef* each being generated by a rolling circle the diameter of which is one quarter the diameter of the pitch circle *acgf*. The second rotor will be in gear at right-angles to the first.

Since the faces and flanks are complete cycloids, the pressure angle will vary from zero, when *g* and *h* coincide, to 90° in the position shown, so that the two spindles must be driven externally by equal gear wheels on the shafts, a very small clearance being allowed between 1 and 2 in the position shown.

The blower is driven at a high speed of revolution, and since the rotors have a comparatively large moment of inertia the teeth of the driving wheels must be cut very accurately, otherwise frequent breakages are experienced.

166. **Friction Between Gear Teeth.** Let P (Fig. 191) be the point at which contact between the teeth of two wheels with involute teeth ends. Then RS will be the arc of recess.

From the geometry of the involute,

$$\text{Arc } BE = \overline{BS}; \quad \text{Arc } AE = \overline{AP}$$

From similar triangles,

$$\overline{AR} = \overline{BS}; \quad \therefore \text{Arc } AB = \overline{RP} = \text{arc } CD$$

$$\therefore \overline{PR} = r_1 \,.\, \cos\alpha \,.\, \theta_1 \quad . \quad . \quad . \quad \text{(IX.27)}$$

It follows from (IX.27) that the sliding velocity v_s is proportional to the angular distance travelled from the line of centres by the tooth.

∴ Average sliding velocity during arc of recess

$$= (\omega_1 + \omega_2) \ \frac{\overline{PR}}{2}$$

$$= \frac{\omega_1 + \omega_2}{2} \,.\, r_1 \,.\, \theta_1 \cdot \cos\alpha$$

Let F = tangential force acting at pitch radius

μ = coefficient of friction between teeth

α = pressure angle

Then, the normal pressure between the teeth acts along AP, and its magnitude is therefore $\frac{F}{\cos\alpha}$,

∴ Resistance to sliding $= \frac{\mu \,.\, F}{\cos\alpha}$

∴ Work done against friction during period of recess

$$= \frac{\mu \,.\, F}{\cos\alpha} \times \text{average sliding velocity} \times \text{time}$$

$$= \frac{\mu \,.\, F}{\cos\alpha} \,.\, \frac{\omega_1 + \omega_2}{2} \,.\, r_1 \cos\alpha \,.\, \theta_1 \,.\, \frac{\theta_1}{\omega_1}$$

$$= \frac{1}{2}\left(1 + \frac{\omega_2}{\omega_1}\right) . \, r_1 \,.\, \theta_1^2 \,.\, \mu F \text{ which is independent of } \alpha.$$

Similarly, work done against friction during period of approach

$$= \frac{1}{2}\left(1 + \frac{\omega_1}{\omega_2}\right) . r_2 . \theta_2^2 . \mu . F$$

∴ Work done against friction for whole period of action,

$$W = \frac{1}{2} (\omega_1 + \omega_2)\left(\frac{r_1}{\omega_1} . \theta_1^2 + \frac{r_2}{\omega_2} . \theta_2^2\right) . \mu F$$

$$= \frac{1}{2} . \frac{r_1 + r_2}{r_1 r_2} . \{(r_1\theta_1)^2 + (r_2\theta_2)^2\} . \mu F \quad . \qquad \text{(IX.28)}$$

Let l_1 = length of arc of approach $= r_2\theta_2$

l_2 = length of arc of recess $= r_1\theta_1$

$l = l_1 + l_2$

Then, from (IX. 28), for a given value of l, W will be a minimum when $l_1^2 + (l - l_1)^2$ is a minimum.

Differentiating with regard to l_1, taking l as constant, and equating to zero, the condition for a minimum becomes

$$l_1 = \frac{l}{2} = l_2$$

Hence, *for a given total arc of action, the work wasted in friction will be least when the arcs of approach and recess are equal.*

167. Limiting Dimensions for no Interference.

Let x_1 = addendum of teeth on wheel 1

Then, if O is the centre of wheel 1 (Fig. 191)

$$x_1 = \overline{OP} - \overline{OR}$$

$$\therefore \overline{OP} = r_1 + x_1$$

also $\overline{PR} = r_1 \cos \alpha . \theta_1$ and angle $PRO = 90° + \alpha$

By a property of the triangle OPR,

$$(r_1 + x_1)^2 = r_1^2 + r_1^2 . \cos^2 \alpha . \theta_1^2 + 2r_1^2 . \cos \alpha . \sin \alpha . \theta_1$$

Neglecting the term in x_1^2, we have

$$2r_1x_1 = r_1^2\{2 \cos \alpha . \sin \alpha . \theta_1 + \cos^2 \alpha . \theta_1^2\}$$

$$\therefore x_1 = \frac{r_1\theta_1}{2}\left\{\sin 2\alpha + \frac{\theta_1}{2}(1 + \cos 2\alpha)\right\} \quad . \quad . \qquad \text{(IX.29)}$$

Similarly,

$$x_2 = \frac{r_2\theta_2}{2}\left\{\sin 2\alpha + \frac{\theta_2}{2}(1 + \cos 2\alpha)\right\} \quad . \quad . \quad \text{(IX.30)}$$

θ_1 and θ_2 are in radians in each case.

If $\alpha = 15°$, (IX.29) and (IX.30) become

$$x_1 = \frac{r_1\theta_1}{4}(1 + 1{\cdot}86\,\theta_1) \quad . \quad . \quad . \quad . \quad \text{(IX.31)}$$

$$x_2 = \frac{r_2\theta_2}{4}(1 + 1{\cdot}86\,\theta_2) \quad . \quad . \quad . \quad . \quad \text{(IX.32)}$$

If no interference is to take place,

Maximum value of $\overline{PR} = RN = r_2 . \sin \alpha$

$\therefore$ If θ_o is the maximum permissible value of θ_1,

$$\theta_o . r_1 \cos \alpha = r_2 . \sin \alpha$$

$$\therefore \theta_o = \frac{r_2}{r_1} . \tan \alpha \quad . \quad . \quad . \quad . \quad \text{(IX.33)}$$

Substituting this for θ_1 in (IX.29), we have for the maximum value of x_1 for no interference,

$$x_1 = \frac{r_2}{2} . \frac{\sin \alpha}{\cos \alpha} . \left\{\cos^2 \alpha . \frac{r_2}{r_1} . \frac{\sin \alpha}{\cos \alpha} + 2 \sin \alpha . \cos \alpha\right\}$$

$$= \frac{r_2}{2}\left(\frac{r_2}{r_1}\sin^2 \alpha + 2 \sin^2 \alpha\right)$$

$$= r_2 . \left(\frac{r_2}{2r_1} + 1\right) . \sin^2 \alpha \quad . \quad . \quad . \quad . \quad \text{(IX.34)}$$

Similarly, maximum value of x_2 for no interference is

$$x_2 = r_1 . \left(\frac{r_1}{2r_2} + 1\right) . \sin^2 \alpha \quad . \quad . \quad . \quad . \quad \text{(IX.35)}$$

Let N_2 = no. of teeth in smaller wheel

P = diametral pitch of teeth

$\frac{1}{P}$ = addendum for each wheel (See Art. 147)

Then $2r_2 = \frac{N_2}{P}$; $\therefore \frac{1}{P} = \frac{2r_2}{N_2}$, and for no interference

$$\frac{2r_2}{N_2} \text{ must not exceed } r_2\left(\frac{r_2}{2r_1}+1\right).\sin^2\alpha$$

$$\text{i.e. } \frac{1}{N_2} \quad ,, \quad ,, \quad ,, \quad \frac{1}{2}\left(\frac{r_2}{2r_1}+1\right).\sin^2\alpha$$

$$\therefore\ N_2 \text{ must not be less than } \frac{2\operatorname{cosec}^2\alpha}{\dfrac{r_2}{2r_1}+1} \quad . \quad . \quad \text{(IX.36)}$$

$$\text{Thus, if } r_1 = 6r_2,\ \alpha = 15^\circ. \quad N_2 \nless \frac{2 \times 15}{1+\frac{1}{12}} \nless 28$$

$$\text{If } \alpha \text{ is increased to } 20^\circ, \quad N_2 \nless \frac{2 \times 8{\cdot}6}{1+\frac{1}{12}} \nless 16$$

EXAMPLES 9.

1. What circumstances determine severally the pitch, breadth, height, and arc of contact of the teeth of spur wheels? Explain how (1) cycloidal, (2) involute, teeth may be formed on a set of wheels so that any two of the wheels will gear together. (L.U.)

2. The following data are given for a single reduction-gear with double helical teeth, to transmit 2,500 h.p.—

No. of teeth in pinion = 23
,, ,, wheel = 260
Revs. per min. of wheel = 340
Circular pitch of teeth = ⅞ in.

Calculate (*a*) the pitch diameters of wheel and pinion, (*b*) the torque on the pinion shaft, (*c*) the load at the pitch line, (*d*) the width of the wheel face if the load per inch width of tooth is not to exceed 500 lb., (*e*) the normal pitch if the spiral angle of the gears is 30°, (*f*) the radial thrust between the wheels if the pressure angle is 22½°, (*g*) the heat generated per minute if the mechanical efficiency is 97 per cent, (*h*) the weight of oil circulated per minute if the specific heat is ·33, and the rise of temperature of the oil is 15° F. (Mechanical equivalent of heat = 778 ft.-lb. per B.Th.U.)

3. Taking a pitch circle diameter of 4 in., draw two cycloidal teeth of 1 in. circular pitch, addendum ·3 in., dedendum ·35 in. (*a*) Rolling circle ¾ in. diameter. (*b*) Rolling circle 1¼ in. diameter. (*c*) Rolling circle 2 in. diameter.

4. Taking a pitch circle diameter of 4 in., draw two involute teeth of 1 in. circular pitch, addendum ·3 in., dedendum ·35 in. (*a*) Pressure angle 15°, (*b*) Pressure angle 22½°.

5. A wheel with 18 teeth of standard proportions, 4 D.P., is to gear with one of 38 teeth. Calculate (*a*) the distance between centres, (*b*) the diameters of the blanks from which the wheels will be cut. What is the smallest pressure angle below which interference is likely to occur?

6. Explain why practically all cut teeth are now of involute form, and state the advantages and disadvantages of this form of tooth as compared with the cycloidal form. Why is it that small inaccuracies in cutting affect the strength of the tooth more at high speeds than at low speeds?

7. Two wheels with standard involute teeth, 5 D.P., are to gear together with a velocity ratio of 4·5, the pressure angle being 15°.

(*a*) Find the least permissible number of teeth in the pinion if interference is not to occur.

(*b*) If the distance between the centres is to be as small as possible, find suitable numbers of teeth for the two wheels and the distance between the wheel centres.

8. Repeat the calculations of question 7, taking the pressure angle to be 20°.

9. A spur wheel with straight and axial teeth is to be cut by means of a triple-threaded worm hob. The teeth are to be ¾ in. circular pitch, and the lead of the worm is to be 6 in. Calculate (*a*) the pitch diameter of the worm, (*b*) the angle at which the worm axis must be set for cutting the teeth.

10. Two shafts in the same plane, with axes inclined at 60°, are to be connected by bevel wheels with teeth of 4 D.P., the velocity ratio being 2·5 : 1. The smaller wheel is to have 20 teeth and the length of face is 2¼ in. Set out the wheels to scale, taking the standard proportions for involute teeth, and state (*a*) the semi-angles of the pitch cones, (*b*) the greatest diameter and cone angle for each blank, (*c*) the cone angle for the bottom of the tooth space for each wheel.

11. The following data are to be satisfied by a worm gear, slight variations being permissible—

Spiral angle of worm = 42°
Velocity ratio = 4·3
Normal pitch = 1¼ in.
Distance between centres = 6 in.

Find (*a*) the pitch diameter, lead, and number of threads per pitch of worm (*b*) pitch diameter of worm wheel.

[Let d = pitch diameter of worm

$$\frac{\text{lead}}{\pi d} = \tan 42° = \cdot 9$$

$$\therefore \text{lead} = 2{\cdot}83d$$

$$\therefore \text{Pitch circumference of wheel} = 4{\cdot}3 \times 2{\cdot}83d$$

$$= 12{\cdot}2d$$

$$\therefore \text{Pitch diameter of wheel} = 3{\cdot}87d = D$$

$$\therefore \frac{3{\cdot}87d + d}{2} = 6$$

$$\therefore \underline{d = 2{\cdot}46 \text{ in.}}$$

$$\underline{D = 3{\cdot}87 \times 2{\cdot}46 = 9{\cdot}54 \text{ in.}}$$

$$\therefore \underline{\text{lead} = 2{\cdot}83d = 6{\cdot}96 \text{ in.}}$$

$$\therefore \text{Threads per pitch} = \frac{\text{lead} \times \cos \theta}{\text{normal pitch}}$$

$$= \frac{6{\cdot}96 \times \cdot 743}{1{\cdot}25} = 4{\cdot}14$$

Taking 4 threads per pitch and lead 7 in.,

$$\therefore \text{Normal pitch} = \frac{7 \times \cdot 743}{4} = 1{\cdot}3 \text{ in.}$$

$$\text{and } \tan \theta = \frac{7}{3{\cdot}14 \times 2{\cdot}46} = \cdot 907$$

$$\therefore \theta = 42{\cdot}2°$$

No. of teeth in worm wheel = 4 × 4·3 = 17·2. The nearest whole number to this is 17, giving a slightly smaller P.C.D. to the wheel.]

12. The worm specified in question 11 is to transmit 35 h.p. at 1,800 r.p.m., the mechanical efficiency being ·92. Calculate (*a*) the axial thrust on the worm, (*b*) the axial thrust on the worm wheel, (*c*) the heat generated per minute due to friction. If the pressure angle of the teeth is 20°, calculate the radial thrust (neglecting friction) due to tooth pressure and the total radial load on the worm bearings.

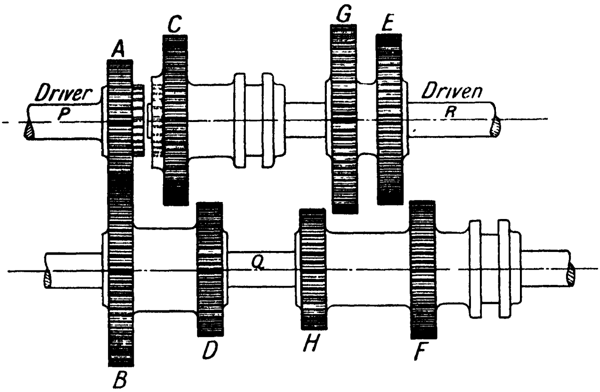


FIG. 192. FOUR-SPEED GEAR BOX

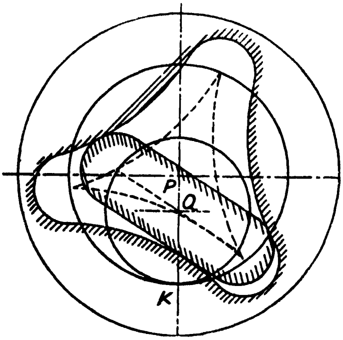


FIG 193 ROTARY PUMP

13. Fig. 192 shows the layout of a four-speed gear box. Wheel *A* is keyed rigidly to *P*. *C* is keyed to *R*, but can slide axially. *E* and *G* are keyed rigidly to *R*. *B* and *D* are keyed rigidly to *Q*. *F* and *H* are keyed to *Q*, but can slide axially.

All teeth are 5 D.P., and the four speeds are attained as follows: 1st speed, *G* and *H* in gear; 2nd speed, *E* and *F* in gear; 3rd speed, *C* and *D* in gear; 4th speed, *C* and *A* in gear (direct drive). The numbers of teeth in the wheels are *A*, 18; *B*, 40; *C*, 24; *E*, 32; *G*, 40. Calculate the speeds of revolution of *R* for each gear when *P* rotates at 1,000 r.p.m.

14. Referring to the type of gear-box in question 13, the distance between the centre lines of *P* and *Q* is to be as before, but a new set of gears is required, the lowest speed of *R* to be about 170 r.p.m. when the speed of *P* is 1,000 r.p.m., the speed ratios to be as nearly as possible in geometrical progression. Find the numbers of teeth required for the wheels.

Assuming that the shaft *P* transmits 18 h.p. at 1,000 r.p.m., calculate the torque on the shaft *R* for each gear, assuming a mechanical efficiency of ·92.

15. A rotary pump has two rotating members of the form shown in full lines in the figure (Fig. 193), which rotate together in a casing with constant

velocity ratio about the fixed centres P and Q. The two pitch circles, whose diameters are in the ratio 3 : 2, roll together, one within the other, as shown at K, without slipping. The actual working profiles shown are curves parallel to two base curves shown dotted, one of which is the diameter of the smaller pitch circle, and the other a 3-cusped hypocycloid within the larger pitch circle.

Show that these two base curves form two ideal profiles which will gear together with constant velocity ratio, the straight line always preserving contact with the three branches of the hypocycloid in the manner indicated. (U.L.)

16. The arrangement of a double reduction gearing set is shown in Fig. 194. Pinion A is 9 in. diameter, and transmits 4,000 h.p. at 2,200 r.p.m. Pinion B transmits the same power at 3,000 r.p.m. The pressure angle of the teeth is 20°. Determine the magnitudes and directions of the loads on the bearings for the main gear and the left-hand intermediate gear. (U.L.)

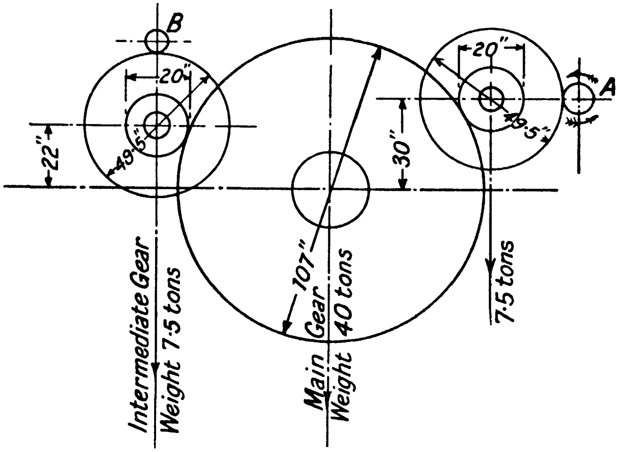


FIG. 194. DOUBLE REDUCTION GEAR

17. A worm reduction gear transmits 30 h.p., with speeds of worm and wheel at 1,600 and 40 r.p.m. respectively. The worm is $3\frac{1}{2}$ in. diameter, 1 in. pitch, single thread. The thrust is taken by a flat collar bearing, mean diameter 5 in. The coefficient of friction for worm and thrust may be taken as ·05. Determine the efficiency of the gear, and show whether the gear is self-locking if the drive is attempted from the wheel side. (U.L.)

18. Show that involute teeth satisfy the necessary conditions for constant velocity ratio between the axes. What is the smallest practicable number of involute teeth to be used on a gear-wheel, if the angle of obliquity of the thrust is limited to $14\frac{1}{2}$° ? (U.L.)

19. Obtain an expression for the velocity of rubbing between the teeth of a couple of spur wheels which gear together with constant velocity ratio. The expression is to be in terms of the velocity v of the pitch line, p the pitch of the teeth, d the distance of the point of contact of the teeth from the point of contact of the pitch circles, and n_1 and n_2 the numbers of teeth of the wheels. (U.L.)

20. The outline of the projection A (Fig. 195) is circular, and gears with

the outline of the recess *B*, the wheels rotating at the same speed in opposite directions. Find graphically the outline of the recess *B*, and explain the reasons for your construction. Find the greatest velocity of sliding between *A* and *B* when the wheels rotate at 120 r.p.m. (U.L.)

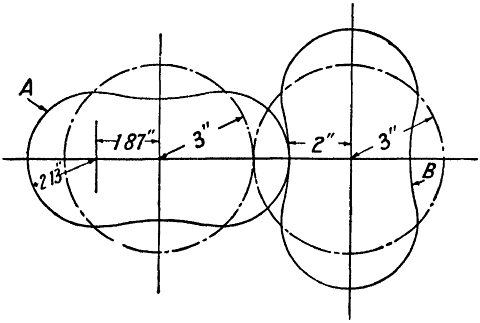


FIG. 195

21. The minute hand of a clock is attached to a spindle and the hour hand to a sleeve rotating loosely on the same spindle. Find a suitable train of wheels *A*, *B*, *C*, *D* to satisfy the requirements, all the teeth to have the same pitch. Each wheel must have more than 9 teeth. Make the total number of teeth as small as possible. (U.L.)

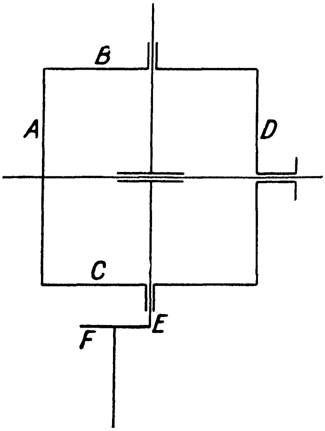


FIG. 196

22. Discuss the defects which lead to noisy working, vibration, or excessive wear of spur-wheel teeth.

23. Sketch any arrangement of change-speed gears for a motor-car which gives at least three speeds and a reverse.

24. Design a suitable train of wheels to give a velocity ratio of 84 : 1 between two parallel shafts. $N \nless 15$.

25. Show in a sketch two epicycloidal teeth gearing together with the teeth shown, (1) when beginning contact, (2) when ending contact. Show the complete locus of the point of contact, and the arcs of approach and recess. Find the value of the circular pitch in terms of the arc of action if one pair of teeth comes into operation just as another pair of teeth cease to make contact.

26. The differential gear shown in the figure (Fig. 196) consists of four bevel wheels, *A*, *B*, *C*, and *D*, of the same size. *A* is keyed to the main shaft, which rotates at 120 r.p.m. *B* and *C* are mounted loosely on an "arm" wheel *E*, which is free to rotate on the main shaft. *E*, which has 64 teeth, is driven by a bevel wheel *F*, which has 12 teeth. The speed of *F* is 800 r.p.m. *D* gears with *B* and *C*, and is also free to rotate on the main shaft. If the directions of motion of *A* and *E* are the same, determine the speed and the direction of rotation of wheel *D*. (I.C.E.)

27. In the epicyclic train shown in Fig. 197, the wheel D is fixed and the wheel A has an angular velocity Ω. If the centre of the compound wheel BC has an angular velocity ω, and if the velocity of rolling of C on A is ω_1, and that of B on D is ω_2, show that numerically

$$\frac{\Omega}{R_2 - R_1} = \frac{\omega\,(r_2 - R_2)}{R_1 r_2} = \frac{\omega_1}{R_2} = \frac{\omega_2}{R_1}$$

(I.C.E.)

[Assume that A is the first wheel and D is the last wheel in the train.] Then

$$\text{Value of train } (e) = \frac{\text{angular vel. of last wheel}}{\text{angular vel. of first wheel}} \text{ (arm fixed)}$$

$$\therefore\ e = \frac{R_1 \,.\, r_2}{r_1 \,.\, R_2} \text{ (negative)}$$

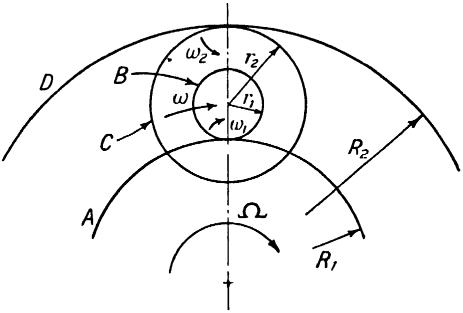


FIG. 197

$$\therefore\ \frac{0 - \omega}{\Omega - \omega} = -\frac{R_1 r_2}{r_1 \,.\, R_2}$$

$$\therefore\ \Omega = \frac{\omega(r_1 R_2 + R_1 \,.\, r_2)}{R_1 \,.\, r_2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From the geometry of the figure

$$R_2 - R_1 = r_1 + r_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\begin{aligned}\therefore\ r_1 R_2 + R_1 r_2 &= r_1 R_1 + r_1^2 + r_1 r_2 + R_1 r_2 \\ &= (r_1 + r_2)(R_1 + r_1) \\ &= (R_2 - R_1)(R_2 - r_2)\end{aligned}$$

$$\therefore\ \text{From (1)} \qquad \Omega = \frac{\omega(R_2 - R_1)(R_2 - r_2)}{R_1 \,.\, r_2}$$

$$\therefore\ \frac{\Omega}{R_2 - R_1} = \frac{\omega(R_2 - r_2)}{R_1 r_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Angular velocity of B relative to arm

$$= -\frac{R_1}{r_1} \times \text{angular vel. of } A \text{ relative to arm}$$

Angular velocity of A relative to arm $= \Omega - \omega$

∴ Angular velocity of B relative to A is

$$\omega_1 = -\frac{R_1}{r_1}(\Omega-\omega)-(\Omega-\omega)$$

$$= -\left(1+\frac{R_1}{r_1}\right)(\Omega-\omega) \quad . \quad . \quad . \quad . \quad (4)$$

From (1) $$\Omega-\omega = \frac{r_1R_2}{R_1r_2}\,.\,\omega$$

$$\therefore \omega_1 = -\frac{R_1+r_1}{r_1}\,.\,\frac{r_1R_2}{R_1r_2}\,.\,\omega$$

$$= -\frac{(R_2-r_2)\,.\,R_2}{R_1r_2}\,.\,\omega$$

$$\therefore \frac{\omega_1}{R_2} = -\frac{(R_2-r_2)\,.\,\omega}{R_1r_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Angular velocity of C relative to arm = angular velocity of B relative to arm

$$= -\frac{R_1}{r_1}(\Omega-\omega)$$

$$= -\frac{R_2}{r_2}\,.\,\omega$$

Angular velocity of D relative to arm $= -\omega$

∴ Angular velocity of C relative to D is given by

$$\omega_2 = -\omega\left(\frac{R_2}{r_2}-1\right)$$

$$\therefore \frac{\omega_2}{R_1} = -\frac{\omega(R_2-r_2)}{R_1r_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)]$$

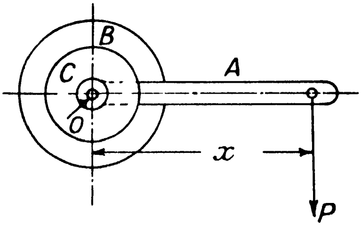


FIG. 198

28. Fig. 198 shows the scheme of an epicyclic train transmission dynamometer, A being an arm carrying a train of wheels, of which B is the driver and C the driven member. O is a fixed centre. The gear ratio of the train of wheels is such that when the arm A is fixed, C revolves twice in the opposite direction to B for one revolution of the latter. The power transmitted is measured by the moment of the force P about O, which is necessary to hold the arm at rest.

Given $P\,.\,x = 300$ lb.-ft., when C revolves at 400 r.p.m., calculate the power transmitted, neglecting friction.

If P acts in the direction shown, what are the directions of rotation of B and C? (U.L.)

29. A scheme for a differential hoisting mechanism is shown in Fig. 199. The $35T$ gear is fixed, the $31T$ gear is connected to the weight sprocket, and the 11 in. diameter hoisting chain sprocket carries the differential pinions, of which there are three sets. Determine the velocity ratio of the hoist, the direction of rotation of the weight sprocket, and the force between the gear teeth when a force of 40 lb. weight is applied to the hoisting chain.

The velocity ratio is regarded as rather too high ; what would be the effect of employing 34 and 30 teeth instead of 35 and 31 in the main gears ? (U.L.)

30. In the epicyclic reduction gear shown in Fig. 200, a shaft A is driven by an arm B, which is fixed to it. B has a pin fixed to its outer end, and two pinions. C, E, revolve together on this pin. C gears with an annular fixed wheel D, and E gears with a pinion F, which is driven by a belt pulley G. The numbers of teeth are as follows : $D = 80$; $C = 15$; $E = 24$; $F = 18$.

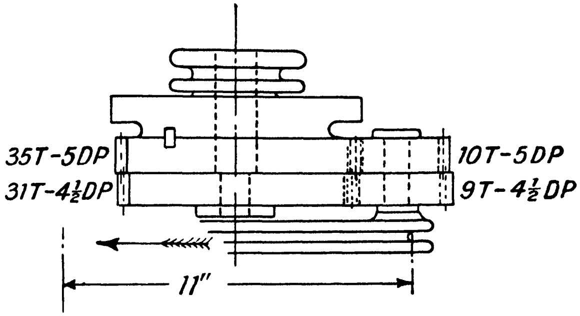


FIG. 199

If the pulley G runs at 240 r.p.m., find the speed of the shaft A. (L.U.)

31. Three spur wheels, A, B, and C in gear are mounted on an arm which makes 60 r.p.m. about the axis of A. A is fixed and has 120 teeth; C has 60 teeth. Find the direction and speed of rotation of C.

A pointer whose length is equal to the distance between the centres of A and C is mounted on the axis of C. Show that its free end moves in a straight line with simple harmonic motion. (U.L.)

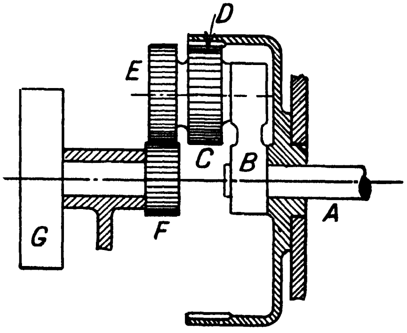


FIG 200. EPICYCLIC REDUCTION GEAR

32. A mechanism for recording the distance travelled by a bicycle is as follows : There is a fixed annular wheel A of 22 teeth, and another annular wheel B of 23 teeth, which revolves loosely on the axis of A. An arm, driven by the bicycle wheel through gearing not described, also revolves freely on the axis of A, and carries on a pin at its extremity two wheels C and D, which are attached to each other. The wheel C, with 19 teeth, meshes with A and the wheel D, with 20 teeth, meshes with B.

The diameter of the bicycle wheel is 28 in. What must be the velocity ratio between the bicycle wheel and the arm if B makes one revolution for each mile covered ? (U.L.)

33. Fig. 201 shows the principle of the epicyclic gear for low speed and reverse of a Ford car. The driving shaft P drives the arm K, and the wheels A, B, C, revolve together on the pin M, gearing with D, E, F respectively. For the low speed, brake drum H is fixed, G being free, and for the reverse G is

fixed, H being free. The numbers of teeth are as follows: A, 27; B, 33; C, 24; D, 27; E, 21; F, 30. Calculate the ratio of the speeds of revolution of P and Q for each gear.

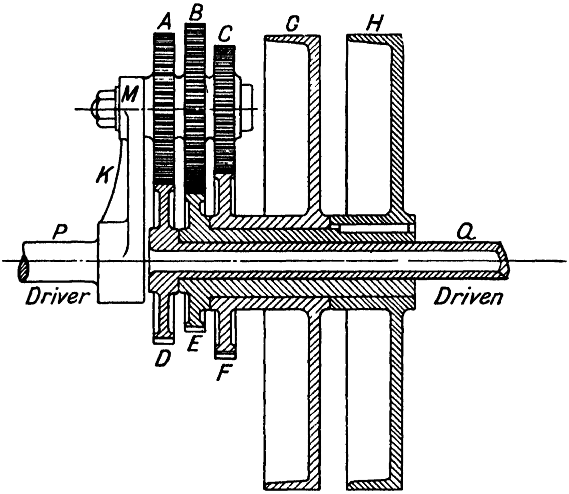


FIG. 201. FORD SPEED GEAR

34. Fig. 202 shows the principle of the compound epicyclic gear used for the Lanchester car (practical details, such as roller bearings, etc., are omitted). The shaft and sleeve A, with external teeth T, drives wheels B and C.

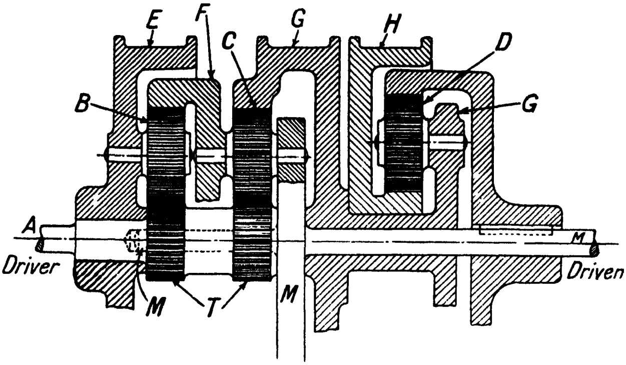


FIG. 202. SCHEME OF LANCHESTER DIFFERENTIAL GEAR

B rotates on a pin attached to E, and drives F. C rotates on a pin attached to both F and M, and drives G. D rotates on a pin attached to G as shown and gears with both H and K. K is keyed to the driven shaft M. Each sun wheel has 28 teeth, and each ring has 80 teeth, the intermediate wheels B, C, and D being all of the same size.

Calculate the ratios of the speeds of A and M when E, G, and H are fixed successively.

[If E is fixed, we get the train A, B, F, C, M.

R.p.m. of F = r.p.m. of $M = -\frac{28}{80} \times$ r.p.m. of A

$$\therefore \frac{\text{r.p.m. of } A}{\text{r.p.m. of } M} = -\frac{80}{28} = -2\tfrac{6}{7} \text{ (reverse)}$$

(2) If G is fixed, we get the train A, C, M.

Taking A as first wheel, M as arm, and G as last wheel, n = r.p.m. of A.

$$\varepsilon = -\frac{28}{80} \quad \therefore \frac{0-a}{n-a} = -\frac{28}{80} \quad \therefore \frac{n}{a} = \frac{108}{28} = 3\tfrac{6}{7}$$

$$\therefore \frac{\text{r.p.m. of } A}{\text{r.p.m. of } M} = 3\tfrac{6}{7}$$

(3) If H is fixed, we get the train A, C, G, D, K.

Considering the train K, D, H, let a = r.p.m. of K = r.p.m. of M; = r.p.m. of G; H = last wheel.

$$\varepsilon = -\frac{80}{28}; \frac{0-b}{a-b} = -\frac{80}{28}. \quad \therefore b = \frac{20}{27}a$$

Considering the train G, C, A. R.p.m. of $A = n$ (last wheel); r.p.m. of $G = \frac{20}{27} \cdot a$; r.p.m. of $M = a$. $e = -\frac{80}{28}$

$$\therefore \frac{n-a}{\frac{20}{27}a - a} = -\frac{80}{28}$$

$$\therefore n = \frac{47}{27} \cdot a \quad \therefore \frac{\text{r.p.m. of } A}{\text{r.p.m. of } M} = \frac{47}{27}$$]

35. In the epicyclic gear shown (Fig. 202A) the wheel B, having 30 teeth, 1¼ in. pitch, is geared to D through the intermediate wheel C which is attached to the arm A. When the wheel B makes 42 r.p.m. in an anticlockwise direction and the arm A makes 6 r.p.m. in an anticlockwise direction, the wheel D is required to make 14 r.p.m. in a clockwise direction.

Find the necessary number of teeth in wheels C and D and the pitch-circle diameter of D. (U.L.C.I.).

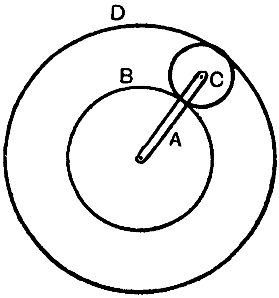


FIG. 202A

CHAPTER X

HIGHER PAIRING *(continued)*—CAMS

168. Cams. The object of a cam is to impart reciprocating motion of a specified character to some portion of a mechanism. There are three elements in the motion, the importance of these depending largely upon the rapidity with which the sequence of operations of the cam is repeated. These are—

1. The displacement of the driven member at any instant
2. The velocity at any instant.
3. The acceleration at any instant.

In the case, for instance, of a cam operating a valve of a high-speed petrol engine, the velocity of the valve gives the rate of increase or decrease of opening and the rubbing velocity of the spindle in its guide, while the force acting at any instant is proportional to the acceleration.

With cams having a slow speed of rotation and without rapid changes of contour, the accelerations are of small importance.

The types of cams and cam followers are so numerous that it will be necessary to confine ourselves in this chapter to the simpler cases of "edge" cams and their analysis.

There are two cases to consider, viz. (1) given the nature of the motion required, to determine the necessary contour of the cam, and (2) given the contour of the cam, to construct curves showing the displacement, velocity, and acceleration of the driven member.

Case 1. To determine the shape of cam to give any desired motion. In this case it is desirable to set out a curve the ordinates of which represent the lift of the follower, and abscissae the corresponding angular displacements of the cam-shaft (Fig. 203). The slope of this curve at any point is proportional to the velocity of the follower. Thus, if h is in inches, and θ is measured in degrees, the velocity in ft. per sec. at any point is

$$v = \cdot 5N \cdot \frac{dh}{d\theta}$$

where N is the speed in revolutions per minute of the cam-shaft.

Proof of the above—

If h is in feet, θ in radians, t in seconds, and v in feet per second, then

$$v = \frac{dh}{dt} = \frac{dh}{d\theta} \cdot \frac{d\theta}{dt} = \omega \cdot \frac{dh}{d\theta}$$

If N is the speed in revolutions per minute, then $\omega = \frac{2\pi N}{60}$.

If θ is measured in degrees, then for $d\theta$ we must write $\frac{\pi}{180} \cdot d\theta$.

If h is measured in inches, then for dh we must write $\frac{dh}{12}$.

In this case, the above equation becomes

$$v = \frac{2\pi N}{60} \cdot \frac{dh}{12} \cdot \frac{180}{\pi \, . \, d\theta} = 0{\cdot}5 \, N \cdot \frac{dh}{d\theta}$$

$\frac{dh}{d\theta}$ at any point is measured by drawing a tangent to the displacement curve (Fig. 203) and measuring its slope. For acceleration, we have

$$\begin{aligned} a &= \frac{dv}{dt} = \frac{dv}{d\theta} \cdot \frac{d\theta}{dt} = \omega \cdot \frac{dv}{d\theta}, \ \theta \text{ in radians} \\ &= \frac{2\pi N}{60} \cdot \frac{180}{\pi} \cdot \frac{dv}{d\theta}, \ \theta \text{ in degrees} \\ &= 6N \, . \, \frac{dv}{d\theta}, \ \theta \text{ in degrees} \end{aligned}$$

$\frac{dv}{d\theta}$ at any point is measured by drawing a tangent to the velocity curve (Fig. 204) and measuring its slope.

Otherwise, $\quad v = {\cdot}5N \frac{dh}{d\theta}$

and $\quad a = 6N \frac{dv}{d\theta} = 3N^2 \frac{d^2h}{d\theta^2}$

Values of $\frac{dh}{d\theta}$ and $\frac{d^2h}{d\theta^2}$ may be found by the methods of Arts. 23, 24 and 25 and the values of v and a calculated from these.

Fig. 204 is a velocity curve derived from Fig. 203 and Fig. 205 is the acceleration curve derived from Fig. 204. It will be noticed that the changes in acceleration are very abrupt, and that the acceleration at 360° would be infinite if the follower were infinitely rigid. As the acceleration at any instant is proportional to the square of the angular velocity, it will be evident that Fig. 203 would be an unsuitable curve for a high-speed cam.

The procedure in setting out the cam would be as follows—

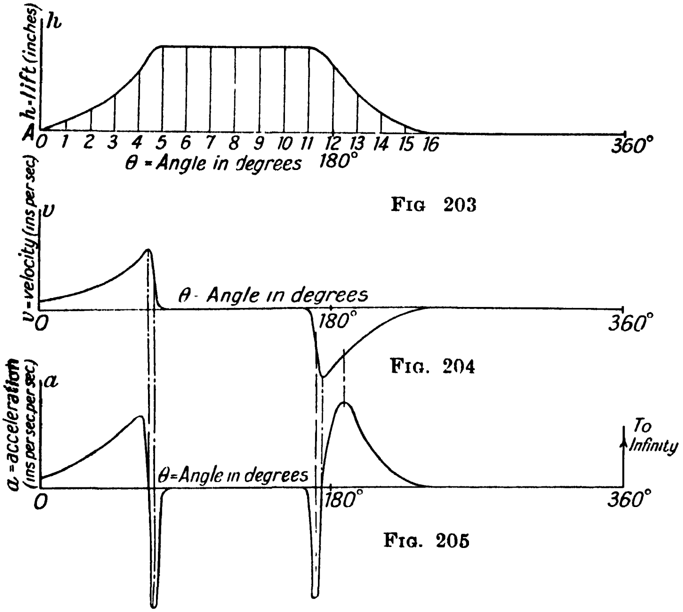


FIG 203

FIG. 204

FIG. 205

LIFT, VELOCITY, AND ACCELERATION GRAPHS

Set up ordinates along *AB* (Fig. 203) at suitable distances (say 15°) apart. Select a radius *CO* for the base circle (Fig. 206) and draw radii *C*0, *C*1, *C*2, etc., at 15° apart. Beyond the base circle, and on the corresponding radii, set out the lengths of the ordinates at 1, 2, 3, etc., and complete the contour as shown in Fig. 206. This gives the curve for a cam, the follower of which makes point contact with the cam surface, the axis of the follower passing through the centre of the cam base circle.

If the follower has a roller at its end in contact with the cam, then the full curve shows the path of the centre of the roller, and the dotted curve, which is the envelope of the small circles shown, shows the correct shape of the cam, the diameter of each of the small circles being that of the roller.

169. **Offset Cam.** Fig. 207 shows how the construction must be modified if the line of stroke *PP′* of the follower passes a distance *a* to the right of the centre of the cam base circle. If an inner circle of radius *a* be drawn, then distances corresponding to ordinates of Fig. 203 must be measured along *tangents* to this inner circle, and the construction will then be as follows—

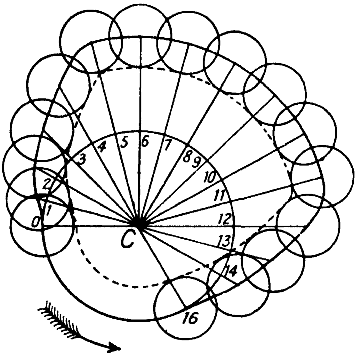


FIG. 206. DIRECT LIFT CAM

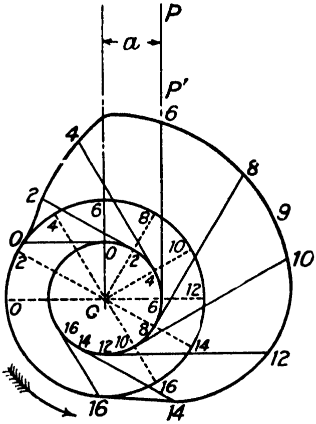


FIG. 207. OFFSET CAM

Set out $C0$, $C1$, $C2$, etc., as in Fig. 206 (intermediate lines have been omitted from Fig. 207 for the sake of clearness). Draw 00, 22, 44. etc., tangential to the inner circle and parallel to $C0$, $C2$. $C4$, etc., respectively, making the distance from the outer base circle to the curve, measured along the tangent, equal to the ordinate of corresponding number in Fig. 203. The resulting curve gives the path of the centre of the roller at the contact end of *PP′*, and the correct cam contour is obtained in the same way as in Fig. 206, the radius of each of the small circles being the radius of the roller.

170. **Given the Shape of the Cam to find the Motion of the Driven Member.** (*Case* 2, Art. 168.) The analysis of the motion of a follower derived from a cam of given contour may be carried out by the aid of drawing paper and tracing paper, as follows (Fig. 208)—

Set out on the drawing paper the contour of the cam, and on the tracing paper draw a line *PQ* and a circle of the radius of the roller having its centre on *PQ*. Divide the base circle into equal angular divisions by radial lines *O*1, *O*2, *O*3, etc. For any radius (e.g. *O*5), set the tracing paper so that *PQ* coincides with *O*5, and the circle touches the cam contour. Mark the position of *O* on *PQ*. In this way, *PQ* will have a series of marks indicating actual distances moved by the follower for equal angular displacements of the cam, and a diagram such as Fig. 203 can be constructed and analysed. If the direction of rotation of the cam is clockwise, *PQ* should be rotated in an anti-clockwise direction.

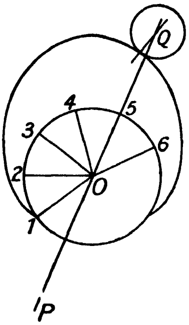


Fig. 208

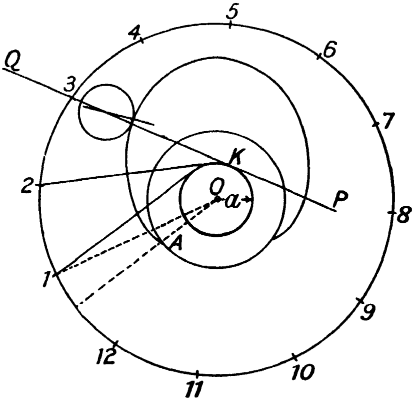


Fig. 209

In the case where there is an offset *a* (Fig. 209), *PQ* must be tangential to the inner circle, and some modification in the method is necessary to ensure reasonable accuracy. With centre *O* draw a circle of large radius. If *A* is the starting point on the cam, draw the radius *OA* and draw a line parallel to *OA* and tangential to the inner circle, cutting the outer large circle at 1. From 1 divide the circumference of the outer circle into equal divisions, 12, 23, 34, etc. Then, for position 3, the line *QP* on the tracing paper must pass through 3 and touch the inner circle, the point of contact *K* being marked on *PQ*. A series of marks will thus be obtained and the displacement diagram can be drawn as before. A series of lines on the tracing paper, perpendicular to *PQ*, will facilitate the determination of *K*.

171. Common Types of Cams. Two types of cams which are of frequent occurrence in practice are shown in Figs. 210 and 211. r_1 is the radius of the base circle, and r_2 is that of the "nose" circle, their centres being a distance d apart. In Fig. 210 these circles are connected by tangents, as shown. and in Fig. 211 they are connected by circular arcs, the latter type being used with a flat-footed follower. The two quantities which are usually specified in each case are (1) the lift l, (2) the total angle of action 2α. In Fig. 210 the maximum obliquity of action between the cam and the roller is of importance, as this determines the thrust at right-angles to the direction of motion of the follower.

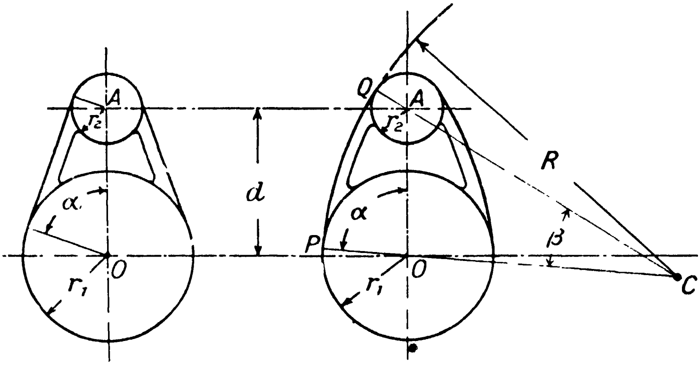


FIG. 210. STRAIGHT-SIDED CAM

FIG. 211. CAM CONTOUR MADE UP OF CIRCULAR ARCS

172. Straight-sided Cam (Fig. 212). Let $BN = r_o$ be the radius of the roller on the follower, and let θ be the angular distance moved by OM from its vertical position (corresponding to the lowest position of the follower).

Since $AO = d$, the total lift is given by

$$l = d - (r_1 - r_2) \qquad (X.1)$$

and since

$$r_1 - r_2 = d \cos \alpha \qquad (X.2)$$

$$\therefore l = d(1 - \cos \alpha) \qquad (X.3)$$

Let y = lift of follower from the lowest position.

Then

$$y = OB - (OM + BN)$$

$$= \frac{r_1 + r_o}{\cos \theta} - (r_1 + r_o)$$

$$\therefore y = (r_1 + r_o)(\sec \theta - 1) \qquad (X.4)$$

Velocity of lift, $v = \frac{dy}{dt} = (r_1 + r_o) \, . \sec \, \theta \, . \tan \theta \, . \frac{d\theta}{dt}$

$$\therefore v = \omega(r_1 + r_o) \, . \sec \theta \, . \tan \theta \quad . \quad \text{(X.5)}$$

where ω is the angular velocity of the cam.

Acceleration of follower,

$$f = \omega(r_1 + r_o) \, (\sec^3\theta + \sec \theta \, . \tan^2\theta) \, . \frac{d\theta}{dt}$$

$$\therefore f = \omega^2(r_1 + r_o) \, . \frac{1 + 2 \tan^2 \theta}{\cos \theta} \quad . \quad . \quad . \quad \text{(X.6)}$$

When $\theta = 0$, $f = \omega^2(r_1 + r_o)$, so that there is a sudden acceleration of $\omega^2(r_1 + r_o)$ applied to the follower as soon as the roller passes on to *M*. For this and other reasons, this form of cam is undesirable for high-speed work.

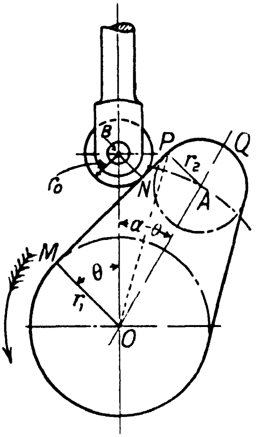


FIG. 212. STRAIGHT-SIDED CAM WITH ROLLER ON FOLLOWER

The above equations only apply while the roller is on the portion *MP* of the cam. The following is the method of finding the angle turned through by *OM* while the roller moves from *M* to *P*—

From Fig. 212, $MP = d \, . \sin \alpha$ (X.7)

also $MN = (r_1 + r_o) \tan \theta$ (X.8)

Let $\theta = \theta_1$ when *N* is at *P*. Then from (X.8),

$$(r_1 + r_o) \, . \tan \theta_1 = d \, . \sin \alpha$$

$$\therefore \tan \theta_1 = \frac{d}{r_1 + r_o} \, . \sin \alpha \quad \text{(X.9)}$$

From (X.2), $r_2 = r_1 - d \cos \alpha \quad . \quad . \quad . \quad$ (X.10)

When the roller is on the nose of the cam, we have the equivalent of a crank of radius d and a connecting rod of length $r_o + r_2 = r_o + r_1 - d \cos \alpha$, the ratio of crank to connecting-rod being

$$\frac{r_o + r_1}{d} - \cos \alpha$$

EXAMPLE.

Given $r_1 = 1{\cdot}5''$; $r_o = 1''$; $a = 60°$; total lift of follower $= 0{\cdot}9''$; speed of revolution of cam-shaft 300 r.p.m. ($\omega = 31{\cdot}4$ rad. sec.).

Then, from (X.3), $d(1 - {\cdot}5) = {\cdot}9$

$$\therefore \underline{d = 1{\cdot}8''}$$

From (X.10), $r_2 = 1{\cdot}5 - (1{\cdot}8 \times {\cdot}5) = \underline{6''}$

From (X.9), $\tan\theta_1 = \dfrac{1{\cdot}8}{1{\cdot}5 + 1} \times {\cdot}866 = {\cdot}623$

$$\therefore \theta_1 = 32^\circ$$

$$\alpha - \theta_1 = 28^\circ$$

Thus the flat side of the cam is in action for 32°, and the nose is responsible for the remainder of the upward travel.

From (X.4), lift from M to $P = 2{\cdot}5\ (1{\cdot}18 - 1) = {\cdot}45''$

$\therefore$ Lift from P to $Q = {\cdot}9 - {\cdot}45 = {\cdot}45''$

From (X.6), acceleration at $M(\theta = o)$

$$= \omega^2 \times \frac{1{\cdot}5 + 1}{12}$$

$$= {\cdot}21\omega^2 \text{ ft.-sec.}^2$$

$$= 207 \text{ ft.-sec.}^2$$

Also, acceleration at P $(\theta = 32^\circ)$

$$= {\cdot}21\omega^2 \times \frac{1 + 2 \times ({\cdot}623)^2}{{\cdot}85}$$

$$= {\cdot}441\omega^2$$

$$= 434 \text{ ft.-sec.}^2$$

The acceleration for points of contact between P and Q can be obtained by means of acceleration image diagrams (Chapter V), from which it will be found that there is a sudden change of acceleration at P from $+\,434$ ft.-sec.^{2} to $-\,310$ ft.-sec.^{2}, changing gradually to $-\,315$ ft.-sec.^{2} at Q. The downward spring pressure on the follower at P must thus be sufficient to cause a downward acceleration of 310 ft.-sec.^{2} if the roller is not to leave the cam at P. If the total weight moved by the cam is 0·5 lb., the downward force required in this case will be $\dfrac{{\cdot}5}{32} \times 310 = 4{\cdot}92$ lb. This force will be proportional to the square of the speed of revolution of the cam, e.g. at 1,800 r.p.m. it would be $4{\cdot}92 \times 36 = 177$ lb.

Side Thrust with Straight-sided Cam. If F is the downward thrust exerted by the follower in the direction BO (Fig. 212), the pressure between the cam and the roller, which is in the direction BN, will be $\frac{F}{\cos\,\theta}$, and the side-thrust on the follower guide will be $F\,.\tan\,\theta$.

To avoid this side-thrust (1) a swinging link is sometimes interposed between the cam and the follower (Fig. 213), (2) a flat-footed follower is used with a cam of the form of Fig. 211. as in Fig. 214.

173. **Cam with Swinging Link** (Fig. 213). The construction for finding the motion of the follower in this case depends upon the facts that for all positions of the cam the angle AOB is constant, and that when the cam moves through an angle AOA' the lift of the follower is $OA' - OA$.

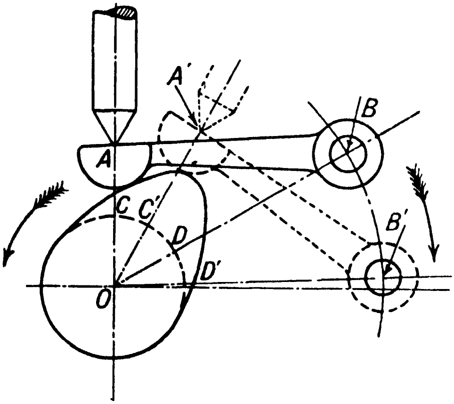


Fig 213 Cam with Swinging Link

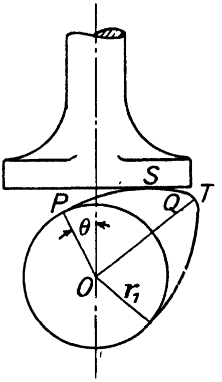


Fig. 214. Cam with Flat-footed Follower

With centre O, radius OB, describe a circle. Set out the angle $AOA' = COC'$ to represent the angular distance moved by the cam, and set out the angle $DOD' = BOB' = COC'$. Take a tracing of the link AB, and set this so that B coincides with B' and the other end of the link touches the cam. Prick through the point A' on OC' produced on to the drawing paper. Repeat this procedure for different angles during one complete revolution of the cam. We may thus obtain a curve showing the distances of A from O for all positions of the cam, and the displacements of the follower can be measured from this curve. The method is applicable to any form of link AB.

174. Cam with Flat-footed Follower. With a flat-footed follower (Fig. 214), β is the angle turned through by the cam while the point of contact moves from P to Q, since the change of inclination of the tangent from P to Q is the angle between the radii at P and Q. The portion PQ of the cam face (Figs. 211 and 214) may be regarded as a portion of an eccentric of "throw" $R - r_1$, R and r_1 having the meanings shown in Fig. 211, so that the lift of the follower for an angular movement θ of the cam from the zero position is given by

$$x = (R - r_1) \,.\, (1 - \cos\theta) \qquad \text{(X.11)}$$

the total lift from P to Q being

$$l_1 = (R - r_1) \,.\, (1 - \cos\beta) \qquad \text{(X.12)}$$

The maximum lift of the follower is

$$l = d - (r_1 - r_2) \qquad \text{(X.13)}$$

From (X.12), the velocity of lift for contact between P and Q is

$$v = \frac{dx}{dt} = \omega(R - r_1)\sin\theta \qquad \text{(X.14)}$$

$$\text{Acceleration} = \frac{dv}{dt} = \omega^2 \,.\, (R - r_1) \,.\, \cos\theta \qquad \text{(X.15)}$$

$$\text{Initial acceleration (at } P) = \omega^2 \,.\, (R - r_1) \qquad \text{(X.16)}$$

When the nose of the cam is in contact, the lift for an angle θ is, from Figs. 211 and 214,

$$x = d \,.\, \cos(\alpha - \theta) + r_2 - r_1 \qquad \text{(X.17)}$$

$$v = \frac{dx}{dt} = d \,.\, \omega \,.\, \sin(\alpha - \theta) \qquad \text{(X.18)}$$

$$a = \frac{dv}{dt} = -d \,.\, w^2 \,.\, \cos(\alpha - \theta). \qquad \text{(X.19)}$$

To find the radius of the arc PQ in the cam of Fig. 211. $AC = R - r_2$; $AO = d$; $CO = R - r_1$.

From the properties of the triangle AOC,

$$\cos\alpha = \frac{\overline{AC}^2 - \overline{AO}^2 - \overline{CO}^2}{2\overline{AO} \,.\, \overline{CO}}$$

$$\therefore \cos a = \frac{(R - r_2)^2 - d^2 - (R - r_1)^2}{2(R - r_1) \,.\, d}$$

$$\therefore\ 2R(r_1 - r_2) - (r_1^2 - r_2^2) - d^2 = 2R \,.\, d \,.\, \cos\alpha - 2r_1 \,.\, d \,.\, \cos\alpha$$

$$\therefore\ R = \frac{r_1^2 - r_2^2 + d^2 - 2r_1 d\cos\alpha}{2(r_1 - r_2 - d\cos\alpha)} \quad . \quad . \quad (X.20)$$

Since $\dfrac{\sin\beta}{\sin\alpha} = \dfrac{AO}{AC} = \dfrac{d}{R - r_2}$

$$\therefore\ \sin\beta = \frac{d}{R - \mathrm{r}_2}\,.\,\sin\alpha \quad . \quad . \quad . \quad . \quad (X.21)$$

EXAMPLE.

Given $r_1 = 1''$; $r_2 = \frac{1}{4}''$; $d = 1\frac{1}{4}''$; $\alpha = 75°$; speed of cam = 600 r.p.m. $\omega = 62{\cdot}8$ rad. per sec.)

From (X.20). $R = \dfrac{1 - \frac{1}{16} + 1\frac{9}{16} - 2{\cdot}5 \times {\cdot}26}{2(1 - \frac{1}{4} - 1{\cdot}25 \times {\cdot}26)} = 2{\cdot}15''$

From (X.21), $\sin\beta = \dfrac{1{\cdot}25}{1{\cdot}9} \times {\cdot}97 = {\cdot}64$

$$\therefore\ \beta = 40° \text{ (approx.)};\ \cos\beta = {\cdot}77$$

From (X.12), lift from P to Q = $1{\cdot}15 \times {\cdot}23 = {\cdot}265''$

From (X.13), total lift = $1{\cdot}25 - {\cdot}75 = {\cdot}5''$

∴ The nose is responsible for ·235″ of the lift (47 per cent).

From (X.16), initial acceleration (at P)

$$= \frac{3930 \times 1{\cdot}15}{12} = 377 \text{ ft.-sec.}^2$$

From (X.15) acceleration at Q ($\theta = 40°$)

$$= \frac{3930 \times 1{\cdot}15 \times {\cdot}77}{12} = 290 \text{ ft.-sec.}^2$$

When the nose is acting—

From (X.19), acceleration at Q ($\theta = 40°$)

$$= -\frac{1{\cdot}25 \times 3930 \times {\cdot}82}{12} = -385 \text{ ft.-sec.}^2$$

Acceleration at $T = -d\,.\,\omega^2 = -410$ ft.-sec.^{2}

Note the sudden change of acceleration at Q.

175. **Modification of above type of Cam for Pause at Top of Stroke (Flat-footed Follower).** By separating the two halves of the cam (Fig. 211) by an interval β (Fig. 215), a pause is obtained from D to D'. If the lift of the follower is

to commence at the same point as before, the equations for the cam with no pause (Fig. 214) must be modified by substituting $\alpha - \frac{\beta}{2}$ for α in each case.

Thus, taking the data of the previous example, and making $\beta = 40°$, so that $\alpha - \frac{\beta}{2} = 55°$, we have

$$R = \frac{1 - \frac{1}{16} + \frac{9}{16} - 2{\cdot}5 \times {\cdot}574}{2(1 - \frac{1}{4} - 1{\cdot}25 \times {\cdot}574)} = 16{\cdot}2''$$

This is an impracticably large radius, as the accelerations at *A* and *B* would be much too large, and the angle *ACA′* would in practice be reduced by the value of β, and, if necessary, a slight clearance allowed between the foot and the portion *AA′* at the bottom of the stroke. This, of course, limits the application of this type of cam in practice. If the cam revolves at the same speed as before the accelerations at points between *A* and *D* will not be altered by the separation of *CD* and *CD′* if the angle *DCA* is not altered.

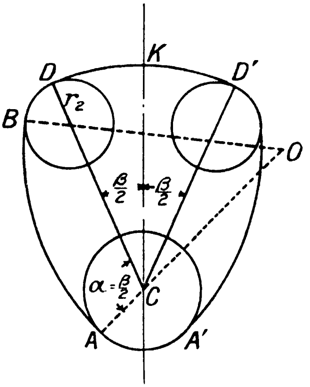


FIG. 215. CAM GIVING PAUSES AT ENDS OF STROKE

Fig. 216 is a special case of Fig. 215, in which a cam is arranged to give a positive drive in both directions of motion of the follower, which is a rectangular yoke in this case. The radius of each of the joining curves is $d + r$, the triangle *ABC* is equilateral, and the minimum width a of the yoke is $2(d + r)$. The angle a in this case is 150° and β is 60°. If the cam rotates in an anti-clockwise direction, and we start from the position when *P′* is in contact with the lower surface of the yoke, the motion is as follows: pause for 60°, from *P′* to *P*; downward movement $\frac{d}{2}$ with simple harmonic motion for 60° from *P* to *Q*; further downward movement of $\frac{d}{2} + r$ with simple harmonic motion for 60° from *Q* to *R*; pause for 60°; upward movement in reverse order.

The initial acceleration at P will be $\omega^2 . d$, and the acceleration at Q will be $\omega^2 . d . \cos 60° = \cdot 5\omega^2 . d$, changing to zero. The acceleration from Q to R will diminish from zero to $-\omega^2 . d . \cos 30°$. When contact is at K, CN is horizontal, and the yoke is at a distance $\overline{KN} - r$ from an extreme position, where KNB is the normal and $\widehat{KNC}$ is a right-angle.

176. **Choice of Displacement Curve for High-speed Cams.**

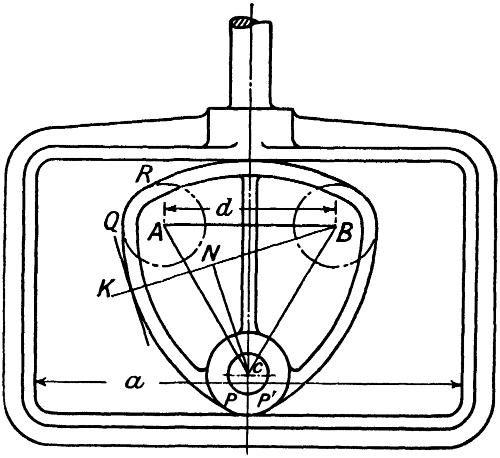


FIG. 216. CAM GIVING POSITIVE DRIVE IN BOTH DIRECTIONS

In settling the contour of a high-speed cam, it is important to avoid, as far as possible, sudden changes of acceleration, if reasonably silent working is to be ensured. At any point of the motion where the acceleration of the follower will be downward if it remains in contact with the cam face, the axial force exerted on the follower by the spring must not be less than the force required to produce such acceleration if the follower is not to separate from the cam. If the displacement curve chosen has a reasonably simple equation, the accelerations can be calculated, and there is the further advantage that the generation of the cam contour in special machines is simplified.

Let y = displacement of follower from lowest position for an angular displacement θ (radians) of the cam

ω = angular velocity of cam (radians per second)

If the equation of the displacement curve is

$$y = a \, . \, \theta^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (X.22)$$

in which a and θ are both positive,

then $$v = \omega \, . \, \frac{dy}{d\theta} = \omega \, . \, n \, . \, a \, . \, \theta^{n-1}$$

$$= \omega \, . \, n \, . \, \frac{y}{\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (X.23)$$

$$f = \text{acceleration} = \omega^2 \, . \, \frac{d^2y}{d\theta^2}$$

$$= \omega^2 \, . \, n \, . \, (n-1) \, . \, a \, . \, \theta^{n-2} \quad . \quad . \quad . \quad . \quad . \quad (X.24)$$

If $n = 2$, the curve is a parabola and the acceleration will be constant throughout.

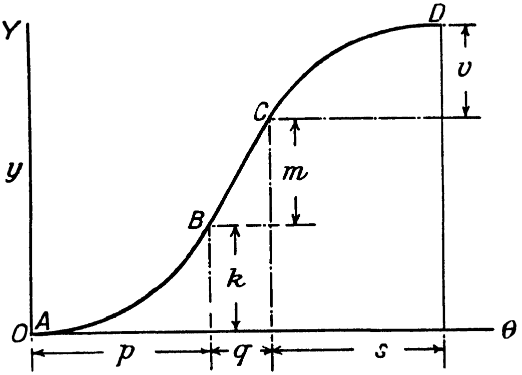


FIG. 217. DISPLACEMENT CURVE

If n is greater than 2, the initial acceleration ($\theta = o$) will be zero and the acceleration will increase as θ increases. n must not be less than 2 if f is to have a finite value when $\theta = o$.

Let $ABCD$ (Fig. 217) be a displacement curve of which AB and CD have equations of the form $y = a \, . \, \theta^n$ (see below, equation of CD), and BC is a straight line. The slope of AB at B and the slope of CD at C are each the same as the slope of BC, which is $\frac{m}{q}$.

Equation of AB.

When $\theta = p$, $y = k$. $$\therefore y = \frac{k}{p^n} \, . \, \theta^n \quad . \quad . \quad (X.25)$$

When $\theta = p, \dfrac{dy}{d\theta} = \dfrac{m}{q}$. $\quad \therefore n \,.\, \dfrac{k}{p} = \dfrac{m}{q}$

$$\therefore n = \frac{m}{q} \,.\, \frac{p}{k} \qquad . \quad . \quad (X.26)$$

Equation of CD.

In this case D is taken as the origin, so that y is measured downward from D, and θ is measured to the left from D.

When $\theta = s, y = v$. $\quad \therefore y = \dfrac{v}{s^n} \,.\, \theta^n \qquad . \quad . \quad$ (X.27)

When $\theta = s, \dfrac{dy}{d\theta} = \dfrac{m}{q}$. $\quad \therefore n \,.\, \dfrac{v}{s} = \dfrac{m}{q}$

$$\text{i.e. } n = \frac{m}{q} \,.\, \frac{s}{v} \qquad . \quad . \quad (X.28)$$

From (X.24), acceleration at B

$$= \omega^2 \,.\, n \,.\, (n-1) \,.\, \frac{k}{p^n} \,.\, p^{n-2}$$

$$= w^2 \,.\, n\,(n - 1) \,.\, \frac{k}{p^2}, \text{ where } n = \frac{m}{q} \,.\, \frac{p}{k} \qquad . \quad (X.29)$$

This changes to zero along BC.

At C the acceleration will decrease from zero to

$$f = -\,\omega^2 \,.\, n(n-1) \,.\, \frac{v}{s^2}, \text{ where } n = \frac{m}{q} \,.\, \frac{s}{v} \qquad . \quad (X.30)$$

If p, q, and s are measured in degrees, then, in the above equations, for p, q, and s we must write $\dfrac{p}{57{\cdot}3}$ $\dfrac{q}{57{\cdot}3}$ and $\dfrac{s}{57{\cdot}3}$ respectively.

EXAMPLE.

Rise of ·34″ for 65°, ·39″ for 35°, ·27″ for 80°.

In this case $k = {\cdot}34''$; $m = {\cdot}39''$; $v = {\cdot}27''$

$$p = \frac{65}{57{\cdot}3};\quad q = \frac{35}{57{\cdot}3};\quad s = \frac{80}{57{\cdot}3}$$

$$= 1{\cdot}136 \qquad = {\cdot}61 \qquad = 1{\cdot}4$$

For curve AB, $n = \dfrac{{\cdot}39}{{\cdot}61} \times \dfrac{1{\cdot}136}{{\cdot}34} = 2{\cdot}13$

$\therefore$ equation of AB is $y = \dfrac{{\cdot}34}{(1{\cdot}136)^{2{\cdot}13}} \,.\, \theta^{2{\cdot}13} = {\cdot}276 \,.\, \theta^{2{\cdot}13}$

For curve CD $\qquad n = \frac{\cdot 39}{\cdot 61} \times \frac{1\cdot 4}{\cdot 27} = 3\cdot 3$

$\therefore$ Equation of CD is $y = \frac{\cdot 27}{(1\cdot 4)^{3\cdot 3}} \,.\, \theta^{3\cdot 3} = \cdot 089 \,.\, \theta^{3\cdot 3}$

Change of acceleration at $B = \omega^2 \times 2\cdot 13 \times 1\cdot 13 \times \frac{\cdot 34}{(1\cdot 136)^2}$

$= \cdot 635\omega^2$ in. sec.^{2}

Change of acceleration at $C = \omega^2 \times 3\cdot 3 \times 2\cdot 3 \times \frac{\cdot 27}{(1\cdot 4)^2}$

$= 1\cdot 046\omega^2$ in. sec.^{2}

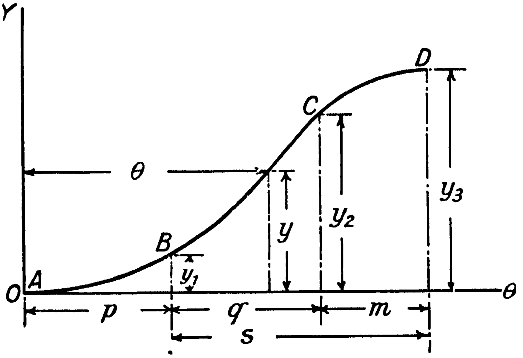


FIG. 218. DISPLACEMENT CURVE

If the speed of revolution is 1,200 per min., then $\omega = 125\cdot 6$, and $\omega^2 = 15{,}800$.

$\therefore$ Change of acceleration at $B = \frac{\cdot 635 \times 15{,}800}{12} = 836$ ft. sec.^{2}

$\therefore$ Change of acceleration at $C = \frac{1\cdot 046 \times 15{,}800}{12}$

$= 1386$ ft. sec.^{2}

If the follower, etc., weigh 1 lb., then the forces acting on the follower at B and C will be 26·1 lb. and 42·8 lb. respectively.

If the acceleration is to increase uniformly from zero at A to a maximum at B (Fig. 218), *and then to decrease uniformly from B to D, C being the point of zero acceleration, until the velocity is zero at D*, the equation of AB will have the form

$$\frac{d^2y}{d\theta^2} = k \,.\, \theta \qquad . \qquad . \qquad . \qquad \text{(X.31)}$$

which is satisfied by $$y = \frac{k}{6} . \theta^3 \quad . \quad . \quad . \qquad (X.32)$$

or $$y = \frac{y_1}{p^3} . \theta^3 \quad . \quad . \quad . \qquad (X.33)$$

Similarly the equation of BCD, with reference to the same axes OX and OY, will have the form

$$\frac{d^2y}{d\theta^2} = k . \theta - r(\theta - p) \quad . \qquad (X.34)$$

where $k = \frac{6y_1}{p^3}$ and r is a constant the value of which is to be determined.

From (X.34), $$\frac{dy}{d} = \frac{k - r}{2} . \theta^2 + r . p . \theta + C \quad . \qquad (X.35)$$

When $\theta = p + s, \frac{dy}{d\theta} = 0$, from which

$$C = \frac{r - k}{2} (p + s)^2 - rp(p + s)$$

$$\therefore \frac{dy}{d\theta} = \frac{r - k}{2} \{(p + s)^2 - \theta^2\} - r . p\{(p + s) - \theta\} . \qquad (X.36)$$

Integrating (X.36), we have

$$y = \frac{r - k}{2} \left\{(p + s)^2 . \theta - \frac{\theta^3}{3}\right\}$$
$$- r . p \left\{(p + s) . \theta - \frac{\theta^2}{2}\right\} + D \quad . \qquad (X.37)$$

When $\theta = p, y = y_1$

$$\therefore y_1 = \frac{r - k}{2} \left\{p (p + s)^2 - \frac{p^3}{3}\right\}$$
$$- r . p \left\{p (p + s) - \frac{p^2}{2}\right\} + D \quad . \qquad (X.38)$$

$$y - y_1 = \frac{r - k}{2} \left\{ (p + s)^2 . (\theta - p) - \frac{\theta^3 - p^3}{3}\right\}$$
$$- r . p \left\{(p + s) (\theta - p) - \frac{\theta^2 - p^2}{2}\right\} \qquad (X.39)$$

From (X.36), since $\frac{dy}{d\theta} = \frac{r - k}{2} \{(p + s)^2 - \theta^2\} - r . p\{(p + s) - \theta\}$

and when $\theta = p$, $\frac{dy}{d\theta} = \frac{k}{2} \, . \, p^2$ (since the tangent at B is common to both AB and BC),

$$\therefore \frac{k}{2} \, . \, p^2 = \frac{r-k}{2} \left\{ (p+s)^2 - p^2 \right\} - r \, . \, p \, . \, s$$

$$k \, . \, p^2 = (r-k)(2ps + s^2) - 2\, r \, . \, p \, . \, s$$

$$\therefore r = \left(\frac{p+s}{s} \right)^2 . \, k \quad . \quad . \quad . \quad (X.40)$$

$$\therefore r - k = \frac{p\,(p + 2s)}{s^2} \quad . \quad . \quad . \quad . \quad (X.41)$$

Since ratios only are involved in the above equations, p, q, m, s and θ can be measured in any units so long as the units are the same for each.

Differentiating (X.36), we have

$$\frac{d^2y}{d\theta^2} = -(r-k) \, . \, \theta + r \, . \, p$$

$$= \frac{p \, . \, k}{s^2} \left\{ (p+s)^2 - (p+2s)\,\theta \right\} \quad . \quad (X.42)$$

$$\therefore \frac{d^2y}{d\theta^2} \text{ is zero when } (p+s)^2 - (p+2s) \, . \, \theta = \theta$$

$$\text{i.e. when } \theta = q = \frac{(p+s)^2}{p+2s} \quad . \quad . \quad . \quad . \quad (X.43)$$

From (X.34) greatest value of $\frac{d^2y}{d\theta^2} = p \, . \, k = \frac{6y_1}{p^2}$. (X.44)

Example.

Given $p = \cdot 3$; $s = \cdot 7$; $y_1 = \cdot 1''$

$$k = \frac{6y_1}{p^3} = \frac{\cdot 6}{\cdot 027} = 22{\cdot}2$$

From (X.40), $r = \left(\frac{\cdot 3 + \cdot 7}{\cdot 7} \right)^2 \times 22{\cdot}2 = 45{\cdot}4$

$$\therefore r - k = 23{\cdot}2$$

From (X.39), putting $y = y_3$ and $\theta = p + s = 1$

$$y_3 - \cdot 1 = 11{\cdot}6 \left\{ (1 - \cdot 3) - \frac{1 - \cdot 027}{3} \right\} - 13{\cdot}6 \left\{ (1 - \cdot 3) - \frac{1 - \cdot 09}{2} \right\}$$

$$= 1{\cdot}03''$$

$$\therefore y_3 = 1{\cdot}13'' \, ; \; \frac{y_3}{y_1} = 11{\cdot}3$$

From (X.43), $\frac{d^2y}{d\theta^2} = 0$ when $q = \frac{(\cdot3 + \cdot7)^2}{\cdot3 + 1\cdot4} = \cdot59$

$\therefore p = \cdot3\,;\ q = \cdot59\,;\ m = \cdot11$

Substituting $\theta = \cdot89$ in (X.39), we obtain

$y_2 = 1\cdot05''$

$\therefore y_1 = \cdot1''\,;\ y_2 = 1\cdot05''\,;\ y_3 = 1\cdot13''$

If we take the equation of AB (Fig. 219) to be $y = \frac{y_1}{p^3} \,.\, \theta^3$ as before (uniform increase of acceleration from A to B), and assume that *the velocity is to decrease uniformly to zero from B to C*, we have

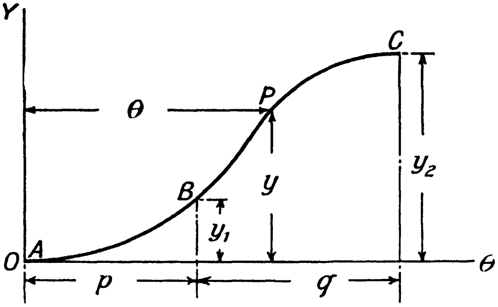


FIG 219. DISPLACEMENT CURVE

At B, $$\frac{dy}{d\theta} = \frac{3y_1}{p^3} \,.\, p^2 = \frac{3y_1}{p} \qquad \text{(X.45)}$$

At any point P between B and C,

$$\frac{dy}{d\theta} = \frac{p+q-\theta}{q} \,.\, \frac{3y_1}{p} = \left(\frac{p+q}{q} - \frac{\theta}{q}\right) . \frac{3y_1}{p} \qquad \text{(X.46)}$$

$\therefore$ the equation to BC is

$$y = \frac{3y_1}{p}\left\{\frac{p+q}{q} \,.\, \theta - \frac{\theta^2}{2q} + C\right\} \qquad \text{(X.47)}$$

When $\theta = p,\ y = y_1$

$$\therefore C = -\frac{p\,(p+2q)}{2q} + \frac{p}{3}$$

$\therefore$ the equation to BC is

$$y = \frac{3y_1}{p}\left\{\frac{p+q}{q} \,.\, \theta - \frac{\theta^2}{2q} - \frac{p(3p+4q)}{6q}\right\} \qquad \text{(X.48)}$$

whence

$$\frac{d^2y}{d\theta^2} = -\frac{3y_1}{p \cdot q} \qquad \text{(X.49)}$$

To find the value of $\frac{y_2}{y_1}$

$$\text{From (X.48)},\ y_2 = \frac{3y_1}{p}\left\{\frac{(p+q)^2}{q} - \frac{(p+q)^2}{2q} - \frac{p(3p+4q)}{6q}\right\}$$

$$= \left(\frac{3q}{2p} + 1\right) \cdot y_1 \qquad \text{(X.50)}$$

$$\therefore \frac{y_2}{y_1} = \frac{3q}{2p} + 1 \qquad \text{(X.51)}$$

EXAMPLE.

Given $p = \cdot 3$; $q = \cdot 7$; $y_2 = 1''$

$$\text{From (X.51)},\ \frac{y_2}{y_1} = \frac{2{\cdot}1}{\cdot 6} + 1 = 4{\cdot}5\ ;\ \therefore y_1 = \frac{1}{4{\cdot}5} = \cdot 222''$$

$$\therefore \text{ equation of } AB \text{ is } y = \frac{\cdot 222}{\cdot 027} \cdot \theta^3 = 8{\cdot}23\theta^3$$

$$\text{equation of } BC \text{ is } y = \frac{\cdot 666}{\cdot 3}\left\{\frac{\theta}{\cdot 7} - \frac{\theta^2}{1{\cdot}4} - \frac{1{\cdot}11}{4{\cdot}2}\right\}$$

$$\text{or } y = 1{\cdot}59\{2\theta - \theta^2 - \cdot 37\}$$

$$\text{At } B \text{ on } AB, \text{ acceleration} = \omega^2 \cdot \frac{d^2y}{d\theta^2} = \frac{6 \times \cdot 222}{\cdot 09} \cdot \omega^2$$

$$= 14{\cdot}82\omega^2$$

$$\text{At } B \text{ on } BC, \text{ acceleration} = -\omega^2 \cdot \frac{3 \times \cdot 222}{\cdot 3 \times \cdot 7} = -3{\cdot}18\omega^2$$

$\therefore$ Change of acceleration at $B = 18\omega^2$.

EXAMPLES 10.

1. The following table gives values of the lift x of a follower, for angular displacements θ of the cam-shaft—

θ(degrees)	0	45	60	90	120	150	180	210	225	240	270	300	315	360
x (inches)	0	0	·02	·15	·37	·56	·70	·80	·82	·78	·53	·10	0	0

The diameter of the roller is to be $\frac{3}{4}$ in. and the minimum distance between centre of roller and centre of cam shaft is to be $1\frac{1}{4}''$.

(*a*) Set out the contour of the cam (i) when the line of stroke of the follower passes through the centre of the cam-shaft, (ii) when the line of stroke of the follower passes $\frac{3}{8}''$ to the right of the centre of the cam-shaft, the direction of motion of the cam-shaft being anti-clockwise in each case.

(*b*) If the cam-shaft rotates at 360 r.p.m., construct curves showing the velocity and acceleration of the follower at every instant during its motion. From your curve find the greatest numerical value of the acceleration.

2. The following particulars of a straight-sided cam (Fig. 210) are given: $r_1 = 1{\cdot}2''$; $r_0 = {\cdot}7''$; $\alpha = 75°$; total lift $= {\cdot}7''$; speed of revolution $=$ 660 r.p.m.

(*a*) Calculate the principal dimensions of the cam.

(*b*) Draw the cam contour full size.

(*c*) Using the method of Art. 170, construct a curve showing the displacements of the follower at every instant. Check the values obtained by using the formulae for displacement.

(*d*) Calculate the accelerations of the follower at *M*, *P*, and *Q* (Fig. 212).

(*e*) Calculate the stiffness (lb. per in.) and the initial compression (in.) of a suitable spring for returning the follower if the weight moved is 12 oz.

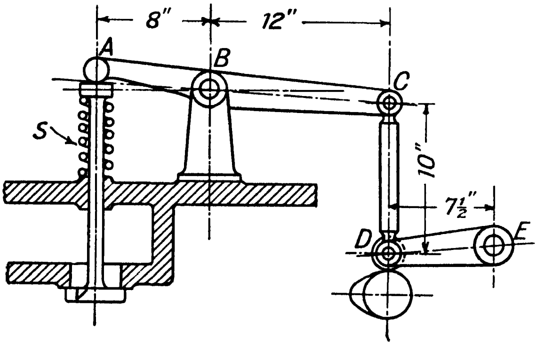


FIG. 220

3. Draw a curve showing the variation of lift of the follower for the cam of Question 2, when the line of stroke of the follower is offset $\frac{1}{2}''$.

4. The following particulars of a curved-sided cam with flat-footed follower (Fig. 214) are given: Total lift $= {\cdot}8''$; $r_1 = 1{\cdot}2''$; $a = 70°$; $R = 4''$; 400 r.p.m.

(*a*) Calculate the principal dimensions of the cam.

(*b*) Draw the cam contour full size.

(*c*) Construct a curve showing the displacement of the follower at every instant.

(*d*) Calculate the accelerations of the follower at *P*, *Q*, and *T* (Fig. 214).

5. A cam is elliptical in shape and is fixed to its shaft so that the axis of the shaft coincides with one of the foci of the ellipse. The axes of the ellipse are 6″ and 4″ respectively. The cam gives a reciprocating motion to a roller and tappet-rod, the axis of which passes through the axis of motion of the cam.

Plot a curve of velocities of the tappet-rod on a time base, assuming that the cam rotates uniformly at 180 r.p.m. (U.L.)

(*Note.* It will be necessary to assume a diameter for the roller, say 1 in., in order to work this question.)

6. Design the outline of a cam which will give harmonic motion to the valve *V* (Fig. 220). Each stroke of the valve is 1″, and the complete movement of the valve occupies one-third of a revolution of the cam-shaft; the

valve is shut during the remaining two-thirds of a revolution. Cylindrical part of cam 4″ diameter, roller at D 2·5″ diameter.

Find the force exerted by the spring to close the valve against the dead weights and inertia forces due to the valve, spindles, and levers. Assume that the weights, radii of gyration and positions of centres of gravity of levers. etc., are known. Speed of cam-shaft is N r.p.m. (U.L.A.)

[Assume the following values—

Distance of C.G. of AC from B towards $C = a_1$ ft.

Wt. of lever $AC = w_1$ lb.; radius of gyration about $B = k_1$ ft.

Corresponding values for DE, including roller, are a_2, w_2, k_2.

Wt. of $CD = w_3$ lb. Wt. of valve and rod $= w_4$ lb.

If θ = displacement of cam-shaft (degrees),

x = displacement of valve (feet),

$$\text{then, } x = \frac{1}{12} . \sin 1{\cdot}5\,\theta.$$

If θ is expressed in radians, then

$$x = \frac{1}{12} . \sin \cdot 157 . N . t \text{ (ft.)}$$

$$\therefore v = \frac{dx}{dt} = \cdot 0131\, N . \cos \cdot 157 . N . t \text{ (ft.-sec.)}$$

$$a = \frac{dv}{dt} = - \cdot 00206 . N^2 . \sin \cdot 157 . N . t \text{ (ft.-sec.}^2\text{)}$$

Since the lift of the valve is small,

$$\text{angular acceleration of } AC = \frac{12a}{8} = 1{\cdot}5a \text{ (rads.-sec.}^2\text{)}$$

$$\text{Linear acceleration of } CD = 1{\cdot}5a \text{ (ft.-sec.}^2\text{)}$$

$$\text{Angular acceleration of } DE = \frac{12 \times 1{\cdot}5a}{7{\cdot}5} = 2{\cdot}4a \text{ (rads.-sec.}^2\text{)}$$

$$\text{Accelerating force on valve and rod} = \frac{w_4}{g} . a \text{ (lb.)}$$

$$\text{Torque on } AC = \frac{w_1}{g} . k_1^2 \times 1{\cdot}5a \text{ (lb.-ft.)}$$

$$\text{Force along } CD = \frac{w_3}{g} \times 1{\cdot}5a \text{ (lb.)}$$

$$\text{Torque on } DE = \frac{w_2}{g} . k_2^2 \times 2{\cdot}4a \text{ (lb.-ft.)}$$

$\therefore$ Force required at A is given by

$$F = w_4 . a + \frac{w_1}{g} . k_1^2 . 2{\cdot}25a + \frac{w_3}{g} . 1{\cdot}5a . \frac{12}{8} + \frac{w_2}{g} . k_2^2 . 3{\cdot}84a - \frac{w_1 \times 12a_1}{8} - w_2 . \frac{12a_2}{7{\cdot}5} . \frac{12}{8}$$

$$\therefore F = \cdot 00206N^2 \sin \cdot 157 . N . t \left\{ w_4 + 2{\cdot}25 . \frac{w_1}{g} . k_1^2 + 2{\cdot}25 . \frac{w_3}{g} + 3{\cdot}84 . \frac{w_2}{g} . k_2^2 \right\} - 1{\cdot}5 . w_1 \; a_1 - 2{\cdot}4w_2 . a_2$$

Since the last two terms are usually small, the ratio $\frac{\text{force}}{\text{displacement}}$ is given by

$$\frac{F}{x} = \cdot 0247N^2 \left\{ w_4 + 2{\cdot}25 . \frac{w_1}{g} . k_1^2 + 2{\cdot}25 . \frac{w_3}{g} + 3{\cdot}84 . \frac{w_2}{g} . k_2^2 \right\}$$

This gives the necessary stiffness of spring at A.]

CHAPTER XI

GOVERNORS

177. Variation of Speed. Governing. Variations in the speed of an engine may be due to (1) variations in the driving torque during a complete cycle, (2) variations in the resisting torque due to changes in the load on the engine. In case (1), the average driving torque remains constant for successive cycles, and is equal to the average resisting torque, and by increasing the moment of inertia of the rotating parts of the engine (flywheel effect) the fluctuation of speed during the cycle can be kept within any desired limits (see Art. 44).

If, however, the average resisting torque changes, it is necessary to alter the supply of working fluid to the engine, so that the average driving torque is equal to the new resisting torque, and this is the function of the governor. The driving torque remaining constant, a change of resisting torque will necessarily result in a change of speed, the rate of change of speed depending upon the difference between the driving and resisting torques and the moment of inertia of the rotating parts of the engine. This change of speed results in a change of configuration of the governor, and by suitable attachments the motion of some part of the governor is communicated to the apparatus controlling the admission of working fluid to the engine.

In the case of steam-engines, governing may be effected—

1. By variation of initial pressure (throttle governing).

2. By variation of expansion ratio due to alteration of travel of valve, or angular advance of eccentric, or both combined.

3. By variation of the point at which tripping takes place with Corliss valves or drop valves.

In the case of internal combustion engines, governing may be effected—

1. By variation of the number of impulses per minute by "cutting out" supplies of the working fluid.

2. By variation of the quantity of working fluid per impulse.

(*a*) By throttling, as in gas engines.

(*b*) By variation of the stroke of the fuel pump, as in some types of oil engines.

(*c*) By cutting off the admission of the fuel earlier or later in the stroke, as in Diesel engines.

In all these cases the governor has no control over any change of speed that may take place between successive admissions of the working fluid, although the method by which the governor controls the average speed may have a very considerable influence upon the nature of such changes.

178. **Centrifugal Governors.** These depend upon the balancing of the " centrifugal " force on a ball rotating with a given angular velocity about a fixed axis, at a given distance from it, by an equal and opposite radial force derived either from the action of gravity (as in the Watt governor), or from a strained spring (as in the Hartnell governor). If these forces are not balanced, the ball moves outward or inward until equilibrium is established and, by suitable connections to the apparatus controlling the admission of the working fluid, the governor is made to control the speed of the engine.

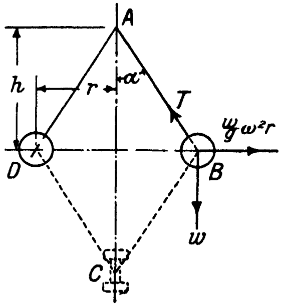


FIG. 221. SIMPLE " WATT" GOVERNOR

For details of the controlling devices, textbooks on heat engines should be consulted.

179. **Simple Watt Governor.** This governor is shown in Fig. 221. *AC* is a rotating shaft driven from the crank-shaft. The links *AD* and *AB* turn on a pivot at *A* carried by the shaft. *D*, *B* are balls attached to the links at their lower ends. When rotating, the balls move inwards, or outwards, as the speed of the engine varies. The dotted links *DC*, *BC* are pivoted at *C*, the pivot being part of a sleeve which slides on the shaft. *C* is connected by levers and rods to the apparatus which controls the admission of the working fluid (Art. 177) and so the speed of the engine is regulated.

Let ω = angular velocity of the balls about the vertical axis *AC* (rads. per sec.)

w = weight of one ball (lb.)

r = horizontal distance of ball from the vertical axis *AC* (ft.)

h = " height " of governor (ft.)

Then, regarding the weights of the arms, etc., as negligible in comparison with the weights of the balls. we may consider the ball B as being in equilibrium under the action of (i) the reversed centripetal force $\frac{w}{g} \cdot r \cdot \omega^2$, (ii) the tension T in the arm AB. (iii) the weight w.

Taking moments about A, we have

$$\frac{w}{g} \cdot r \cdot \omega^2 \cdot h = w \cdot r$$

$$\therefore h = \frac{g}{\omega^2} \quad . \quad . \quad . \quad . \quad \text{(XI.1)}$$

The height of the governor thus depends upon the speed

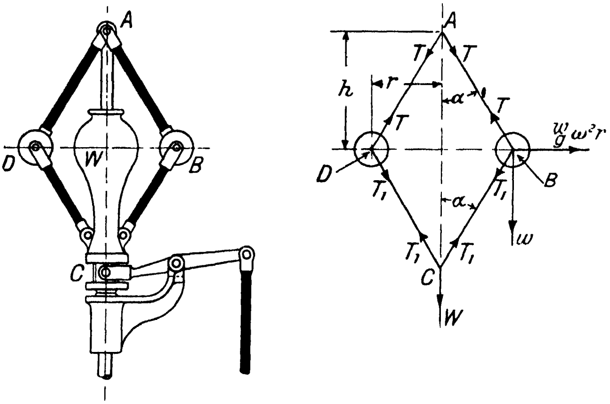


FIG. 222. LOADED "WATT" OR PORTER GOVERNOR

only. If H is the height in inches and N is the speed in revs. per min. (XI.1) reduces to

$$H = \frac{35{,}300}{N^2} \quad . \quad . \quad . \quad \text{(XI.2)}$$

This governor is obviously unsuited for high revolution speeds. For example, if $N = 200$, $H = 0{\cdot}882''$. Even at slow speeds close governing is impossible, for reasons to be stated later.

180. **Loaded Watt Governor, or "Porter" Governor** (Fig. 222). This is the simple Watt governor, with the addition of a weight W lb. rising and falling with the sleeve C. The ball B is now subjected to an additional force T_1 due to the tension in BC. T_1 is found from the equation

$$2T_1 \cdot \cos \alpha = W$$

obtained by considering the equilibrium of the forces acting at C.

$$\therefore T_1 = \frac{W}{2 \cos \alpha} \quad . \quad . \quad . \qquad \text{(XI.3)}$$

Resolving vertically the forces acting at B,

$$T \cos \alpha = T_1 \cos \alpha + w = \frac{W}{2} + w \quad . \quad . \quad . \qquad \text{(XI.4)}$$

Resolving horizontally,

$$T \sin \alpha + T_1 \sin \alpha = \frac{w}{g} \, . \, r \, . \, \omega^2$$

$$\therefore T \sin \alpha = \frac{w}{g} \, . \, r \, . \, \omega^2 - \frac{W}{2} \, . \tan \alpha \qquad \text{(XI.5)}$$

By division,

$$\frac{\frac{w}{g} \, . \, r \, . \, \omega^2 - \frac{W}{2} \, . \tan \alpha}{\frac{W}{2} + w} = \tan \alpha$$

$$\therefore \frac{w}{g} \, . \, r \, . \, \omega^2 = (W + w) \, . \tan \alpha \quad . \qquad \text{(XI.6)}$$

But
$$\tan \alpha = \frac{r}{h}$$

$\therefore$ from (XI.6),
$$h = \frac{W + w}{w} \cdot \frac{g}{\omega^2} \quad . \quad . \qquad \text{(XI.7)}$$

From (XI.7),

$$\frac{dh}{d\omega} = - \frac{W + w}{w} \cdot \frac{2g}{\omega^3}$$

$$\therefore dh = - \frac{2g(W + w)}{w} \cdot \frac{d\omega}{\omega^3}$$

$$\therefore \frac{dh}{h} = - \frac{2g(W + w)}{w} \cdot \frac{d\omega}{\omega^3} \cdot \frac{w}{W + w} \cdot \frac{\omega^2}{g}$$

$$\therefore \frac{dh}{h} = - \frac{2d\omega}{\omega} \quad . \quad . \quad . \quad . \quad . \qquad \text{(XI.8)}$$

or the fractional change of h for a small change of speed is twice the fractional change of speed. Since the load W increases h in the ratio $\frac{W + w}{w}$, the actual change of h will be increased in the same ratio.

If f is the friction of the moving parts of the governor, expressed as an equivalent load on the sleeve, the corrected expression for (XI.7) is

$$h = \frac{W + w \pm f}{w} \cdot \frac{g}{\omega^2} \quad . \quad . \quad (\text{XI.9})$$

the positive sign being taken when the sleeve is rising and the negative sign when the sleeve is falling.

For a given value of h, the two values of ω between which the governor will not act are given by

$$\omega = \sqrt{\frac{W + w \pm f}{w} \cdot \frac{g}{h}} \quad . \quad . \quad (\text{XI.10})$$

The effect of adding W is thus to increase the " sensitiveness " of the governor.

181. **Increase of Axial Force on Sleeve due to Sudden Change of Speed.** Suppose the angular velocity to increase suddenly to ω_1 while the sleeve is in the equilibrium position corresponding to angular velocity ω.

Then, increase of radial force on each ball $= \frac{w}{g} \cdot r \cdot (\omega_1^2 - \omega^2)$.

Let $T_2 =$ increase of tension in AB and BC due to this.

Resolving horizontally,

$$2T_2 \cdot \sin\alpha = \frac{w}{g} \cdot r \cdot (\omega_1^2 - \omega^2)$$

$$\therefore T_2 = \frac{w}{2g} \cdot r \cdot \frac{\omega_1^2 - \omega^2}{\sin\alpha} \quad . \quad . \quad . \quad (\text{XI.11})$$

$\therefore$ Increased axial force on sleeve

$$= 2T_2 \cdot \cos\alpha$$

$$= \frac{w}{g} \cdot r \cdot (\omega_1^2 - \omega^2) \cdot \cot\alpha$$

$$= \frac{w}{g} \cdot (\omega_1^2 - \omega^2) \cdot h \quad . \quad . \quad (\text{XI.12})$$

Substituting the value of h in (XI.7), we have

Increased axial force in sleeve due to change of speed

$$= (W + w) \cdot \left\{\left(\frac{\omega_1}{\omega}\right)^2 - 1\right\} \quad . \quad (\text{X.13})$$

If this force is greater than f, the sleeve will rise with increase of speed from ω to ω_1. The effect of the load W is thus, (1) to render the governor more practicable for high speeds of revolution, (2) to increase the sensitiveness of the governor.

If $\omega_1 = 1{\cdot}01\omega$, then $\left(\frac{\omega_1}{\omega}\right)^2 = 1{\cdot}02$.

$\therefore$ Extra force on sleeve due to an increase of speed of 1 per cent

$$= \cdot 02\,(W + w)$$

If the speed increases from ω to $\omega + a$ (a small), then

$$\omega_1^2 = \omega^2 + 2a\,.\,\omega \text{ (approx.)}$$

$$\therefore \frac{\omega_1^2}{\omega^2} - 1 = \frac{2a}{\omega}$$

$\therefore$ Extra force on sleeve

$$= (W + w)\,.\,\frac{2a}{\omega}$$

Sensitiveness of Porter Governor. Let

x = rise of sleeve in feet

ω_1 = angular velocity at lowest position in radians per sec.

ω_2 = angular velocity at highest position in radians per sec.

then $$x = h_1 - h_2 = k\,.\,g\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right)$$

where $$k = \frac{W + w \pm f}{w}$$

$$\therefore \quad \frac{1}{\omega_2} = \sqrt{\frac{1}{\omega_1^2} - \frac{x}{k\,.\,g}}$$

$$\therefore \quad \omega_2 - \omega_1 = \omega_1\left\{\frac{1}{\sqrt{1 - \dfrac{x\,.\,\omega_1^2}{kg}}} - 1\right\}$$

$\therefore$ If s is the sensitiveness of the governor (Art. 186)

$$\frac{1}{s} = \frac{\omega_2 - \omega_1}{\omega_1} = \frac{1}{\sqrt{1 - \dfrac{x\,.\,\omega_1^2}{kg}}} - 1 \quad . \quad . \qquad \text{(XI.14)}$$

If x and ω_1 are the same for two governors under comparison, then the one with the larger value of k will have the greater value of s.

Example.

If $x = 0{\cdot}1$ ft., $\omega_1 = 24$ (229 r.p.m.), $k = 10$, then, from (XI.14), $s = 9{\cdot}43$. If k is increased to 15, then s is increased to 15·15.

From (XI.10),

$$\omega = \sqrt{\frac{m \pm n}{h}}$$

where $m = \dfrac{W + w}{w} \cdot g$; $n = \dfrac{f}{w} \cdot g$; $\dfrac{n}{m} = \dfrac{f}{W + w}$.

Let ω_2 = highest speed for a given value of h

ω_1 = lowest speed for a given value of h

then $\omega_2 = \sqrt{\dfrac{m + n}{h}}$; $\omega_1 = \sqrt{\dfrac{m - n}{h}}$

∴ Possible fractional variation of speed from the mean for a given value of h

$$= \pm \frac{\frac{1}{2}(\omega_2 - \omega_1)}{\frac{1}{2}(\omega_2 + \omega_1)}$$

$$= \pm \frac{\sqrt{m + n} - \sqrt{m - n}}{\sqrt{m + n} + \sqrt{m - n}}$$

$$= \pm \frac{2m - 2\sqrt{m^2 - n^2}}{2n}$$

$$= \pm \frac{m}{n}\left\{1 - \sqrt{1 - \frac{n^2}{m^2}}\right\}$$

$$= \pm \frac{m}{n}\left\{1 - 1 + \frac{n^2}{2m^2}\right\} \text{ approx., if } \frac{n}{m} \text{ is small}$$

$$= \pm \frac{n}{2m}$$

$$= \pm \frac{f}{2(W + w)} \quad . \quad . \quad . \quad . \quad \text{(XI.15)}$$

The quantity $\frac{f}{2(W + w)}$, which is independent of h, might be regarded as the *speed fluctuation factor*, as distinct from *sensitiveness.*

182. **Effect of Crossing Arms. Isochronism.** In the governor shown in Fig. 222, the speed must increase or decrease as the sleeve is raised or lowered, so that the speed of the engine will increase with reduction of resisting torque on the engine. In cases where such variation of speed is undesirable, some arrangement is necessary in which h is constant for all positions of the sleeve. A governor which satisfies this condition is said to be "isochronous."

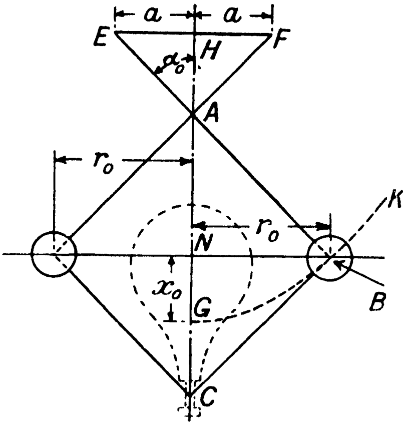


FIG. 223. CROSSED-ARM LOADED GOVERNOR

The type of governor described above can be made isochronous by causing the balls to move in a parabolic path such as GBK (Fig. 223), since the subnormal of a parabola is of constant length. The equation of the parabola GBK is

$$y^2 = 2h \, . \, x^* \quad . \quad . \quad . \quad \text{(XI.16)}$$

where h is the constant " height " of the governor as given by (XI.7), and x is measured from the vertex G along GA.

The position of G is given by

$$r_o^2 = 2h \, . \, x_o \quad . \quad . \quad . \quad \text{(XI.17)}$$

A more convenient arrangement in practice is to cross the arms, as shown (Fig. 223), making the direction of EB normal to the parabola GBK in the normal working position of B, and its length equal to the radius of curvature of the parabola at this point. This is given by

$$\text{Radius of curvature} = l = \overline{EB} = \frac{h^*}{\cos^3 \alpha_o} \quad . \quad . \quad \text{(XI.18)}$$

---

* See Appendix A.

Thus, if the least value of α is 30°, $l = \frac{h}{\cdot 65} = 1{\cdot}54h$.

The distance a (Fig 223) is given by

$$a = h \,.\, \tan^3 \alpha_0 \quad . \quad . \quad . \qquad \text{(XI.19)}$$

$$\text{or } a = l \,.\, \sin^3 \alpha_0 \quad . \quad . \quad . \qquad \text{(XI.20)}$$

For any other inclination α_1 of the arm EB we have

$$\overline{HN} = \overline{EB} \,.\, \cos \alpha_1$$

$$\therefore h_1 = \overline{HN} - \overline{HA} = l \,.\, \cos \alpha_1 - a \,.\, \cos \alpha_1$$

$$\therefore \frac{h_1}{h} = \frac{\cos \alpha_1}{\cos^3 \alpha_0} - \frac{\tan^3 \alpha_0}{\tan \alpha_1} \quad . \qquad \text{(XI.21)}$$

Thus, if $\alpha_0 = 30°$, $\alpha_1 = 36°$, then $\frac{h_1}{h} = \cdot 98$

and if $\alpha_0 = 30°$, $\alpha_1 = 42°$, then $\frac{h_1}{h} = \cdot 93$

In this case there would be a reduction in h of 7 per cent, and a consequent increase in speed of 3½ per cent from the lowest to the highest position of the sleeve if α varies from 30° to 42°, and of 1 per cent only if α varies from 30° to 36°.

Let $\overline{KC} = y$.

$$\begin{aligned} \text{Then } y &= \overline{KA} + 2\overline{AB} \,.\, \cos \alpha \\ &= a \,.\, \cot \alpha + 2(l - AE) \,.\, \cos \alpha \\ &= a \quad \cot \alpha + 2 \left(l - \frac{a}{\sin \alpha} \right) . \cos \alpha \\ &= 2l \cos \alpha - a \,.\, \cot \alpha \quad . \quad . \quad . \quad . \qquad \text{(XI.22)} \end{aligned}$$

$\therefore$ Rise of sleeve for a change of α from α_0 to α_1

$$= 2l(\cos \alpha_0 - \cos \alpha_1) - a(\cot \alpha_0 - \cot \alpha_1) \quad . \qquad \text{(XI.23)}$$

Example.

Given $h_0 = 7''$; $\alpha_0 = 30°$; $N_0 = 300$ r.p.m.

Then, from (XI.18) $l = 7 \times 1{\cdot}54 = 10{\cdot}78''$ (say $10\frac{3}{4}''$)

From (XI.20) $a = 10{\cdot}78 \times {\cdot}125 = 1{\cdot}347''$ (say $1\frac{3}{8}''$)

$\therefore$ Rise of sleeve from 30° to 36°

$$= 21{\cdot}5(\cos 30° - \cos 36°) - 1{\cdot}375\,(\cot 30° - \cot 36°)$$

$$= 21{\cdot}5({\cdot}866 - {\cdot}809) - 1{\cdot}375(1{\cdot}732 - 1{\cdot}376)$$

$$= 1{\cdot}225 - {\cdot}489$$

$$= {\cdot}736 \text{ in.}$$

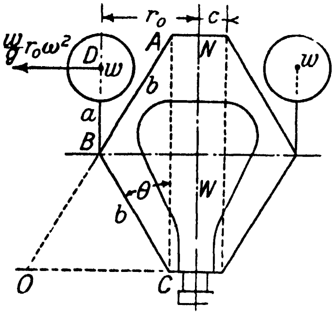


Fig. 224. Proell Governor

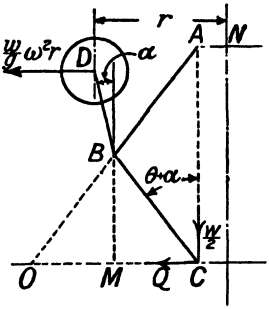


Fig. 225. Force Diagram for Proell Governor

From (XI.7) and (XI.2) we have

$$h_o \text{ (in.)} = \frac{W + w}{w} \cdot \frac{35{,}300}{N^2}$$

$$\therefore \frac{W + w}{w} = \frac{7 \times (300)^2}{35{,}300} = 17{\cdot}84$$

$$\therefore W = 16{\cdot}84w$$

The choice of w will depend upon the sensitiveness required and the value of f. [See (XI.9).]

183. **Proell Governor.** In this governor, DBC is a rigid lever pivoted to the link AB at B, and to the sleeve at C. Fig. 224 shows the configuration when BD is vertical, the arms AB and BC each making an angle θ with the vertical. Fig. 225 shows the configuration when the arm BD has moved outward through an angle α.

Let $BD = a$; $BA = BC = b$; $AN = c$.

Then, neglecting the effect of gravity on the balls (Fig. 225), the forces acting on the lever DBC are as follows—

1. A force $\frac{w}{g} . r . \omega^2$ acting horizontally at D.
2. A force P, acting along BA.
3. A vertical force $\frac{W}{2}$ at C.
4. A horizontal force Q at C.

We may eliminate Q and P by taking moments about O, and the following equation is then obtained—

$$\frac{w}{g} . r . \omega^2 \{a . \cos\alpha + \overline{BM}\} = \frac{W}{2} . 2\overline{CM}$$

$$\therefore \frac{w}{g} . r . \omega^2 \{a . \cos\alpha + b . \cos(\theta+\alpha)\}$$
$$= W . b . \sin(\theta+\alpha) \quad . \quad . \quad . \quad (XI.24)$$

Since α is small, $\cos\alpha = 1$ (approx.) and $\sin^2\alpha$ is negligible;

also $r = a . \sin\alpha + b . \sin(\theta+\alpha) + c$

$$= a . \sin\alpha + b(\sin\theta + \cos\theta . \sin\alpha) + c \quad . \quad (XI.25)$$

$\therefore$ (XI.24) becomes

$$\frac{w}{g} . \omega^2 \{a . \sin\alpha + b(\sin\theta + \cos\theta . \sin\alpha) + c\}$$
$$\times \{a + b(\cos\theta - \sin\theta . \sin\alpha)\}$$
$$= W . b . (\sin\theta + \cos\theta . \sin\alpha) \quad . \quad (XI.26)$$

The value of ω for any given value of α can be obtained from (XI.26), since the remaining quantities are known constants.

When $\alpha = 0$, the value of ω is obtained from

$$\frac{w}{g} . \omega^2 . (b . \sin\theta + c)(a + b . \cos\theta) = W . b . \sin\theta \quad (XI.27)$$

If approximate isochronism is required, then (XI.26) must be true for all values of α, ω being constant. In this case the co-efficients of $\sin\alpha$ on each side of the equation must be equal and the terms independent of α on each side of the equation must also be equal.

From these conditions we derive the equations

$$\frac{w}{g} \cdot \omega^2 \{2ab\cos\theta + a^2 - bc\sin\theta + b^2(\cos^2\theta - \sin^2\theta)\}$$
$$= W \cdot b \cdot \cos\theta \quad . \quad . \quad . \quad . \quad (XI.28)$$

$$\text{and}\frac{w}{g} \cdot w^2\{a \cdot c + ab\sin\theta + bc\cos\theta + b^2\cos\theta \cdot \sin\theta\}$$
$$= W \cdot b \cdot \sin\theta \quad . \quad . \quad . \quad . \quad (XI.29)$$

Given values of ω, a, b, and θ, the necessary values of $\frac{W}{w}$ and c can be found from (XI.28) and (XI.29).

(*Note.* Since, in the expression $\frac{w}{g} \cdot r \cdot \omega^2$, r must be in feet, it will be advisable to take all measurements in *feet* in the above equations.)

EXAMPLE.

Given $a = 4''$; $b = 9''$; $c = x''$; $\theta = 30°$; $N = 135$ r.p.m. ($\omega^2 = 199$)

From (XI.28), $$\frac{199}{32\cdot2}\left\{\frac{72 \times \cdot866 + 16 - 4\cdot5x + 40\cdot5}{144}\right\} = \frac{W}{w} \times \frac{7\cdot79}{12}$$

$$\therefore \frac{W}{w} = 7\cdot9 - 0\cdot3x \quad . \quad . \quad . \quad . \quad (A)$$

From (XI.29), $$\frac{199}{32\cdot2}\left\{\frac{4x + 18 + 7\cdot79x + 35}{144}\right\} = \frac{W}{w} \times \frac{4\cdot5}{12}$$

$$\therefore \frac{W}{w} = 1\cdot35x + 6\cdot1 \quad . \quad . \quad . \quad . \quad (B)$$

$$\therefore 1\cdot35x + 6\cdot1 = 7\cdot9 - 0\cdot3x$$
$$1\cdot65x = 1\cdot8$$
$$\underline{x = 1\cdot1''}$$

From (A) $$\frac{W}{w} = 7\cdot9 - 0\cdot33 = \underline{7\cdot57}$$

Using (XI.27) as a check, and substituting the above values, we obtain

$$\frac{199}{32\cdot2}\left(\frac{4\cdot5 + 1\cdot1}{12}\right)\left(\frac{4 + 7\cdot79}{12}\right) = \frac{7\cdot57 \times 4\cdot5}{12}$$
$$2\cdot83 = 2\cdot84$$

Calculations such as the above will enable preliminary dimensions for a design to be settled. For accurate calculations, however, the weights of the revolving links, etc., must be taken into account, each link being replaced by its equivalent dynamical system (page 46). Moments may be taken about O as before.

184. Hartnell Governor. This type of governor may be represented diagrammatically as in Fig. 226. The sleeve A, with its attachments, slides on the rotating spindle B, to which a framework carrying the pins OO' is attached. The " centrifugal " forces due to the balls rotating at a radius r ft. at a speed of ω rad. per sec., acting through the bell crank levers ROQ, $R'O'Q'$, tend to raise the sleeve A, and this tendency is resisted by the force due to the compression of the spring and the weight of the sleeve and attachments.

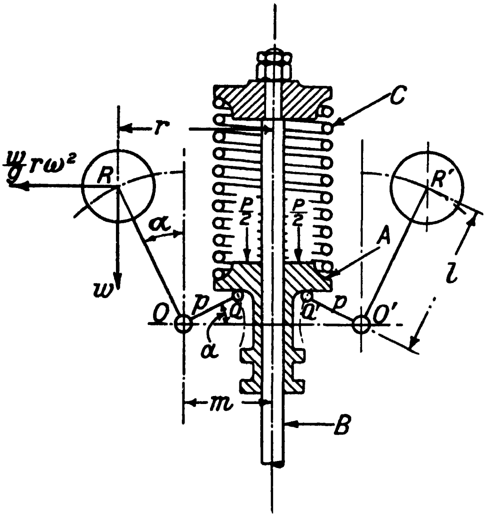


FIG. 226. HARTNELL GOVERNOR

For a given speed ω (radians per second), let α be the inclination of OR to the vertical when the sleeve is in equilibrium under all the forces acting.

Let P = total downward force on sleeve due to compression of spring and dead load

P_0 = total downward force when $\alpha = 0$
s = stiffness of spring (lb. per ft. compression)
l = length of OR (ft.)
p = length of OQ (ft.)
w = weight of each ball in lb. + allowance for arm OR
m = distance of O from axis of rotation (ft.)

Taking moments about O, we have

$$\frac{w}{g} \cdot r \cdot \omega^2 \cdot l \cdot \cos\alpha + w \cdot l \cdot \sin\alpha = \frac{P}{2} \cdot p \cdot \cos\alpha \quad . \quad . \quad \text{(XI.30)}$$

In practice $w \cdot l \cdot \sin\alpha$ is negligible in comparison with the other terms, so that a close approximation is obtained from

$$\frac{w}{g} \cdot r \cdot \omega^2 \cdot l \cos a = \frac{P}{2} \cdot p \cdot \cos\alpha \quad . \quad . \quad \text{(XI.31)}$$

$$\therefore \omega^2 = \frac{p \cdot g}{2l \cdot w} \cdot \frac{P}{r} \quad . \quad . \quad . \quad \text{(XI.32)}$$

Now $P = P_0 + s \cdot p \cdot \sin\alpha$

and $r = m + l \cdot \sin\alpha$

$$\therefore \omega^2 = \frac{p \cdot g}{2l \cdot w} \cdot \frac{P_0 + s \cdot p \sin\alpha}{m + l \cdot \sin\alpha} \quad . \quad . \quad . \quad \text{(XI.33)}$$

If the governor is to be isochronous, ω must be constant for all values of α, i.e.

$$P_0 + s \cdot p \cdot \sin\alpha = \frac{2 \cdot l \cdot w}{p \cdot g} \cdot \omega^2 \cdot (m + l \cdot \sin\alpha) \quad . \quad \text{(XI.34)}$$

where ω is the designed speed of rotation.

If (XI.34) is to be true for all values of α, then

$$P_0 = \frac{2l \cdot m \cdot w}{p \cdot g} \cdot \omega^2 \quad . \quad . \quad . \quad \text{(XI.35)}$$

and $$s \cdot p \cdot \sin a = \frac{2l^2 \cdot w \cdot \omega^2}{p \cdot g} \cdot \sin\alpha$$

i.e. $$s = 2 \cdot \frac{l^2}{p^2} \cdot \frac{w}{g} \cdot \omega^2 \quad . \quad . \quad . \quad \text{(XI.36)}$$

Equations (XI.36) and (XI.35) give the necessary particulars of the stiffness of spring and initial compression in order that the governor may be isochronous.

If N is the speed in revolutions per minute, then

$$\omega = \cdot 1047N \; ; \; \omega^2 = \cdot 011N^2$$

so that the above equations become

$$P_o = \frac{\cdot 000684 \, . \, l \, . \, m \, . \, w}{p} \, . \, N^2 \quad . \quad . \quad \text{(XI.37)}$$

and

$$s = \cdot 000684 \, . \, \frac{l^2}{p^2} \, . \, w \, . \, N^2 \quad . \quad . \quad \text{(XI.38)}$$

$$\therefore \frac{P_o}{s} = \frac{p}{l} \, . \, m \quad . \quad . \quad . \quad . \quad . \quad \text{(XI.39)}$$

EXAMPLE.

Given $l = 8'' = \cdot 67$ ft. ; $p = 4'' = \cdot 333$ ft $m = 6'' = \cdot 5$ ft. ; $w = 6$ lb.; $N = 240$ r.p.m.

Then, from (XI.38),

$$s = \cdot 000684 \times 4 \times 6 \times 57{,}600 = 946 \text{ lb. per ft.}$$
$$= 78 \cdot 8 \text{ lb. per in.}$$

From (XI.37),

$$P_o = \cdot 000684 \times 2 \times \cdot 5 \times 6 \times 57{,}600 = 236 \cdot 5 \text{ lb.}$$

Thus, if the weight of sleeve and attachments amounts to 25 lb., the amount of compression of the spring when $\alpha = o$ must be

$$\frac{236 \cdot 5 - 25}{78 \cdot 8} = 2 \cdot 69 \text{ in.}$$

If the equation (XI.33) be differentiated, it will be found that the condition for $\frac{d(\omega^2)}{d\alpha}$ to be positive is

$$P_o < \frac{s \, . \, p \, . \, m}{l} \text{ i.e. } P_o < \frac{2l \, . \, m \, . \, w}{p \cdot g} \, . \, \omega^2$$

Thus, adding weight to the sleeve of an isochronous Hartnell governor will render it unstable, since the value of ω will *decrease* as α *increases.*

If the governor is to be stable, and we assume two values of α for the highest and lowest positions of the sleeve and the corresponding values of N, equation (XI.33) will enable us to determine the necessary values of P_o and s.

EXAMPLE.

Given $l = 8'' = \cdot 67$ ft.; $p = 4'' = \cdot 333$ ft.; $m = 6'' = \cdot 5$ ft ; $w = 6$ lb.
When $\alpha = 10°$, $N = 244$ r.p.m. ($\omega^2 = 652$).
When $\alpha = -10°$, $N = 236$ r.p.m. ($\omega^2 = 613$).

From (XI.33), $$\frac{\cdot 333 \times 32\cdot 2}{1\cdot 34 \times 6} \cdot \frac{P_o + \cdot 058s}{\cdot 5 + \cdot 115} = 652 \quad . \quad . \quad (A)$$

$$\frac{\cdot 333 \times 32\cdot 2}{1\cdot 34 \times 6} \cdot \frac{P_o - \cdot 058s}{\cdot 5 - \cdot 115} = 613 \quad . \quad . \quad (B)$$

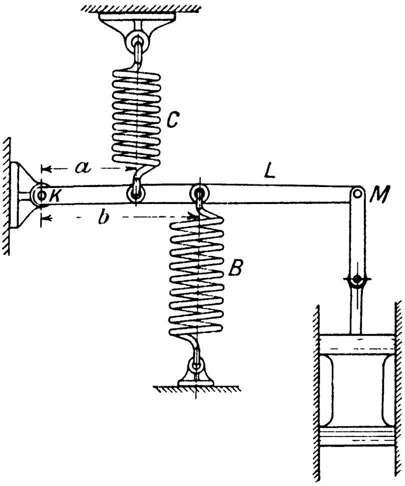


FIG. 227

From (A), $P_o + \cdot 058s = 299$
From (B), $P_o - \cdot 058s = 175\cdot 6$
$\therefore P_o = 237\cdot 3$ lb.
$s = 1060$ lb. per ft.
$= 88\cdot 3$ lb. per in.

The rise of the sleeve from the lowest to the highest position in this case will be $2p \,.\, \sin 10° = 8 \times \cdot 174 = 1\cdot 392''$.

If in addition to the main compression spring C (Fig. 226), an auxiliary tension spring B be fixed to the lever KM, as shown in Fig. 227, the effect of B may be investigated as follows—

Let s_1 = stiffness of C (lb. per ft.)
s_2 = ,, ,, B (lb. per ft.)

The effect of B will be the same as if a spring of stiffness $\frac{b^2}{a^2} . s_2$ were attached direct to the sleeve.

∴ Total force on sleeve for a rise of 1 in.

$$= \frac{1}{12}\left(s_1 + \frac{b^2}{a^2} . s_2\right)$$

This is equivalent to substituting $s_1 + \frac{b^2}{a^2} . s_2$ for s in (XI.33). P_o may also be altered by altering the initial tension in spring B.

Friction.

If f is the friction of the governor, expressed as a load at the sleeve, (XI.32) becomes

$$\omega^2 = \frac{p . g}{2l . w} . \frac{P_o \pm f + s . p . \sin\alpha}{m + l . \sin\alpha} \quad . \quad \text{(XI.40)}$$

In the case of an isochronous governor, from (XI.35),

$$\omega = \sqrt{\frac{p . g}{2l . m . w} . (P_o \pm f)} \quad . \quad . \quad \text{(XI.41)}$$

Thus, if f is 2 per cent of P_o, the governor cannot prevent a fluctuation of ± 1 per cent in the speed.

In one modification of the Hartnell governor for direct attachment to the shaft of a high-speed engine, the spring C (Fig. 226) is replaced by a spring, each end of which is attached to one of the balls R, R'. The axis being horizontal in this case, the effect of gravity is negligible, and, using the same notation as before, the equation for equilibrium is

$$\frac{w}{g} . r . \omega^2 = P_o + 2l . \sin\alpha . s \quad . \quad . \quad . \quad \text{(XI.42)}$$

also $$r = m + l . \sin\alpha$$

$$\therefore \omega^2 = \frac{g}{w} . \frac{P_o + 2l . \sin\alpha . s}{m + l . \sin\alpha} \quad . \quad . \quad \text{(XI.43)}$$

Since $l . \sin\alpha = r - m$, an alternative expression for ω^2 is

$$\omega^2 = \frac{g}{w} . \frac{(P_o - 2m . s) + 2s . r}{r} \quad . \quad . \quad \text{(XI.44)}$$

The conditions for this governor to be isochronous (i.e. ω independent of r) are

$$P_o = 2m . s \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XI.45)}$$

$$s = \frac{w}{2g} . \omega^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XI.46)}$$

185. Controlling Force. The radial force F required at each ball in order to maintain the balls at an axial distance r ft. from the central axis when the governor is stationary is known as the "controlling force." In the case of the "Porter" governor this is given by $(W + w) . \frac{r}{h}$, or $(W + w). \frac{r}{\sqrt{l^2 - r^2}}$ where l is the length of the link AB (Fig. 222).

Let PSQ (Fig. 228) be the curve showing values of F (plotted vertically) for various values of r (plotted horizontally). Then

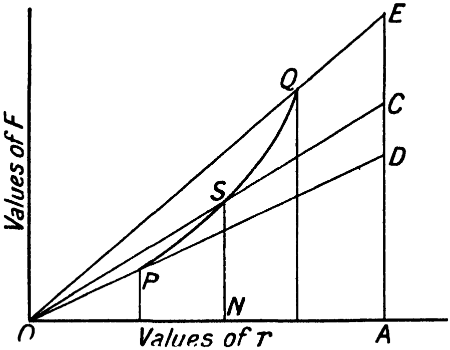


FIG. 228. CURVE OF CONTROLLING FORCE

for the point S on the curve, the speed at which the governor is in equilibrium (neglecting friction) is given by

$$\frac{w}{g} . \overline{ON} . \omega^2 = \overline{NS}$$

$$\therefore \omega^2 = \frac{g}{w} . \frac{\overline{NS}}{\overline{ON}} = \frac{g}{w} . \frac{\overline{AC}}{\overline{OA}} \quad . \quad . \quad . \quad . \quad (XI.47)$$

If OA is any convenient known radius r_o, then

$$\omega^2 = \frac{g}{w . r_o} . \overline{AC} \quad . \quad . \quad . \quad . \quad . \quad . \quad (XI.48)$$

If N is the speed in revolutions per minute, then AC (plotted to the same scale as F) is given by

$$\overline{AC} = \frac{w . r_o . \omega^2}{g} = \frac{w . r_o . N^2}{2910} \quad . \quad . \quad . \quad . \quad (XI.49)$$

AE may be graduated for values of N, and the range of speed from P to Q will be the values corresponding to $\overline{AD}$ to $\overline{AE}$.

Additional curves may be plotted above and below PQ corresponding to $W \pm f + w$, so that the effect of friction at any value of r can be investigated. It will be evident that the slope of OS must always be less than the slope of the curve PQ if the governor is to be stable over its whole range.

For the crossed-arm governor (Fig. 223) the controlling force is given by

$$F = (W + w) \cdot \frac{r + a}{\sqrt{l^2 - (r + a)^2}} \quad . \quad \text{(XI.50)}$$

In the case of the Hartnell governor (Fig. 226) the controlling force is given by

$$F = \frac{p}{l} \cdot \frac{P_0 + s \cdot p \cdot \sin \alpha}{2} \quad . \quad . \quad \text{(XI.51)}$$

Since $\sin a = \frac{r - m}{l}$, then

$$F = \frac{p}{2l} \left\{ P_0 + \frac{p}{l} \cdot s \cdot (r - m) \right\}$$

$$= \frac{p}{2l} \left\{ \left(P_0 - \frac{p \cdot m}{l} \cdot s \right) + \frac{p \cdot s}{l} \cdot r \right\} \quad \text{(XI.52)}$$

Equation (XI.52) shows that PQ will be a straight line, the position and slope depending upon the values of P_0 and s.

P_1Q_1 (Fig. 229) is the graph for an unstable governor ; P_2Q_2 shows that for an isochronous governor, and P_3Q_3 shows that for a stable governor.

The effect of increasing P_0 only will be to raise the line PQ, but not to alter its slope. The effect of increasing s will be to lower the point of intersection of QP produced with OY and to increase the slope of PQ.

The use of curves such as the above was first suggested by Hartnell (*Proc. I. Mech. E.*, 1882).

186. **Hunting.** The object of a governor is so to adjust the admission of the working fluid that the mean driving torque (taken over a complete working cycle) exactly balances the mean resisting torque. Some change of speed is necessary in order that the governor may act, and the ideal governor should alter the admission to that required for the new torque immediately after the resisting torque has changed, and should remain in this position until a fresh change of conditions takes place. This condition is impossible of exact fulfilment, and

the difficulty of approximating to the ideal increases as the governor approaches isochronism, i.e. as its sensitiveness increases. If the speed changes from ω_1 to ω_2 for the full range of movement of the sleeve, then the sensitiveness may be expressed as $\frac{\omega}{\omega_2 - \omega_1}$. where w is the mean angular speed.

With a fully isochronous governor, the effect of an exceedingly small increase of speed is to move the sleeve to its extreme position for *minimum* opening of the valve. The driving torque is now too small, and the speed falls below the normal, so that the sleeve now moves to the full open position of the valve, causing the speed to rise above the normal, so that the speed of the driven machine rises and falls continuously, the magnitude of the oscillations depending upon the moment of inertia of the driven machine, the inertia of the governor, and the variation of area of valve opening with lift of sleeve. The effect is accentuated by solid friction, and is known as "hunting."

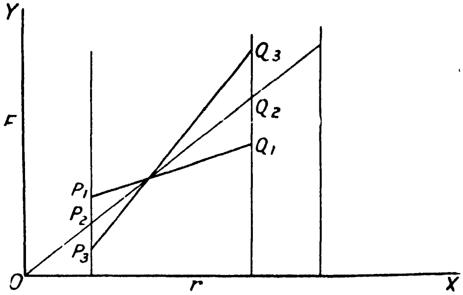


Fig. 229

The inertia of the governor itself may cause a similar effect in the case of a stable governor with a high degree of sensitiveness as defined above and, in the case of an engine with cyclic variations of speed, it will be seen that a high degree of sensitiveness is undesirable.

The fitting of a dash-pot to the sleeve or its attachments introduces fluid friction and eventually damps out the oscillations (Art. 52), but in the case of a large change of resisting torque, taking place suddenly, a very undesirable change of speed may occur before the sleeve finally assumes the correct position for the new load. In the case of steam turbines, a governor of high sensitiveness and small inertia is employed, operating a relay which admits oil under pressure to a piston which moves the throttle valve. The shapes of the valve and its seating are so arranged that approximately equal increments of load correspond to equal increments of lift of the valve.

EXAMPLE.

An engine or turbine, whose moving parts have a total moment of inertia I about the shaft axis, is controlled by a governor in such a manner that the driving torque M varies with the speed according to the relation

$$M = F + G(\omega_0 - \omega)$$

where ω_0 is the steady speed at no load, ω the instantaneous speed, F the friction moment (supposed constant at all speeds), and G is a constant depending upon the sensitiveness of the governor.

If the moment of the load on the engine is suddenly changed from a steady value R_1 to another steady value R_2, show that the subsequent speed of the engine is

$$\omega = \omega_1 + \frac{R_1 - R_2}{G}\left(1 - \varepsilon^{-\frac{G}{I}t}\right)$$

where ω_1 is the steady speed corresponding to the load R_1, and t is the time from the change of load. (L.U.A.)

Accelerating torque at any instant $= M - R_2 = I \cdot \dfrac{d\omega}{dt}$

$$\therefore I \cdot \frac{d\omega}{d} = F + G(\omega_0 - \omega) - R_2$$

$$\therefore I \cdot \frac{d\omega}{dt} + G \cdot \omega = F + G \cdot \omega_0 - R_2 \quad . \quad . \quad . \quad \text{(A)}$$

At the steady load R_1,

$$R_1 = F + G(\omega_0 - \omega_1)$$

$$\therefore F + G \cdot \omega_0 = R_1 + G \cdot \omega_1$$

$\therefore$ From (A),

$$I \cdot \frac{d\omega}{dt} + G \cdot \omega = R_1 - R_2 + G \cdot \omega_1 \quad . \quad . \quad . \quad \text{(B)}$$

The solution of this equation is of the form

$$\omega = A + B \cdot \varepsilon^{nt} \quad . \quad . \quad . \quad . \quad \text{(C)}$$

Substituting this in (B),

$$InB\varepsilon^{nt} + GA + GB\varepsilon^{nt} = R_1 - R_2 + G\omega_1$$

from which we obtain on equating coefficients and simplifying,

$$n = -\frac{G}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(D)}$$

and
$$A = \frac{R_1 - R_2}{G} + \omega_1 \quad . \quad . \quad . \quad . \quad \text{(E)}$$

When $t = o$, $\omega = \omega_1$,

$$\therefore \omega_1 = A + B$$

$$\therefore B = \omega_1 - A = -\frac{R_1 - R_2}{G} \quad . \quad . \quad . \quad \text{(F)}$$

∴ From (C),

$$\omega = \omega_1 + \frac{R_1 - R_2}{G} - \frac{R_1 - R_2}{G} \cdot \varepsilon^{-\frac{G}{I}t}$$

$$\therefore \omega = \omega_1 + \frac{R_1 - R_2}{G}\left(1 - \varepsilon^{-\frac{G}{I}t}\right).$$

The ratio $\frac{G}{I}$ should be large if $\varepsilon^{-\frac{G}{I}\cdot t}$ is to diminish rapidly to a negligible quantity.

187. **Inertia Governors.** Centrifugal governors operate through changes of *speed*. "Inertia" governors operate through changes of *acceleration* (linear or angular), which precede changes of speed, and are particularly useful for marine engines, especially internal combustion engines, where the resisting torque in a seaway may vary rapidly and continuously. The best known example of this type is the Aspinall Emergency Governor, the principle of action of which is shown in Fig. 230.

The governing arrangement in this case is operated by changes in linear acceleration of the engine crosshead.

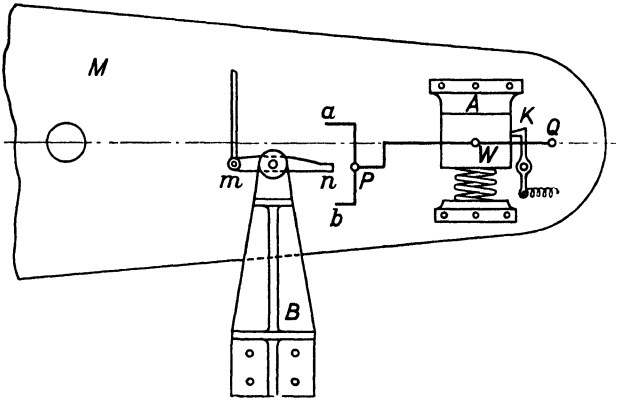


Fig. 230

Referring to Fig. 230, an oscillating arm *M* is connected by links to the engine crosshead, part of the mechanism oscillating with the arm. *P* and *Q* are fixed on *M*. The bracket *B* is fixed to the engine column and supports the lever *mn*. From *m* a rod operates to regulate the supply of fuel to the cylinder. *A* is a stop fixed to the oscillating arm and the weight *W* is maintained in contact with *A* by means of a spring so long as the downward inertia force on W is not greater than the compressive force exerted by the spring.

When W is in contact with A, a can contact with end n of lever mn, while b is out of contact. If W moves downward relative to A, b comes into contact with n and closes the fuel supply. If the acceleration is sufficiently reduced on the return stroke, b moves out of contact and a again opens the fuel valve.

If the increase of acceleration is abnormal (e.g. owing to a broken propeller shaft), the spring is compressed sufficiently to bring the stop K into operation, and the lever mn remains permanently in the "closed" position. The pins P and Q are placed near to the centre of oscillation of M, so that the amplitudes are relatively small.

Let W = weight in contact with A
w = extra load to be moved when W leaves the stop
s = stiffness of spring
c_0 = initial compression of spring
r = amplitude of oscillation of A
n = connecting rod/crank ratio of engine
ω_0 = crank speed at which W first leaves stop
y = specified movement of W when speed increases from ω_0 to $k \,.\, \omega_0$ = additional compression of spring

Then,
$$\frac{W}{g} \,.\, r \,.\, \omega_0^2\left(1 + \frac{1}{n}\right) = s \,.\, c_0 - W \qquad . \quad . \quad . \quad (1)$$

$$\frac{W + w}{g} \,.\, r \,.\, k^2 \,.\, \omega_0^2\left(1 + \frac{1}{n}\right) = s(c_0 + y) - W \qquad . \quad (2)$$

From (1) and (2)
$$s = \frac{r}{y} \,.\, \frac{\omega_0^2}{g} \,.\left(1 + \frac{1}{n}\right)\left\{(k^2 - 1)W + k^2 \,.\, w\right\} \quad (3)$$

$$c_0 = \frac{r \,.\, \dfrac{\omega_0^2}{g} \,.\left(1 + \dfrac{1}{n}\right) + 1}{r \,.\, \dfrac{\omega_0^2}{g} \,.\left(1 + \dfrac{1}{n}\right)\left\{k^2\left(1 + \dfrac{w}{W}\right) - 1\right\}} \,.\, y \qquad . \quad . \quad (4)$$

EXAMPLE.

Take $\omega_0 = 25$; $k = 1{\cdot}1$; $W = 12$ lb.; $w = 4$ lb.; $r = 0{\cdot}25$ ft.; $y = 0{\cdot}08$ ft.; $n = 4$.

Then
$$s = \frac{0{\cdot}25}{0{\cdot}08} \times \frac{625}{32{\cdot}2} \times 1{\cdot}25\,(0{\cdot}21 \times 12 + 1{\cdot}21 \times 4)$$

$$= 556 \text{ lb./ft. } (46\cdot3 \text{ lb./in.})$$

$$c_0 = -\frac{0\cdot25 \times \dfrac{625}{32\cdot2} \times 1\cdot25 + 1}{0\cdot25 \times \dfrac{625}{32\cdot2} \times 1\cdot25\,\{1\cdot21 \times 1\cdot33 - 1\}} \times 0\cdot08$$

$$= 0\cdot153 \text{ ft. } (1\cdot84 \text{ in.})$$

In another type of inertia governor a disc which is keyed to a rotating shaft drives a flywheel, moment of inertia I (engineers' units), which is co-axial with the disc, by means of a coil spring or its equivalent, the torque exerted by the spring being $P + Q\,.\,\theta$, where θ is the angular displacement of the flywheel relative to the disc. The movement of the flywheel relative to the disc gives axial movement to a sleeve which actuates a governor mechanism.

The following example will illustrate the application—

EXAMPLE.

A multi-cylinder internal combustion engine, generating 60 h.p. at 240 r.p.m., has the ratio of maximum to mean torque of 1·3. The weight of the rotating parts of the engine (including flywheel and proportion of reciprocating parts) is 2250 lb., the radius of gyration being 1·3 ft. An inertia governor as above is to be insensitive to changes of engine torque during a cycle while the mean speed and load remain constant. The governor is geared to four times the engine speed, and its flywheel weighs 20 lb., with a radius of gyration of 6 in. The relative angular movement of the flywheel is to be 25° when the external resisting torque diminishes quickly by 20 per cent, the driving torque remaining unaffected during this movement. Find P and Q for the spring.

$$\text{Mean torque of engine} = \frac{33{,}000 \times 60}{6\cdot28 \times 240} = 1360 \text{ lb.-ft.}$$

$$\text{Maximum excess torque} = 0\cdot3 \times 1360 = 408 \text{ lb.-ft.}$$

$$I \text{ of rotating parts of engine} = \frac{2250 \times 1\cdot69}{32\cdot2} = 118 \text{ engineers' units.}$$

∴ Maximum angular acceleration of rotating parts of engine at 240 r.p.m.

$$= \frac{400}{118} = 3\cdot46 \text{ rad. sec.}^2$$

$$I \text{ of governor flywheel} = \frac{20 \times 0\cdot25}{32\cdot2} = 0\cdot156.$$

If θ is the displacement of the flywheel relative to the disc, then when $\theta = 0$, $\alpha = 3\cdot46 \times 4 = 13\cdot84$ rad. sec.^{2}

$$\therefore\ P = 0\cdot156 \times 13\cdot84 = \underline{\underline{2\cdot16 \text{ lb.-ft.}}}$$

When the resisting torque is reduced, the temporary maximum excess torque is increased to $408 + 272 = 680$ lb.-ft.

$\therefore$ Maximum angular acceleration of crank-shaft

$$= \frac{680}{118} = 5{\cdot}77 \text{ rad. sec.}^2$$

$$\text{Torque on governor flywheel} = 0{\cdot}156 \times 5{\cdot}77 \times 4 = 3{\cdot}6 \text{ lb.-ft.}$$

$$\text{Increase of torque} = 3{\cdot}6 - 2{\cdot}16 = 1{\cdot}44 \text{ lb.-ft.}$$

$$\text{Angular displacement} = \frac{25}{57{\cdot}3} = 0{\cdot}435 \text{ rad.}$$

$$\therefore\ Q = \frac{1{\cdot}44}{0{\cdot}435} = 3{\cdot}31 \text{ lb.-ft. per radian}$$

When the spring is away from the stop, the governor flywheel is subjected to a variation of angular acceleration of $\pm$ 13·84 rad. sec.^{2} with frequency equal to that of the torque variation of the engine. If the engine has 6 cylinders, this frequency is

$$\frac{6 \times 240}{60} = 24 \text{ per sec.}$$

The natural frequency of the governor flywheel is

$$\frac{1}{2\pi}\sqrt{\frac{Q}{I}} = \frac{1}{6{\cdot}28}\sqrt{\frac{3{\cdot}31}{0{\cdot}156}}$$

$= 0{\cdot}733$ per sec. In this case the governor flywheel would be practically unaffected by the cyclical variations of engine torque.

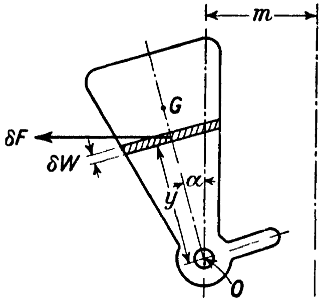


FIG. 231

Note. The usual assumption that the mass of a governor arm may be taken as concentrated at its centre of gravity may lead to appreciable errors in the case where the mass is distributed as, for instance, in Fig. 231.

Let δW = wt. of an element of the arm, distant y from O

α = inclination of OG to axis of rotation

δF = centrifugal force on arm due to δW.

Then $$\delta F = \frac{\delta W}{g}(m + y \,.\, \sin\alpha) \,.\, \omega^2.$$

If

$$M = \text{total moment of C.F. about } O,$$

$$\delta M = \delta F \times y \cos\alpha$$

$$= \frac{\delta W}{g}(m \,.\, y + y^2 \,.\, \sin\alpha) \,.\, \cos\alpha \,.\, \omega^2$$

$$\therefore \quad M = \frac{\cos\alpha \,.\, \omega^2}{g}\{m\Sigma y\,\delta W + \sin\alpha\Sigma y^2\,\delta W\}$$

$$= \frac{\cos\alpha \,.\, \omega^2}{g}\{m \,.\, W \,.\, OG + \sin\alpha \,.\, W \,.\, k_0^2\}$$

i.e.

$$M = \frac{W}{g}\left\{m + \frac{k_0^2\sin\alpha}{OG}\right\} . \,\omega^2 \,.\, OG\cos\alpha$$

If the whole mass is considered as concentrated at G, the quantity in brackets is $m + OG \,.\, \sin\alpha$.

EXAMPLE.

$m = 0{\cdot}167$ ft.; $OG = 0{\cdot}333$ ft.; $k_0 = 0{\cdot}4$ ft.; $\sin\alpha = \frac{1}{6}$.

$$m + \frac{k_0^2}{OG} \,.\, \sin\alpha = 0{\cdot}247$$

$$m + OG \,.\, \sin\alpha = 0{\cdot}222 \text{ (10\% too small).}$$

If the arm is allowed to oscillate freely under gravity about a knife edge at O, then $T = 2\pi\sqrt{\dfrac{k_0^2}{g \,.\, OG}}$. Thus the value of $\dfrac{k_0^2}{OG}$ can be found experimentally.

The above investigation is valid only if the mass is symmetrically disposed about OG. This is not the case in a number of shaft governors and a step-by-step calculation then becomes necessary.

EXAMPLES 11.

1. Explain the function of a governor. How does it fulfil this function? Sketch any type of governor and explain its action.
2. Find the relation between the height of the simple Watt governor and the speed. Plot a graph showing the height at different speeds. What are the disadvantages of this type of governor, and how may these disadvantages be reduced?
3. Explain the effects of friction on the action of governors. Show how to find the range of speeds for any configuration of the governor.
4. What is an "isochronous" pendulum governor? Show that in such a governor the balls should each move along a parabolic arc having as axis the axis of rotation.
5. A loaded link governor of the Porter type has equal arms and links each 12 in long. The rotating balls each weigh 4 lb., and the central weight is 60 lb. The drag on the sleeve due to friction, valve resistances, etc., is 3 lb. If the limits to the movements of the arms are 45° and 30° respectively

to the vertical, calculate the revolutions per minute between which this governor is operative.

Also, find the range of speed for a half-way position of the arms, viz., 37½° to the vertical. (L.U.)

6. In a loaded Porter governor the arms are equal and 9 in. long, and are pivoted at the axis The central load weighs 25 lb., and each rotating ball weighs 5 lb. The balls move in a circle of radius 4 in. at the lowest speed, and in a circle of radius 6 in. at the highest speed. Neglecting friction, find the speeds for these two positions and the range of speed.

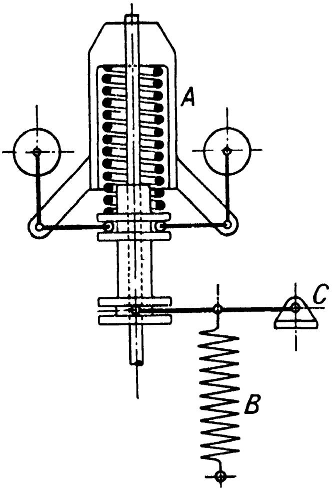


FIG. 232

7. Find the maximum and minimum speeds for a crossed arm Watt governor, neglecting friction The apex angle (*EAF*, Fig. 223) is 60° at its lowest value, and 120° at its highest. Length of arms 10 in. The suspension pins are 1·5 in. from the axis of rotation.

If the weight of each ball is 8 lb., and the friction is equivalent to a force of 1 lb. on the sleeve, find the range of speed for each of the two extreme positions,

8. A crossed arm Watt governor has arms 10 in. long, and each arm is pivoted 1¼ in. from the axis of evolution. Neglecting the weights of the arms, and assuming each ball to weigh 10 lb., draw the controlling force diagram of the governor for radii between 5 in. and 7 in. (I.C.E.)

9. Show how to find the approximate speed of a Proell governor, Fig. 224, at mid-stroke of sleeve *C*, when the arm *BD* is vertical, assuming the weight of each revolving ball = w_1 lb., central weight = W lb., each pin at B, = w_p lb. weight of each arm uniform = w_a lb. per inch run. Speed of governor = ω radians per sec. The arm *DBC* is in one piece, and the arm *AB* is assumed straight. (U.L.A.)

10. In a Hartnell governor the ball radius is 6 in. when the ball arm, which has a length of 5 in., is vertical, and the governor is running at the mean speed of 300 r.p.m. The sleeve arm has a length of 4½ in., and the balls each weigh 5 lb. The sleeve can move 1 in. from its mean position either up or down, and the maximum steam supply is obtained when the speed falls 4 per cent below the mean. Neglecting friction, find—

(*a*) The spring stiffness.

(*b*) The maximum speed of the governor.

If friction is equivalent to a force of 3 lb. at the sleeve, find the speeds at which the governor will begin to open the steam-valve when the sleeve is at the top of its stroke, and to close the valve when the sleeve is at the bottom. (U.L.A.)

11. In the governor of Fig. 221, Watt governor, the two balls are connected by horizontal springs, which exert a radial pull on each ball. The balls each weigh 25 lb. and are carried by links 12 in. long. Each spring requires a total force of 25 lb. to stretch it 1 in. When the balls are in their lowest position they move in a circle of radius 5 in., and each spring is then 2 in. longer than its unstretched length. Find, neglecting friction, weights of spring and links, etc.

(*a*) The speed at which the sleeve begins to lift.

(*b*) The speed when the radius of the circle described by the balls is 7 in.

12. The Hartnell type spring-loaded governor, shown in Fig. 232, has two balls of 10 lb. each, which revolve in a circle of 16 in. diameter when the sleeve is at mid-travel. The total movement of the sleeve is 1 in., which is the same as that of the balls. An adjustable load is applied to the sleeve by means of the spring *B* attached to the centre of a lever moving about a fixed centre *C* in the manner shown, the tension in *B* being adjusted by hand. When there is no tension in the spring *B*, the sleeve just commences to rise from its lowest position at a speed of 200 r.p.m., and reaches the upper limit of its travel at 208 r.p.m. Find the load extension scale of the spring *A*, and also for the spring *B*, which will make these two limiting speeds 220 and 232 r.p.m. Neglect the effect of gravity on the balls. (U.L.)

13. The effect of the spring in constraining the balls of a spring governor is found by observing the force to be applied radially at each ball to keep the mechanism in any required position. This force is equal to $15r - 30$ lb. where r is the radius of the ball path in inches.

Find the working radius when the speed is 270 r.p.m. if the weight of each ball is 6 lb. Also find by how much the initial force in the spring must be increased in order to make the governor isochronous. What will then be the working speed ? (U.L.A.)

14. A centrifugal governor rotates at a mean speed of 300 r.p.m. when the radius of revolution of the revolving masses is 7·5 in. Each mass is 12 lb, and is connected to a spring in compression by means of a bell crank lever, with unequal arms, the arm of the revolving mass being 6 in. long and that of the spring 5 in long.

Calculate the stiffness of the spring so that an increase of 1 in. in the radius of revolution of the masses corresponds to an increase of speed of 1 per cent, and find the period of oscillation of the masses about their mean position when the governor is running at the given speed. (U.L.A.)

CHAPTER XII

ENGINE BALANCING

188. **Objects of Balancing.** In the following investigations the framing and foundations of an engine are regarded as infinitely rigid, i.e. distortion of the whole or of separate parts due to the forces acting is regarded as negligible. In the case of certain types of high-speed engines, however, vibration is possible, due to, for example, torsional distortion of the crankshaft.

From Newton's First Law of Motion, it follows that the centre of gravity of a body at rest cannot be displaced unless an external force acts on the body, this force being given by

$$\text{Force} = \text{mass of body} \times \text{linear acceleration of the centre of gravity} \quad . \quad . \quad . \quad \text{(XII.1)}$$

Similarly, an angular acceleration of any portion of the body, the motion of which does not involve a displacement of the C.G., can only be effected either by an angular acceleration of the remainder of the body in an opposite direction, or the application of an external couple, such that

$$\text{Couple} = \text{M.I. of displaced portion about the C.G.} \times \text{angular acceleration} \quad . \quad . \quad \text{(XII.2)}$$

Thus, the objects of " balancing " an engine are to ensure—

(1) That the C.G. of the whole remains stationary during a complete revolution of the crankshaft.

(2) That the couples involved in accelerations of different moving parts balance each other.

The framing and foundations of an engine may vibrate in any of the following ways—

(1) Vertical translational vibrations due to unbalanced vertical forces.

(2) Horizontal translational vibrations due to unbalanced horizontal forces.

(3) Rotary oscillations in a vertical or horizontal plane due to unbalanced couples in that plane.

(4) Vibrations round an axis parallel to the axis of the crankshaft, due to variations of torque and the " surging " couple caused by the motion of the connecting-rod.

(1), (2), and (3) are unaffected by variations of steam or gas pressure in the cylinders, since at any instant the force on the piston is balanced by an equal and opposite force on the cylinder, the nett force on the frame being zero.

Since the positive torque on the crank-shaft is balanced by an equal negative torque on the engine frame, a varying torque on the shaft will tend to produce varying angular acceleration of the engine round the shaft. In the case of certain types of internal combustion engines, where the variation of torque from the mean is considerable, very appreciable vibration may result. For this reason, crank arrangements for multi-crank internal combustion engines differ from those of steam-engines, which show much smaller torque variations.

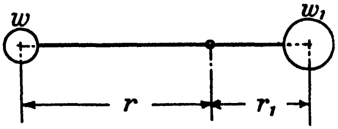


FIG. 233. BALANCING A ROTATING MASS

So far as inertia forces are concerned, in an engine we have two kinds of forces to balance : (1) Constant forces, varying in direction, due to rotating parts ; (2) forces of varying magnitude and sense but constant in direction, due to reciprocating parts. Since both kinds of forces act usually in different planes, couples varying in magnitude and direction may arise.

189. **Balancing of Rotating Parts.** The centrifugal force due to a weight w, the C.G. of which is at radius r from the axis of rotation, rotating with angular velocity ω, is given by

$$F = \frac{w}{g} . r . \omega^2 \quad . \quad . \quad . \quad . \qquad \text{(XII.3)}$$

This may be balanced by a weight w_1 at radius r_1 (Fig. 233), opposite to the first, rotating at the same speed, so that

$$\frac{w}{g} . r . \omega^2 = \frac{w_1}{g} . r_1 . \omega^2$$

i.e. $$w . r = w_1 . r_1 \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(XII.4)}$$

Thus, in the balancing of forces due to rotating parts, values of $w . r$ may be substituted for centrifugal forces, and the balance will be correct for all speeds of rotation. In what follows, the value of $w . r$ will be referred to as the *weight-arm product*, and the balancing value of $w . r$ as the *balancing product.*

The determination of the magnitude and angular position of the balancing product for a number of weights, w_1, w_2, w_3, etc., rotating in the same plane about an axis through O perpendicular to the plane of the weights is accomplished as follows (Fig. 234).

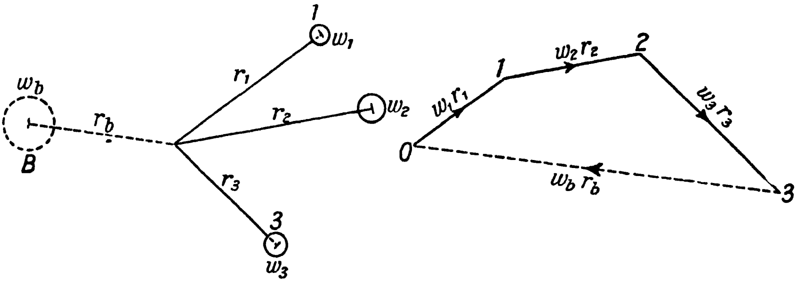


Fig. 234. Polygon of Balancing Products

Set out $O1$ parallel to r_1 and equal to the weight-arm product w_1r_1. From 1, set out 12 parallel to r_2 and equal to w_2r_2, and so on for the remainder of the weights. In the case of the three weights shown, the closing line 30 is the balancing product required.

Draw r_b parallel to 30, and choose convenient values of r_b and w_b to produce the required value of w_br_b.

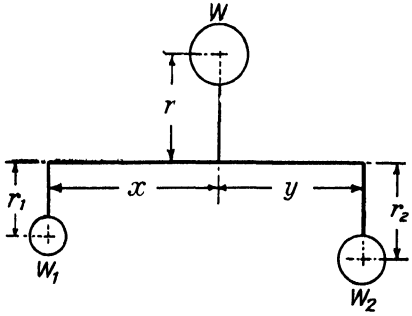


Fig. 235. Two Masses Balancing a Single Mass

The weight W (Fig. 235), rotating at radius r, can be completely balanced by weights w_1 and w_2 at radii r_1 and r_2 respectively, provided the following conditions are satisfied—

(1) $w_1 . r_1 + w_2 . r_2 = W . r$
(2) $w_1 . r_1 . x = w_2 . r_2 . y$
(3) r, r_1, and r_2 are in the same plane.

From (1) and (2) we obtain

$$\left.\begin{aligned} w_1 r_1 &= \frac{y}{x+y} \,.\, W \,.\, r \\ \text{and} \qquad w_2 r_2 &= \frac{x}{x+y} \,.\, W \,.\, r \end{aligned}\right\} \quad . \quad . \quad \text{(XII.5)}$$

Weights Whose Arms are not Coplanar. Two weights, W_1 and W_2, rotating about the axis CD (Fig. 236) at the ends of arms AP, BQ which are not in the same plane, can be balanced by single weights in the planes R and S by the following method, (a) being a perspective view and (b) and (c) being end elevations of the planes R and S, looking in the direction of the arrow X—

Draw Cp, Dp_1, parallel to AP, and Cq, Dq_1, parallel to BQ. Then W_1r_1 can be balanced by $\overrightarrow{Cf} = \frac{y+z}{x+y+z} \times W_1r_1$ on pC produced, and $\overrightarrow{Dh} = \frac{x}{x+y+z} \,.\, W_1r_1$ on p_1D produced. Similarly W_2r_2 can be balanced by $\overrightarrow{Ce} = \frac{z}{x+y+z} \,.\, W_2r_2$ on qC produced, and $\overrightarrow{Dg} = \frac{x+y}{x+y+z} \,.\, W_2 \,.\, r_2$ on q_1D produced.

In the plane R, the resultant balancing product will be $\overrightarrow{Cm}$, and in plane S the resultant balancing product will be $\overrightarrow{Dn}$.

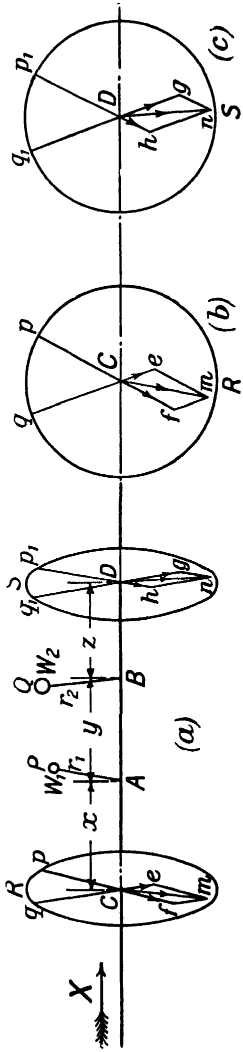


FIG. 236. WEIGHTS WHOSE ARMS ARE NOT IN THE SAME PLANE

General Method. Any number of rotating weights in different planes can be balanced in this way by two revolving weights in two given planes, a polygon replacing the parallelogram for finding the resultant balancing product in either of the two planes. The calculations are, however, somewhat tedious, and Dalby's method, using force and couple polygons, is preferred in many cases. (*Proc. I. Mech. E.*, Nov., 1901.)

Dalby's method depends upon the replacement of a force in a given plane by an equal force in a plane parallel to the first, and a couple (see Art. 29). Thus, the single weight-arm product $w \,.\, r$ at C, which may be treated as a force, may be replaced by a force $w \,.\, r$ at B and a clockwise couple $w \,.\, r \,.\, x$. This clockwise couple may be represented by forces $\dfrac{w \,.\, r \,.\, x}{x + y}$ at A and B, as shown (Fig. 237), the net force at B being $\dfrac{w \,.\, r \,.\, y}{x + y}$. The necessary balancing products are equal and opposite to these, and are the same as those obtained by the previous method.

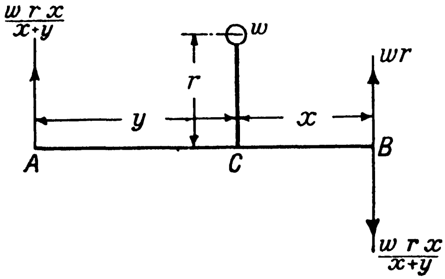


FIG. 237. DALBY'S METHOD

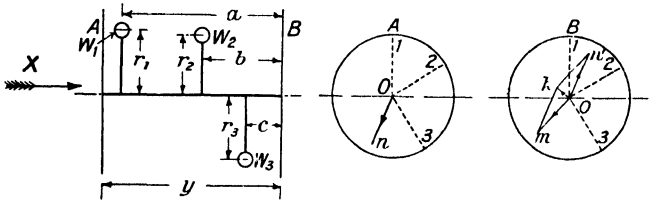


FIG. 238

The following procedure may be adopted for finding the magnitudes and angular positions of the balancing products in planes A and B respectively of any number of weight-arm products (Fig. 238).

(i) Tabulate as below—

Weight (1)	Radius (2)	Distance from reference plane B (3)	Weight-arm product (4)	Couple (5)
W_1	r_1	a	W_1r_1	W_1r_1a
W_2	r_2	b	W_2r_2	W_2r_2b
W_3	r_3	c	W_3r_3	W_3r_3c

(ii) Construct a " force polygon " (Fig. 239), using the values in column (4), drawing the " forces " in directions parallel to 01, 02, 03 respectively (Fig. 238), where 01, 02, 03 represent the angular positions of r_1, r_2, r_3.

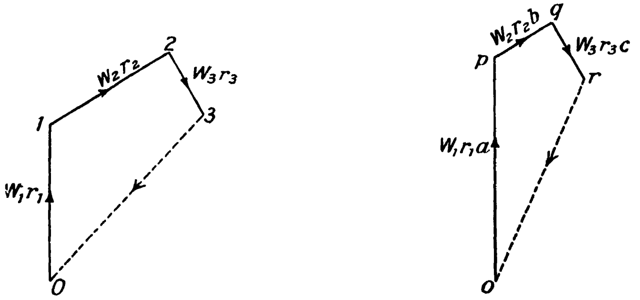


Fig. 239 Fig. 240

Force and Couple Polygons

(iii) Construct a " couple polygon " (Fig. 240), using the values in column (5) and plotting in the directions $\overrightarrow{01}$, $\overrightarrow{02}$, $\overrightarrow{03}$ as before.

(iv) The balancing product in plane B is given by the closing line $\overrightarrow{30}$ (Fig. 239), and is shown by $\overrightarrow{Om}$ in plane B (Fig. 238).

The balancing " couple " is given by the closing line $\overrightarrow{ro}$ (Fig. 240), and is equivalent to two products $\overrightarrow{On}$ in plane A and $\overrightarrow{On'}$ in plane B, as shown in Fig. 238, each equal to $ro \div y$.

W_1, W_2, and W_3 may therefore be completely balanced by the product $\overrightarrow{On}$ in the plane A and the resultant $\overrightarrow{ok}$ in the plane B, in each case looking in the direction of the arrow X.

The method is simple and straightforward, and can be applied to any number of rotating weights attached to the same shaft.

190. **Balancing of Reciprocating Parts.** The unbalanced force due to one line of reciprocating parts is given by the approximate expression

$$F = \frac{W}{g} \,.\, \omega^2 \,.\, r \left\{ \cos\theta + \frac{\cos 2\theta}{n} \right\} \text{ (see II.33)}$$

where $W =$ total weight of reciprocating parts

$\omega =$ angular velocity of crank (radians per second)

$r =$ crank radius (ft.)

$\theta =$ angular displacement of crank from inner dead centre

$n =$ connecting-rod/crank

This may be written

$$F = \frac{W}{g} \,.\, \omega^2 \,.\, r \,.\, \cos\theta + \frac{W}{g} \,.\, \omega^2 \,.\, \frac{r}{n} \,.\, \cos 2\theta \,. \qquad \text{(XII.6)}$$

Considering, for convenience, that the direction of motion of W is vertical, the first term represents the vertical component of the centrifugal force due to a weight W at radius r rotating at crank speed, and the second term represents the vertical component of the centrifugal force of a weight W rotating at radius $\frac{r}{4n}$ at twice crank speed. The first term of (XII.6) is known as the *primary force*, and the second term as the *secondary force*.

If there are a number of sets of reciprocating parts of equal weight, the cranks making angles of α, β, γ, etc., with the first (Fig. 241), then the net primary force will be

$$\begin{aligned} F_1 &= \frac{W}{g} \,.\, \omega^2 \,.\, r \{\cos\theta + \cos(\theta + \alpha) + \cos(\theta + \beta) + \text{etc.}\} \\ &= \frac{W}{g} \,.\, \omega^2 \,.\, r \{\cos\theta\,(1 + \cos\alpha + \cos\beta + \text{etc.}) \\ &\qquad - \sin\theta\,(\sin\alpha + \sin\beta + \text{etc.})\} \quad . \quad . \qquad \text{(XII.7)} \end{aligned}$$

Similarly, the net secondary force will be

$$\begin{aligned} F_2 = \frac{W}{g} \,.\, \omega^2 \,.\, \frac{r}{n} \{&\cos 2\theta\,(1 + \cos 2\alpha + \cos 2\beta + \text{etc.}) \\ &- \sin 2\theta\,(\sin 2\alpha + \sin 2\beta + \text{etc.})\} \quad . \qquad \text{(XII.8)} \end{aligned}$$

For complete force balance at all values of θ, the following four conditions must be satisfied—

$$\left.\begin{aligned}&(1)\ 1+\cos\alpha+\cos\beta+\text{etc.} &&= 0\\ &(2)\ \sin\alpha+\sin\beta+\text{etc.} &&= 0\end{aligned}\right\}\text{Primary force balance}$$

$$\left.\begin{aligned}&(3)\ 1+\cos 2\alpha+\cos 2\beta+\text{etc.} &&= 0\\ &(4)\ \sin 2\alpha+\sin 2\beta+\text{etc.} &&= 0\end{aligned}\right\}\begin{aligned}&\text{Secondary force}\\ &\text{balance}\end{aligned} \quad . \quad (\text{XII.9})$$

For *two cranks at* 180° ($\alpha = 180°$)—

$$\begin{aligned}&(1)\ 1+\cos\alpha &&= 0\\ &(2)\ \quad\sin\alpha &&= 0\\ &(3)\ 1+\cos 2\alpha &&= 2\\ &(4)\ \quad\sin 2\alpha &&= 0\end{aligned}$$

Thus the primary forces are balanced, but there is a secondary force of $\frac{W}{g} \,.\, \omega^2 \,.\, \frac{r}{n} \,.\, 2\cos 2\theta$ which is unbalanced.

For *three cranks at* 120° ($\alpha = 120°$; $\beta = 240°$)

$$\begin{aligned}&(1)\ 1+\cos 120°+\cos 240° = 1-\cos 60°-\cos 60° &&= 0\\ &(2)\ \sin 120°+\sin 240° = \sin 60°-\sin 60° &&= 0\\ &(3)\ 1+\cos 240°+\cos 480° = 1-\cos 60°-\cos 60° &&= 0\\ &(4)\ \sin 240°+\sin 480° = -\sin 60°+\sin 60° &&= 0\end{aligned}$$

Thus, both primary and secondary forces are balanced.

For *complete inertia balance,* a further necessary condition is that the sum of the moments of the forces at any instant, taken about any plane at right-angles to the axis of the engine shaft, shall be zero.

Thus, if x is the distance of any set of reciprocating parts from the reference plane XY (Fig. 241), this condition becomes

$$\frac{W}{g} \,.\, \omega^2 \,.\, r\left[x_1\left\{\cos\theta+\frac{\cos 2\theta}{n}\right\} + x_2\left\{\cos(\theta+\alpha)+\frac{\cos 2(\theta+\alpha)}{n}\right\}+\text{etc.}\right] = 0 \quad . \quad (\text{XII.10})$$

From (XII.10) we derive the following additional conditions for *couple balance*—

$$\left.\begin{aligned}&(1)\ x_1+x_2\,.\,\cos\alpha+x_3\,.\,\cos\beta+\text{etc.} &&= 0\\ &(2)\ x_2\sin\alpha+x_3\sin\beta+\text{etc.} &&= 0\end{aligned}\right\}\begin{aligned}&\text{Primary couple}\\ &\text{balance}\end{aligned}$$

$$\left.\begin{aligned}&(3)\ x_1+x_2\,.\,\cos 2\alpha+x_3\,.\,\cos 2\beta+\text{etc.} &&= 0\\ &(4)\ x_2\,.\,\sin 2\alpha+x_3\sin 2\beta+\text{etc.} &&= 0\end{aligned}\right\}\begin{aligned}&\text{Secondary couple}\\ &\text{balance}\,(\text{XII.11})\end{aligned}$$

Taking the case of *three equidistant cranks at* 120°.
($\alpha = 120°$; $\beta = 240°$; $x_1 = 2a + c$; $x_2 = a + c$; $x_3 = c$) and inserting values in (XII.11), we obtain

(1) $(2a + c) + (a + c) \cos 120° + c \,.\, \cos 240°$

$$= 2a + c - \frac{a+c}{2} - \frac{c}{2} = \frac{3}{2} \,.\, a$$

(2) $(a + c) \,.\, \sin 120° + c \,.\, \sin 240°$

$$= \frac{\sqrt{3}}{2}(a + c - c) = \frac{\sqrt{3}}{2} \,.\, a$$

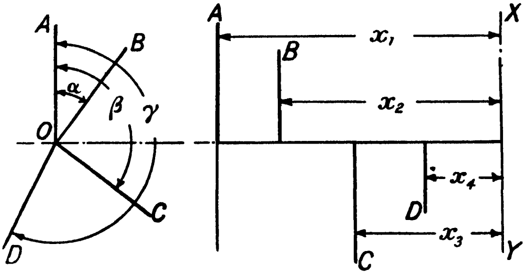


FIG. 241. BALANCING OF RECIPROCATING PARTS

(3) $(2a + c) + (a + c) \,.\, \cos 240° + c \,.\, \cos 480°$

$$= 2a + c - \frac{a+c}{2} - \frac{c}{2} = \frac{3}{2} \,.\, a$$

(4) $(a + c) \,.\, \sin 240° + c \,.\, \sin 480°$

$$= \quad \frac{\sqrt{3}}{2}(a + c) + \frac{\sqrt{3}}{2} \,.\, c = -\frac{\sqrt{3}}{2} \,.\, a$$

There is thus an unbalanced couple C acting in the plane of the axes of the cylinders, so that

$$C = \frac{W}{g} \,.\, \omega^2 \,.\, r \left\{ \frac{3}{2} \,.\, a \,.\, \cos \theta - \frac{\sqrt{3}}{2} \,.\, a \sin \theta \right\}$$

$$+ \frac{W}{g} \,.\, \omega^2 \,.\, \frac{r}{n} \left\{ \frac{3}{2} a \,.\, \cos 2\theta + \frac{\sqrt{3}}{2} \,.\, a \,.\, \sin 2\theta \right\}$$

$$= \frac{W}{g} \,.\, \omega^2 \,.\, r \,.\, \sqrt{3}\, a \left\{ \frac{\sqrt{3}}{2} \,.\, \cos \theta - \frac{1}{2} \sin \theta + \frac{\frac{\sqrt{3}}{2} \,.\, \cos 2\theta + \frac{1}{2} \sin 2\theta}{n} \right\}$$

$$= \frac{W}{g} \,.\, \omega^2 \,.\, r \,.\, \sqrt{3}\, a \left\{ \cos(\theta + 30°) + \frac{1}{n} \,.\, \cos(2\theta - 30°) \right\}$$

If the weights of reciprocating parts are respectively W_1, W_2, W_3, etc., and are not equal, the conditions for complete balance are as follows—

$$\left.\begin{array}{ll}(1)\ W_1 + W_2 \cos\alpha + W_3 \cos\beta + \text{etc.} & = 0\\ (2)\ W_2 \sin\alpha + W_3 \sin\beta + \text{etc.} & = 0\end{array}\right\}\text{Primary force balance}$$

$$\left.\begin{array}{ll}(3)\ W_1 + W_2 \cos 2\alpha + W_3 \cos 2\beta + \text{etc.} & = 0\\ (4)\ W_2 \sin 2\alpha + W_3 \sin 2\beta + \text{etc.} & = 0\end{array}\right\}\text{Secondary force balance}$$

$$\left.\begin{array}{ll}(5)\ W_1 \,.\, x_1 + W_2 x_2 \cos\alpha + W_3 x_3 \cos\beta + \text{etc.} & = 0\\ (6)\ W_2 x_2 \sin\alpha + W_3 x_3 \sin\beta + \text{etc.} & = 0\end{array}\right\}\text{Primary couple balance}$$

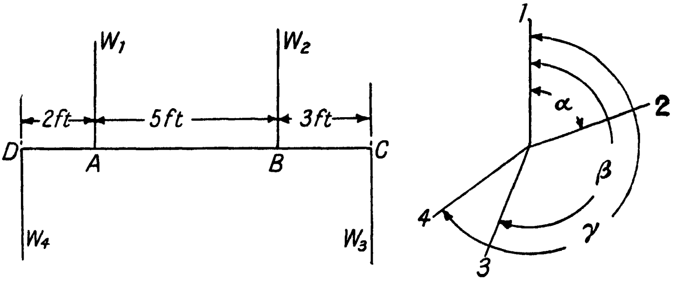


FIG. 242

$$\left.\begin{array}{ll}(7)\ W_1 x_1 + W_2 x_2 \cos 2\alpha + W_3 x_3 \cos 2\beta + \text{etc.} & = 0\\ (8)\ W_2 x_2 \sin 2\alpha + W_3 x_3 \sin 2\beta + \text{etc.} & = 0\end{array}\right\}\text{Secondary couple balance}$$

(XII.12)

EXAMPLE.

Two lines of reciprocating parts at A and B (Fig. 242) are to be balanced for primary forces and couples by two lines of reciprocating parts at C and D. Given $W_1 = 500$ lb.; $W_2 = 750$ lb.; $\alpha = 80°$, to find the necessary values of W_3, W_4, β_1 and γ.

The reference plane for couples will be taken at C for convenience.

Referring to equations (XII.12), the necessary conditions to be satisfied are—

(1) $500 + 750 \times \cdot 174 + W_3 \cos\beta + W_4 \,.\, \cos\gamma = 0$

(2) $750 \times \cdot 985 + W_3 \sin\beta + W_4 \,.\, \sin\gamma = 0$

(5) $500 \times 8 + 750 \times 3 \times \cdot 174 + W_3 \times 0 + W_4 \times 10 \cos\gamma = 0$

(6) $750 \times 3 \times \cdot 985 + W_3 \times 0 + W_4 \times 10 \sin\gamma = 0$

$\therefore$ (1) $W_3 \cdot \cos\beta + W_4 \,.\, \cos\gamma = -630$

(2) $W_3 \,.\, \sin\beta + W_4 \,.\, \sin\gamma = -740$

(5) $W_4 \cos \gamma = -439$

(6) $W_4 \sin \gamma = -222$

From (5) and (6), $\tan \gamma = \dfrac{222}{439} = \cdot 506$

Since $\sin \gamma$ and $\cos \gamma$ are both negative, γ must be between 180° and 270°.

$$\therefore \gamma = 180° + 27° = 207°$$

$\therefore$ From (6) $\quad W_4 = \dfrac{222}{\cdot 454} = 490$ lb.

From (1) $\quad W_3 \cos \beta - 439 = -630$

$$\therefore W_3 \cos \beta = -191 \quad . \quad . \quad . \quad (7)$$

From (2) $\quad W_3 \sin \beta - 222 = -740$

$$\therefore W_3 \sin \beta = -518 \quad . \quad . \quad . \quad (8)$$

$\therefore$ From (7) and (8) $\quad \beta = 250°$

$$W_3 = 562 \text{ lb.}$$

For unbalanced secondary forces we have—

(3) $W_1 + W_2 \cos 2\alpha + W_3 \cos 2\beta + W_4 \cos 2\gamma$

$= 500 - 750 \times \cdot 94 - 562 \times \cdot 766 + 490 \times \cdot 59$

$= -347$ lb.

(4) $W_2 \sin 2\alpha + W_3 \sin 2\beta + W_4 \sin 2\gamma$

$= 750 \times \cdot 34 + 562 \times \cdot 643 + 490 \times \cdot 81$

$= 1015$ lb.

The unbalanced secondary force would thus be

$$F = \frac{\omega^2}{g} \cdot \frac{r}{n} \cdot \{-347 \cos 2\theta - 1015 \sin 2\theta\} \text{ lb.}$$

$$= -1073 \cdot \frac{\omega^2}{g} \cdot \frac{r}{n} \cdot \cos (2\theta - 71°)$$

The maximum value of F will thus be $\dfrac{1073}{n} \cdot \dfrac{\omega^2}{g} \cdot r$

For unbalanced secondary couples we have

(7) $W_1 x_1 + W_2 x_2 \cos 2\alpha + W_3 x_3 \cos 2\beta + W_4 x_4 \cos 2\gamma$

$= 4000 - 2115 - 0 + 578$

$= 2463$ lb.-ft.

$$(8)\quad W_2x_2 \sin 2\alpha + W_3x_3 \sin 2\beta + W_4x_4 \sin 2\gamma$$
$$= 765 + 0 + 796$$
$$= 1561 \text{ lb.-ft.}$$

The unbalanced secondary couple would thus be

$$C = \frac{\omega^2}{g} \cdot \frac{r}{n} \{2463 \cos 2\theta - 1561 \sin 2\theta\}$$
$$= 2920 \cdot \frac{\omega^2}{g} \cdot \frac{r}{n} \cdot \cos (2\theta + 32°) \text{ lb.-ft.}$$

It will be evident that in this case, although the primary forces and couples are completely balanced, there are secondary forces and couples which are by no means of negligible magnitude, and it would be necessary to alter the value of a in order to reduce them.

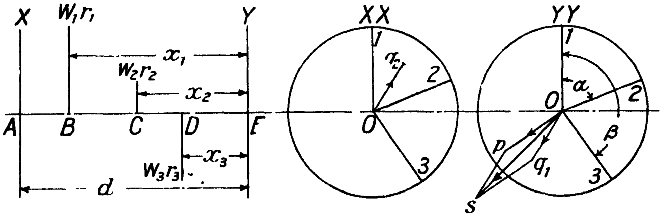


FIG. 243

191. Graphical Method. Considerable time and labour will be saved by using a graphical method such as the following, using the principle of Dalby's method—

In the event of the radii of the cranks driving the various sets of reciprocating parts being unequal, balancing products may be substituted for weights, or, if preferred, equivalent weights at a given crank radius may be used. The method is similar to that used for rotating masses, as the reciprocating mass (for primary balancing) is that mass, revolving at crank radius, the vertical component of the centrifugal force of which is equal to the unbalanced primary force.

Fig. 243 shows the reciprocating masses W_1, W_2, W_3, driven by cranks 01, 02, 03 respectively of radii r_1, r_2, r_3. These are to be balanced for primary forces and couples by reciprocating masses in planes XX and YY, driven by cranks whose angular positions are to be determined.

Tabulate as follows—

Reciprocating weight (1)	Radius (2)	Distance from YY (3)	Product $W . r$ (4)	Couple $W . r . x$ (5)
W_1 W_2 W_3	r_1 r_2 r_3	x_1 x_2 x_3	W_1r_1 W_2r_2 W_3r_3	$W_1r_1x_1$ $W_2r_2x_2$ $W_3r_3x_3$

Draw the "force" polygon (Fig. 244) as before, using the values in column 4.

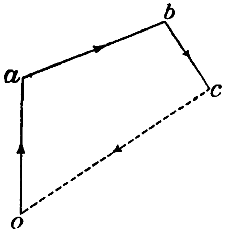


Fig. 244. Force Polygon

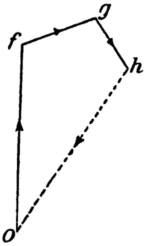


Fig. 245. Couple Polygon

Draw the "couple" polygon (Fig. 245), using the values in column 5.

In plane YY, draw Op equal and parallel to co, the closing line in Fig. 242, and oq_1, parallel to ho, the closing line in Fig. 245, and equal to $\frac{\overline{ho}}{d}$.

In plane XX, draw oq_2 equal and opposite to oq_1 in plane YY. Let $\overrightarrow{Os}$ be the resultant of $\overrightarrow{Op}$ and $\overrightarrow{Oq_1}$.

Then Oq_2 and Os are the angular positions of the cranks which are to be placed in the planes XX and YY respectively. If r_0 is the crank radius in each of these planes, the weight of reciprocating parts in plane XX will be $\frac{\overline{Oq_2}}{r_0}$, and in the plane YY will be $\frac{\overline{Os}}{r_0}$.

192. **Examination for Unbalanced Secondary Forces and Couples.** Let α, β, γ, δ, be the angles which the cranks $O2$, $O3$, Os, Oq_2 will make with $O1$ (Fig. 243).

Set out OP, OQ, OR, OS, OT such that the angles made with OP are 2α, 2β, 2γ, 2δ respectively (Fig. 246).

Using the tabular values above, draw " force " and " couple " polygons as before (Figs. 247 and 248). The couple in the plane YY will be zero, so that $\overline{r_1s_1}$ (Fig. 248) is zero.

Then the unbalanced secondary force is given by

$$F_2 = \frac{\omega^2}{g} \cdot \frac{\overrightarrow{ot}}{n} \cdot \cos 2\theta \quad . \quad . \quad . \quad . \quad . \quad \text{(XII.13)}$$

where ot is the closing line in Fig. 247.

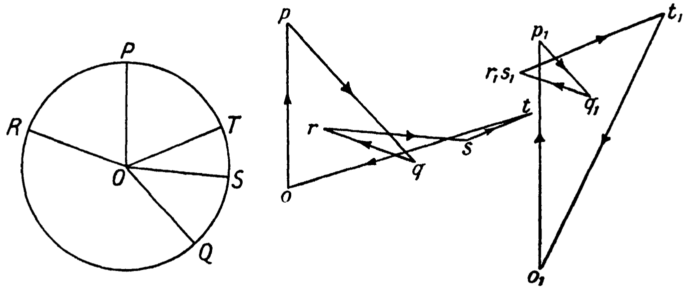


FIG. 246 FIG. 247 FIG. 248

UNBALANCED SECONDARY FORCES AND COUPLES

The unbalanced secondary couple is given by

$$C_2 = \frac{\omega^2}{g} \cdot \frac{\overrightarrow{o_1t_1}}{n} \cdot \cos 2\theta \quad . \quad . \quad . \quad . \quad . \quad \text{(XII.14)}$$

where t_1o_1 is the closing line in Fig. 248.

If " primary " balancing is to produce reasonably satisfactory results, it would evidently be necessary to alter the crank angles at C and D (Fig. 243) until $\overline{ot}$ (Fig. 247) and $\overline{o_1t_1}$ (Fig. 248) are reasonably small.

193. **Balancing the Connecting-rod.** As stated in Art. 41, it is usual in balancing calculations to replace the connecting-rod by weights w_1 and w_2, so that

$$\left.\begin{aligned} w_1 + w_2 &= W \\ w_1l_1 &= w_2l_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(XII.15)}$$

where W is the total weight of the rod and l_1 and l_2 are the distances of w_1 and w_2 from the centre of gravity.

This assumes that the single force due to the inertia of the rod passes through the centre of the crank-shaft, which is not the case in ordinary practice (see Art. 41). For complete dynamical replacement of the rod by w_1 and w_2, the following further condition must be satisfied, i.e.

$$w_1 . l_1^2 + w_2 . l_2^2 = W . k^2 \quad . \quad . \quad . \quad \text{(XII.16)}$$

where k is the radius of gyration of the rod about G.

If F is the single force due to the connecting-rod (Art. 29) and a is its perpendicular distance from the centre of the crank-shaft, then the unbalanced couple with the above replacement will be

$$C = F . a \quad . \quad . \quad . \quad \text{(XII.17)}$$

tending to produce torsional oscillations of the frame of the engine round the shaft. The magnitude of C is proportional to the value of $(k^2 - l_1 l_2) . \omega^2$, and would be reduced to zero if, by extending the mass of the rod beyond the crank-pin, we made $k^2 = l_1 l_2$. This is not convenient in practice, owing to the extra clearance required in the crank-case and the increase in the weight of the rod.

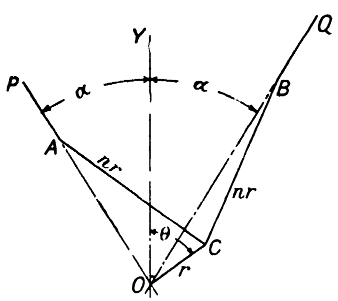


FIG. 249. THE VEE ENGINE

194. **The Vee Engine.** Let OP, OQ be the lines of stroke of two cylinders driving a common crank OC by means of connecting-rods AC, BC respectively. OP and OQ are inclined to the vertical OY at an angle α, as shown (Fig. 249).

Let θ = inclination of the crank OC to the vertical OY at any instant.

Then, inertia force along PO (towards O) is given by

$$F_1 = \frac{W}{g} . r . \omega^2 \left\{ \cos(\theta + \alpha) + \frac{1}{n} . \cos 2(\theta + \alpha) \right\} \quad \text{(XII.18)}$$

Similarly, inertia force along QO (towards O) is

$$F_2 = \frac{W}{g} . r . \omega^2 \left\{ \cos(\theta - \alpha) + \frac{1}{n} . \cos 2(\theta - \alpha) \right\} \quad . \quad \text{(XII.19)}$$

Vertical component of $F_1 = F_1 \cos\alpha$

$$= \frac{W}{g} . r \quad \omega^2 \left\{ \cos(\theta + \alpha) . \cos\alpha \right.$$
$$\left. + \frac{1}{n} \quad \cos 2(\theta + \alpha) . \cos\alpha \right\} \quad . \quad . \quad . \quad \text{(XII.20)}$$

Vertical component of $F_2 = F_2 \,.\, \cos \alpha$

$$\begin{aligned} &= \frac{W}{g} \,.\, r \,.\, \omega^2 \biggl\{ \cos(\theta - \alpha) \,.\, \cos \alpha \\ &\qquad + \frac{1}{n} \cos 2(\theta - \alpha) \,.\, \cos \alpha \biggr\} \qquad . \quad . \quad . \qquad \text{(XII.21)} \end{aligned}$$

∴ *Net vertical force* (downward) is

$$\begin{aligned} F_V &= \frac{W}{g} \,.\, r \,.\, \omega^2 \,.\, \cos \alpha \biggl[\{ \cos(\theta + \alpha) + \cos(\theta - \alpha) \} \\ &\qquad + \frac{1}{n} \{ \cos 2(\theta + \alpha) + \cos 2(\theta - \alpha) \} \biggr] \\ &= \frac{W}{g} \,.\, r \,.\, \omega^2 \,.\, \cos \alpha \left(2 \cos \theta \,.\, \cos \alpha + \frac{2}{n} \,.\, \cos 2\theta \,.\, \cos 2x \right) \\ &= \frac{W}{g} \,.\, r \,.\, \omega^2 \biggl\{ \cos \theta \, (1 + \cos 2\alpha) \\ &\qquad + \frac{1}{n} \cos 2\theta \, (\cos \alpha + \cos 3\alpha) \biggr\} \qquad . \quad . \qquad \text{(XII.22)} \end{aligned}$$

Horizontal component of F_1 (to right) $= F_1 \sin \alpha$

$$\begin{aligned} &= \frac{W}{g} \,.\, r \,.\, \omega^2 \biggl\{ \cos(\theta + \alpha) \,.\, \sin \alpha \\ &\qquad + \frac{1}{n} \,.\, \cos 2(\theta + \alpha) \,.\, \sin \alpha \biggr\} \qquad . \quad . \quad . \qquad \text{(XII.23)} \end{aligned}$$

Horizontal component of F_2 (to left) $= F_2 \sin \alpha$

$$\begin{aligned} &= \frac{W}{g} \,.\, r \,.\, \omega^2 \biggl\{ \cos(\theta - \alpha) \,.\, \sin \alpha \\ &\qquad + \frac{1}{n} \,.\, \cos 2(\theta - \alpha) \,.\, \sin \alpha \biggr\} \qquad . \quad . \quad . \qquad \text{(XII.24)} \end{aligned}$$

∴ *Net horizontal force* (to right) is

$$\begin{aligned} F_H &= \frac{W}{g} \,.\, r \,.\, \omega^2 \,.\, \sin \alpha \biggl[\{ \cos(\theta + \alpha) - \cos(\theta - \alpha) \} \\ &\qquad + \frac{1}{n} \{ \cos 2(\theta + \alpha) - \cos 2(\theta - \alpha) \} \biggr] \\ &= -\frac{W}{g} \,.\, r \,.\, w^2 \,.\, \sin \alpha \left(2 \sin \theta \,.\, \sin \alpha + \frac{2}{n} \,.\, \sin 2\theta \,.\, \sin 2\alpha \right) \\ &= -\frac{W}{g} \,.\, r \,.\, \omega^2 \biggl\{ \sin \theta \, (1 - \cos 2\alpha) \\ &\qquad + \frac{1}{n} \,.\, \sin 2\theta \, (\cos \alpha - \cos 3x) \biggr\} \qquad . \quad . \quad . \qquad \text{(XII.25)} \end{aligned}$$

The resultant of F_H and F_V is equivalent to the resultant of the inertia forces due to—

(i) A weight W at crank radius, revolving with the crank.

(ii) A weight $W \, . \cos 2\alpha$ at crank radius, revolving in the opposite direction to the crank.

(iii) A weight $\frac{W}{n} . \cos \alpha$ at crank radius, revolving in the same direction as the crank at twice crank speed.

(iv) A weight $\frac{W}{n} . \cos 3\alpha$ at crank radius, revolving in the opposite direction to the crank at twice crank speed.

If $\alpha = 30°$

$$F_V = \frac{W}{g} . r . \omega^2 \left[\frac{3}{2} . \cos\theta + \frac{\sqrt{3}}{2n} . \cos 2\theta\right]$$

$$F_H = -\frac{W}{g} . r . \omega^2 \left[\frac{1}{2} \sin\theta + \frac{\sqrt{3}}{2n} . \sin 2\theta\right]$$

If $\alpha = 45°$

$$F_V = \frac{W}{g} . r . \omega^2 . \cos\theta$$

$$F_H = -\frac{W}{g} . r . \omega^2 \left[\sin\theta + \frac{\sqrt{2}}{n} . \sin 2\theta\right]$$

In this case the primary forces can be balanced by a weight W at crank radius, opposite to the crank OC.

If $\alpha = 90°$ (opposed cylinders, line of stroke horizontal)

$$F_V = 0$$

$$F_H = -\frac{2W}{g} . r . \omega^2 \sin\theta$$

In this case the engine could be balanced by a weight $2W$ reciprocating horizontally with simple harmonic motion due to a crank equal and opposite to OC.

195. **More Exact Expression for Inertia Force.** The inertia force due to a single line of reciprocating parts may be expressed as a Fourier series (see Appendix B) as follows—

$$F = \frac{W}{g} . r . \omega^2 \{\cos\theta + A_2 \cos 2\theta + A_4 \cos 4\theta + A_6 \cos 6\theta + \text{etc.}\} \quad . \quad \text{(XII.26)}$$

where A_2, A_4, A_6, etc., are functions of n, the ratio of connecting-rod to crank.

If $n = 4$, $A_2 = 0{\cdot}254$, $A_4 = {\cdot}0041$, $A_6 = {\cdot}000074$.*

A_4, A_6, etc., are very small, but in the case of high-speed automobile and aeroplane engines, where ω is large, the effects of the higher harmonics are not always negligible, and equation (XII.26) will be used in investigating the unbalanced forces in radial engines.

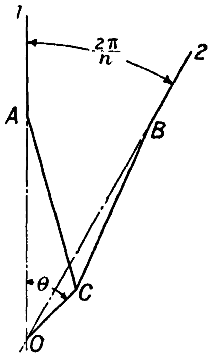


FIG. 250. RADIAL ENGINE

196. **Radial Engine.** Consider an engine with N cylinders, spaced radially at angular intervals $\frac{2\pi}{N}$ round a common crank (Fig. 250).

Primary Unbalanced Forces. By summing the vertical components of inertia forces for each line of reciprocating parts, we obtain for the net vertical primary force

$$F_V = \frac{W}{g} \cdot r \cdot \omega^2 \left\{ \cos\theta \cos 0 + \cos\left(\theta - \frac{2\pi}{N}\right) \cos\frac{2\pi}{N} + \cos\left(\theta - \frac{4\pi}{N}\right) \cdot \cos\frac{4\pi}{N} + \ldots \text{ to } N \text{ terms} \right\} \quad \text{(XII.27)}$$

Since $\cos X \cdot \cos Y = \frac{1}{2}\left\{\cos(X+Y) + \cos(X-Y)\right\}$

$$\therefore \cos\theta \cos 0 = \tfrac{1}{2}\left\{\cos\theta + \cos\theta\right\}$$

$$\cos\left(\theta - \frac{2\pi}{N}\right) \cdot \cos\frac{2\pi}{N} = \tfrac{1}{2}\left\{\cos\theta + \cos\left(\theta - \frac{4\pi}{N}\right)\right\}$$

$$\cos\left(\theta - \frac{4\pi}{N}\right) \cdot \cos\frac{4\pi}{N} = \tfrac{1}{2}\left\{\cos\theta + \cos\left(\theta - \frac{8\pi}{N}\right)\right\}$$

. . . .

. . . .

$$\cos\left\{\theta - \frac{2(N-1)}{N} \cdot \pi\right\} \cdot \cos\frac{2(N-1)\pi}{N} = \tfrac{1}{2}\left[\cos\theta + \cos\left(\theta - \frac{4(N-1)\pi}{N}\right)\right]$$

The sum of the first terms in the brackets on the right-hand sides of the above is $N \cdot \cos\theta$.

---

* See Rowell, *Proc.I.A.E.*, Vol. XVIII, 1924, and Appendix B.

The sum of the second terms on the right-hand side is*

$$\frac{\cos\left\{\theta - \frac{N-1}{N} . 2\pi\right\} . \sin 2\pi}{\sin \frac{2\pi}{N}} = 0, \text{ when } N > 2$$

$$\therefore \underline{F_V = \frac{N}{2} . \frac{W}{g} . r . \omega^2 . \cos\theta} \text{ (downward)} \quad . \quad . \quad \text{(XII.28)}$$

Similarly, $$F_H = \frac{W}{g} . r . \omega^2 \left\{\cos\left(\theta - \frac{2\pi}{N}\right) . \sin\frac{2\pi}{N} + \cos\left(\theta - \frac{4\pi}{N}\right) . \sin\frac{4\pi}{N} + \ldots\right\} \quad . \quad . \quad \text{(XII.29)}$$

From which as above

$$F_H = \frac{N}{2} . \frac{W}{g} . r . \omega^2 . \sin\theta \text{ (to the left)} \quad . \quad . \quad \text{(XII.30)}$$

The resultant of F_V and F_H is therefore

$$= \frac{N}{2} . \frac{W}{g} . r . \omega^2 \sqrt{\cos^2\theta + \sin^2\theta}$$

$$= \frac{N}{2} . \frac{W}{g} . r . \omega^2$$

The primary forces can therefore be balanced by a weight $\frac{N}{2} . W$ opposite the crank at crank radius, rotating with the crank.

197. **Unbalanced Secondary, etc., Forces.** Reasoning as in Art. 196, the net vertical force due to the sum of the terms arising from the term $A_{2k} \cos 2k\,\theta$ in (XII.26) (where k is any positive integer) is

$$F_{V(2k)} = \frac{W}{g} r\, \omega^2 A_{2k} \left\{\cos 2k\theta \cos 0 + \cos 2k\left(\theta - \frac{2\pi}{N}\right) . \cos\frac{2\pi}{N} + \cos 2k\left(\theta - \frac{4\pi}{N}\right) . \cos\frac{4\pi}{N} + \cos 2k\left(\theta - \frac{6\pi}{N}\right) . \cos\frac{6\pi}{N} + \ldots \text{ to } N \text{ terms}\right\} \quad . \quad . \quad . \quad . \quad \text{(XII.31)}$$

But $2 \cos 2k\,\theta . \cos 0 = \cos 2k\theta + \cos 2k\theta$

$$2 \cos 2k\left(\theta - \frac{2\pi}{N}\right) . \cos\frac{2\pi}{N}$$

$$= \cos\left\{2k\theta - (2k-1) . \frac{2\pi}{N}\right\} + \cos\left\{2k\theta - (2k+1) . \frac{2\pi}{N}\right\}$$

---

* See footnote, page 355.

$$2\cos 2k\left(\theta-\frac{4\pi}{N}\right).\cos\frac{4\pi}{N}$$

$$=\cos\left\{2k\theta-(4k-2).\frac{2\pi}{N}\right\}+\cos\left\{2k\theta-(4k+2).\frac{2\pi}{N}\right\}$$

$$2\cos 2k\left(\theta-\frac{6\pi}{N}\right).\cos\frac{6\pi}{N}$$

$$=\cos\left\{2k\theta-(6k-3).\frac{2\pi}{N}\right\}+\cos\left\{2k\theta-(6k+3).\frac{2\pi}{N}\right\}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$2 \times N^{th}$ term

$$=\cos\left\{2k\theta-(N-1)(2k-1).\frac{2\pi}{N}\right\}$$

$$+\cos\left\{2k\theta-(N-1)(2k+1).\frac{2\pi}{N}\right\}$$

Using the formula for the sum of a series of cosines,* we obtain

$$F_{\mathrm{v}(2k)}=\frac{1}{2}\left[\frac{\cos\left\{2k.\theta-\frac{N-1}{2N}.(2k-1).2\pi\right\}.\sin(2k-1)\pi}{\sin\frac{2k-1}{N}.\pi}\right.$$

$$\left.+\frac{\cos\left\{2k.\theta-\frac{N-1}{2N}.(2k+1).2\pi\right\}.\sin(2k+1)\pi}{\sin\frac{2k+1}{N}.\pi}\right]\qquad\text{(XII.32)}$$

Since $\sin(2k \pm 1)\pi$ is zero for all integral values of k, $F_{\mathrm{v}(2k)}$ will be zero unless $\sin\dfrac{2k\pm 1}{N}.\pi$ is also zero, which will be the case if $\dfrac{2k\pm 1}{N}$ is an integer.

Now because $2k \pm 1$ is an odd number $\dfrac{2k\pm 1}{N}$ cannot be an integer if N is an even number, so that *vertical forces other than primary are completely balanced if the number of cylinders is even.*

---

* $\cos\alpha+\cos(\alpha+\delta)+\cos(\alpha+2\delta)+\ldots\ \cos(\alpha+\overline{n-1}\delta)$

$$=\frac{\cos(\alpha+\frac{1}{2}\overline{n-1}\delta).\sin\frac{1}{2}n\delta}{\sin\frac{\delta}{2}}$$

By similar reasoning, we obtain

$$F_{H(2k)} = \tfrac{1}{2}\left[\frac{\cos\left\{2k\,.\,\theta - \dfrac{N-1}{2N}\,.\,(2k-1)\,.\,2\pi\right\}.\,\sin\,(2k-1)\pi}{\sin\dfrac{2k-1}{N}\,.\,\pi} - \frac{\cos\left\{2k\,.\,\theta - \dfrac{N-1}{2N}\,.\,(2k+1)\,.\,2\pi\right\}.\,\sin\,(2k+1)\pi}{\sin\dfrac{2k+1}{N}\,.\,\pi}\right] \qquad \text{(XII.33)}$$

Thus, *horizontal and vertical forces, other than primary, are completely balanced if the number of cylinders is even.* If the number of cylinders is odd this is not the case, the harmonics which are not balanced being found from the relation $\dfrac{2k \pm 1}{N}$ = integer.

In the following table, the harmonics which are not balanced (i.e. the values of $2k$ for which $\dfrac{2k \pm 1}{N}$ is an integer) are indicated by crosses in corresponding columns.

Values of N	Values of $2k$ for which harmonics are not balanced									
	2	4	6	8	10	12	14	16	18	20
1	×	×	×	×	×	×	×	×	×	×
3	×	×		×	×		×	×		
5		×	×				×	×		
7			×	×						×
9				×	×					
11					×	×				
13						×	×			
15							×	×		
17								×	×	
19									×	×
21										×

It will be seen that the lowest unbalanced harmonic is of order $2k$, where $2k = N - 1$.

In the above investigation, it is assumed that the line of thrust of the connecting-rod passes through the centre of the crank-pin in each case. This is not the case in practice, and slight variations of the effective values of r and n occur as a consequence during the revolution of the crank.

198. **Direct and Reverse Cranks.** In the case of a single cylinder engine, the primary force $\left(\frac{W}{g} . r . \omega^2 . \cos\theta\right)$ of (XII.26) could be produced by two masses $\frac{W}{2}$ (Fig. 251A) at crank radius, revolving with angular velocity ω in opposite directions. The components of the centrifugal forces at right-angles to the line of stroke would cancel, while the total component along the line of stroke would be $2 \times \frac{W}{2g} . r . \omega^2 . \cos\theta$ $= \frac{W}{g} . r . \omega^2 . \cos\theta$.

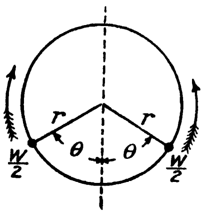
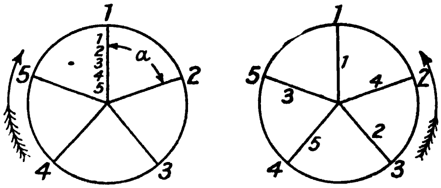


FIG. 251A. DIRECT AND REVERSE CRANKS



Direct Reverse

FIG. 251B. PRIMARY BALANCING, FIVE-CYLINDER RADIAL ENGINE

Similarly, the first harmonic, i.e. the 2θ term, could be produced by two masses at crank radius $\left(\text{each } \frac{A_2}{2} . W\right)$, revolving at angular velocity 2ω in opposite directions. In each case, the mass rotating with the crank is known as the *direct crank*, and the other as the *reverse crank*.

This leads to an easy method of investigating the balance of a radial engine with any number of cylinders for primary and secondary forces. The method can be extended without difficulty to the higher harmonics.

Let Fig. 251B show the directions of axes of the cylinders of a radial engine with any number N of cylinders ($N = 5$ in Fig. 251B), spaced at equal angular distances α apart.

Let α be the angle between the axes of any two adjacent cylinders, i.e. $\alpha = \frac{2\pi}{N}$ radians. Let the cylinders be numbered 1, 2. 3, 4, 5, etc., as shown by numbers outside the circles. When the crank is vertical, i.e. when it is in the inner dead-centre position for cylinder 1, the positions of the direct and reverse cranks for the primary forces due to the various cylinders are as shown in Fig. 251B, the corresponding cylinder numbers being shown inside the circles. For any other crank position the relative positions of each set of cranks will remain unaltered as all the cranks in each set are rotating at the same angular speed.

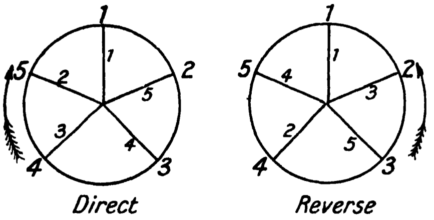


FIG. 252. SECONDARY BALANCING, FIVE-CYLINDER RADIAL ENGINE

Primary unbalanced forces are dealt with as follows: For cylinder line 1, the direct crank will be in line with the cylinder line 1. For cylinder line 2, the direct crank will be an angle α behind cylinder line 2. For line 3, the direct crank will be 2α behind line 3, and so on. On examination of Fig. 251B it will be seen that all the N primary cranks lie on cylinder line 1, and as each primary crank represents a mass $\frac{W}{2}$, the resultant direct crank is a mass of $\frac{NW}{2}$ units moving with the crank. The reverse cranks are dealt with in a similar manner, the angles 0, α, 2α, 3α, etc., in this case being set off on the opposite side of the respective cylinder lines. The result is that the secondary cranks are in the positions shown on the right of Fig. 251B, one

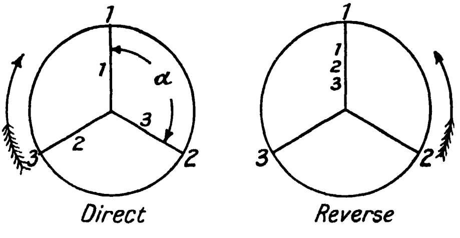


FIG. 253. SECONDARY BALANCING, THREE-CYLINDER RADIAL ENGINE

crank on each cylinder line, giving complete balance. Thus the primary forces are equivalent to a mass of $\frac{NW}{2}$ units moving with the engine crank, and complete primary balance is obtained by placing a mass $\frac{NW}{2}$ at crank radius, opposite to the crank and rotating at the same speed as the crank and in the same sense.

Secondary unbalanced forces are dealt with in a similar manner. The speeds of rotation of the direct and reverse cranks will be twice the crank speed, so that the angles between the cylinder lines and the direct and reverse cranks will be twice those in the case of the primary forces. The reader should have no difficulty in locating the positions of the direct and reverse secondary cranks which are shown in Fig. 252. It will be seen that if there are more than three cylinders, both the direct and reverse secondary cranks are balanced. In the case of a three-cylinder engine (Fig. 253) the direct secondary cranks balance, but the reverse secondary cranks all coincide. Thus the secondary forces can be balanced by a mass $\frac{3W}{2}$ at crank radius rotating in the opposite sense to the crank at twice the crank speed.

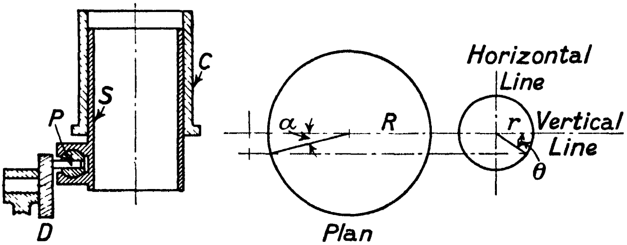


FIG. 254

199. **Balancing Sleeve Valves.** The sleeve valve consists of a sleeve S which is given an oscillating sliding motion in the cylinder combined with an oscillating rotary motion about the axis of the cylinder. The sleeve is driven by a pin P attached to a wheel D which rotates at half the speed of the engine crank-shaft. A spherical bearing in which P is fitted drives the sleeve and allows freedom of movement of P (Fig. 254). Since the majority of high-powered reciprocating aero engines are

now fitted with sleeve valves, an investigation of their effect upon engine balance is desirable.

Let R = external radius of sleeve.

Since the sleeve is thin and the projecting lug has to be allowed for, R may be taken as approximately equal to the radius of gyration about the axis of the cylinder.

Let θ (Fig. 254) be the angular displacement of the driving crank, radius r, from the vertical (assuming the cylinder axis to be vertical) and α the corresponding angular displacement of S (Fig. 254). Then, vertical displacement of S from central position $= r \,.\, \cos\theta$.

Vertical acceleration of $S = -r \,.\, \omega^2 \,.\, \cos\theta$

$$\sin\alpha = \frac{r}{R} \,.\, \sin\theta\,; \quad \cos\alpha = 1 - \frac{r^2}{2R^2}\sin^2\theta \text{ (approx.)}$$

$$\cos\alpha \,.\, \frac{d\alpha}{dt} = \omega \,.\, \frac{r}{R} \,.\, \cos\theta\,; \quad \frac{d\alpha}{dt} = \omega \,.\, \frac{r}{R} \,.\, \cos\theta \,.\, \sec.\ \alpha$$

$$\sec\alpha = 1 + \frac{r^2}{2R^2}\sin^2\theta \text{ (approx.)}$$

$$= \left(1 + \frac{r^2}{4R^2}\right) - \frac{r^2}{4R^2}\cos 2\theta \text{ (approx.)}$$

$$\text{If } r = \frac{R}{3}, \text{ then } \frac{r^2}{4R^2} = 0{\cdot}028.$$

$$\text{Thus, } \frac{d\alpha}{dt} = \omega \,.\, \frac{r}{R} \,.\, \cos\theta\left\{1 - \frac{r^2}{4R^2} \,.\, \cos 2\theta\right\} \text{ (approx.)}$$

$$= \omega \,.\, \frac{r}{R}\left\{\cos\theta - \frac{r^2}{8R^2}(\cos\theta + \cos 3\theta)\right\} \text{ (approx.)} \quad . \quad \text{(XII.34)}$$

$$\text{and } \frac{d^2\alpha}{dt^2} = -\omega^2 \,.\, \frac{r}{R}\left\{\sin\theta - \frac{3}{8} \,.\, \frac{r^2}{R^2} \,.\, \sin 3\theta\right\} \text{ (approx.)} \quad . \quad \text{(XII.35)}$$

If W = weight of sleeve,

$$\text{Vertical inertia force } F = \frac{W}{g} \,.\, r \,.\, \omega^2 \,.\, \cos\theta \quad . \quad . \quad \text{(XII.36)}$$

$$\text{Taking } I = \frac{W}{g} \,.\, R^2,$$

[(XII.37)

$$\text{Horizontal inertia torque } T = \frac{W}{g} \,.\, r \,.\, R \,.\, \omega^2\left\{\sin\theta - \frac{3}{8} \,.\, \frac{r^2}{R^2} \,.\, \sin 3\theta\right\}$$

If ω_c is the angular velocity of the main crank, then $\omega = \frac{\omega_c}{2}$.

Taking 4 *cylinders in line, with main cranks at* 180° *in pairs.*

Since the angular velocity of D is one-half that of the crank-shaft (XII.36) becomes

$$F = \frac{W}{g} . r . \omega^2 \{\cos \theta + \cos (\theta + 90°)$$
$$+ \cos (\theta + 180°) + \cos (\theta + 270°)\} = 0 \quad . \text{(XII.38)}$$

$$T = \frac{W}{g} . r . R . \omega^2[\{\sin \theta + \sin (\theta + 90°)$$
$$+ \sin (\theta + 180°) + \sin (\theta + 270°)\}$$
$$- \frac{3}{8} . \frac{r^2}{R^2} \{\sin 3\theta + \sin (3\theta + 270°)$$
$$+ \sin (3\theta + 180°) + \sin (3\theta + 90°)\}] = 0 \quad . \text{(XII.39)}$$

The sleeves are therefore balanced for vertical forces and horizontal couples.

With 6 *cylinders in line, main cranks at* 120°.

$$F = \frac{W}{g} . r . \omega^2 \{\cos \theta + \cos (\theta + 60°) + \cos (\theta + 120°)$$
$$+ \cos (\theta + 180°) + \cos (\theta + 240°) + \cos (\theta + 300°)\}$$
$$= 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XII.40)}$$

$$T = \frac{W}{g} . r . R . \omega^2 [\{\sin \theta + \sin (\theta + 60°) + \sin (\theta + 120°)$$
$$+ \sin (\theta + 180°) + \sin (\theta + 240°) + \sin (\theta + 300°)\}$$
$$- \frac{3}{8} . \frac{r^2}{R^2} \{\sin 3\theta + \sin (3\theta + 180°) + \sin (3\theta + 360°)$$
$$+ \sin (3\theta + 180°) + \sin 3\theta + \sin (3\theta + 180°)\}]$$
$$T = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XII.41)}$$

The sleeves are therefore balanced both for forces and horizontal couples.

Radial Engine, N cylinders spaced $\frac{2\pi}{N}$ apart round a central axis.

Vertical component of forces

$$F_v = \frac{W}{g} . r . \omega^2 \left\{\cos \theta + \cos \left(\theta + \frac{\pi}{N}\right) . \cos \frac{2\pi}{N}\right.$$
$$\left. + \cos \left(\theta + \frac{2\pi}{N}\right) . \cos \frac{4\pi}{N} + . \text{ to } N \text{ terms}\right\}$$

Proceeding as in Par. 196, we obtain

$$F_v = \frac{W}{g} \cdot r \cdot \frac{\omega^2}{2} \left\{ \frac{\cos\left(\theta - \frac{3}{2} \cdot \frac{N-1}{N} \cdot \pi\right) \cdot \sin\frac{3\pi}{2}}{\sin\frac{3\pi}{2N}} + \frac{\cos\left(\theta - \frac{N-1}{2N} \cdot \pi\right) \cdot \sin\frac{\pi}{2}}{\sin\frac{\pi}{2N}} \right\}$$

$$= \frac{W}{g} \cdot r \cdot \frac{\omega^2}{2} \left\{ \frac{\cos\left(\theta - \frac{N-1}{2N} \cdot \pi\right)}{\sin\frac{\pi}{2N}} - \frac{\cos\left(\theta + \frac{3}{2} \cdot \frac{N-1}{N} \cdot \pi\right)}{\sin\frac{3\pi}{2N}} \right\} \quad . \text{(XII.42)}$$

Similarly,

$$F_B = \frac{W}{g} \cdot r \cdot \frac{\omega^2}{2} \left\{ \frac{\cos\left(\theta - \frac{N-1}{2N} \cdot \pi\right)}{\sin\frac{\pi}{2N}} + \frac{\cos\left(\theta + \frac{3}{2} \cdot \frac{N-1}{N} \cdot \pi\right)}{\sin\frac{3\pi}{2N}} \right\} \quad . \text{(XII.43)}$$

Torque in vertical plane.

$$T_v = \frac{W}{g} \cdot r \cdot R \cdot \omega^2 \Bigg[\bigg\{ \sin\theta + \sin\left(\theta + \frac{\pi}{N}\right) \cdot \cos\frac{2\pi}{N} + \sin\left(\theta + \frac{2\pi}{N}\right) \cdot \cos\frac{4\pi}{N} + \ldots \text{ to } N \text{ terms} \bigg\}$$

$$- \frac{3}{8} \cdot \frac{r^2}{R^2} \bigg\{ \sin 3\theta + \sin\left(3\theta + \frac{3\pi}{N}\right) \cdot \cos\frac{2\pi}{N} + \sin\left(3\theta + \frac{6\pi}{N}\right) \cdot \cos\frac{4\pi}{N} + \ldots \text{ to } N \text{ terms} \bigg\} \Bigg]$$

$$
\begin{aligned}
T_V = \frac{W}{g} \,.\, r \,.\, R \,.\, \frac{\omega^2}{2} \Bigg[& \left\{ \frac{\sin\left(\theta + \frac{3}{2} \,.\, \frac{N-1}{N} \,.\, \pi\right)}{\sin \frac{3\pi}{2N}} \right. \\
& \left. + \frac{\sin\left(\theta - \frac{N-1}{2N}\, \pi\right)}{\sin \frac{\pi}{2N}} \right\} + \frac{3}{8} \,.\, \frac{r^2}{R^2} \left\{ \frac{\sin\left(3\theta + \frac{5}{2} \,.\, \frac{N-1}{N} \,.\, \pi\right)}{\sin \frac{5\pi}{2N}} \right. \\
& \left. + \frac{\sin\left(3\theta + \frac{N-1}{2N} \,.\, \pi\right)}{\sin \frac{\pi}{2N}} \right\} \Bigg] \quad . \quad . \quad . \quad \text{(XII.44)}
\end{aligned}
$$

Torque in horizontal plane.

$$
\begin{aligned}
T_H = \frac{W}{g} \,.\, r \,.\, R \,.\, \omega^2 \Bigg[& \left\{ \sin\theta \,.\, \sin 0^\circ + \sin\left(\theta + \frac{\pi}{N}\right) \,.\, \sin \frac{2\pi}{N} \right. \\
& \left. + \sin\left(\theta + \frac{2\pi}{N}\right) \,.\, \sin \frac{4\pi}{N} \,.\,.\,.\, \text{ to } N \text{ terms} \right\} \\
& - \frac{3}{8} \,.\, \frac{r^2}{R^2} \left\{ \sin 3\theta \,.\, \sin 0^\circ + \sin\left(3\theta + \frac{3\pi}{N}\right) \,.\, \sin \frac{2\pi}{N} \right. \\
& \left. + \sin\left(3\theta + \frac{6\pi}{N}\right) \,.\, \sin \frac{4\pi}{N} \,.\,.\,.\, \text{ to } N \text{ terms} \right\} \Bigg] \\
= \frac{W}{g} \,.\, r \,.\, R \,.\, \frac{\omega^2}{2} \Bigg[& \left\{ \frac{\cos\left(\theta - \frac{N-1}{2N} \,.\, \pi\right)}{\sin \frac{\pi}{2N}} \right. \\
& + \frac{\cos\left(\theta + \frac{3}{2} \,.\, \frac{N-1}{N} \,.\, \pi\right)}{\sin \frac{3\pi}{2N}} - \frac{3}{8} \,.\, \frac{r^2}{R^2} \left\{ \frac{\cos\left(3\theta + \frac{N-1}{2N} \,.\, \pi\right)}{\sin \frac{\pi}{2N}} \right. \\
& \left. - \frac{\cos\left(3\theta + \frac{5}{2} \,.\, \frac{N-1}{N} \,.\, \pi\right)}{2 \sin \frac{5\pi}{2N}} \right\} \Bigg] \quad . \quad . \quad . \quad \text{(XII.45)}
\end{aligned}
$$

If $N = 9$, values of forces and couples are as follows—

$$F_V = \frac{W}{g} \cdot r \cdot \frac{\omega^2}{2} \left\{ \frac{\cos(\theta - 80^\circ)}{\sin 10^\circ} - \frac{\cos(\theta + 240^\circ)}{\sin 30^\circ} \right\}$$

$$= 2{\cdot}21 \cdot \frac{W}{g} \cdot r \cdot \omega^2 \cos(\theta + 63^\circ)$$

$$F_H = \frac{W}{g} \cdot r \cdot \frac{\omega^2}{2} \left\{ \frac{\cos(\theta - 80^\circ)}{\sin 10^\circ} + \frac{\cos(\theta + 240^\circ)}{\sin 30^\circ} \right\}$$

$$= 3{\cdot}88 \cos(\theta + 90^\circ)$$

$$\begin{aligned} T_V &= \frac{W}{g} \cdot r \cdot R \cdot \omega^2 \left[\left\{ \frac{\sin(\theta - 80^\circ)}{2 \sin 10^\circ} - \frac{\sin(\theta + 240^\circ)}{2 \sin 30^\circ} \right\} \right. \\ &\qquad \left. - \frac{3}{8} \cdot \frac{r^2}{R^2} \left\{ \frac{\sin(3\theta + 40^\circ)}{2 \sin 50^\circ} + \frac{\sin(3\theta + 80^\circ)}{2 \sin 10^\circ} \right\} \right] \end{aligned}$$

$$= \frac{W}{g} \cdot r \cdot R \cdot \omega^2 \left\{ 2{\cdot}21 \cos(\theta + 208^\circ) - 1{\cdot}25 \frac{r^2}{R^2} \cdot \cos(3\theta - 17^\circ) \right\}$$

$$\begin{aligned} T_H &= \frac{W}{g} \cdot r \cdot R \cdot \omega^2 \left[\left\{ \frac{\cos(\theta - 80^\circ)}{2 \sin 10^\circ} + \frac{\cos(\theta + 240^\circ)}{2 \sin 30^\circ} \right\} \right. \\ &\qquad \left. - \frac{3}{8} \cdot \frac{r^2}{R^2} \left\{ \frac{\cos(3\theta + 80^\circ)}{2 \sin 10^\circ} - \frac{\cos(3\theta + 40^\circ)}{2 \sin 50^\circ} \right\} \right] \end{aligned}$$

$$= \frac{W}{g} \cdot r \cdot R \cdot \omega^2 \left\{ 3{\cdot}7 \cos(\theta + 74{\cdot}5^\circ) - 1{\cdot}28 \frac{r^2}{R^2} \cos(3\theta + 90^\circ) \right\}$$

In this case it will be seen that the sleeves are not balanced either for forces or couples, and parts of the airplane structure which are responsive to vibration of frequency equal to one-half crank speed and 1·5 of crank speed must be stiffened or damped. It must, however, be remembered that the mass of the engine is relatively large, and consequently only a small fraction of the forces and couples will be transmitted to the airplane structure.

200. **Locomotive Balancing.** The usual practice with two-crank engines is to balance the whole of the revolving parts and from one-third, or less, to two-thirds of the reciprocating parts by revolving weights in the planes of the wheels. The problem is thus reduced to finding the balancing products in

the planes of the wheels required to balance a number of rotating weight-arm products in planes passing through the lines of stroke of the cylinders, eccentrics, etc., and this may be solved by the methods given previously.

The effect of the unbalanced portion of the reciprocating parts is to produce alternating forces and couples in a horizontal plane, the forces producing variations in the draw-bar pull, and tending to produce fore-and-aft oscillations of the locomotive on the springs at certain speeds, while the couples tend to produce " nosing " of the locomotive at high speeds.

The effect of balancing the horizontal components of the primary forces and couples due to reciprocating parts by rotating masses is to introduce vertical forces and couples of equal magnitudes, causing variations of the pressures between wheels and rails and, at certain speeds, actual lifting of the wheel with the corresponding " hammer blow " on its return.*

If $w \, . \, r$ is the weight-arm product required in each wheel plane to balance the selected proportion of the reciprocating parts, then the variation of total wheel pressure in each plane is given by $\pm \frac{w}{g} r\omega^2$.

In the case of an engine with coupled wheels, this weight-arm product may be divided between the coupled wheels, thus reducing the variation of individual wheel pressures in proportion to the number of coupled axles.

If W_1 = load on any wheel

wr = balancing product in that wheel for *reciprocating parts only*,

then the wheel will be about to lift when $\frac{w}{g} r\omega^2 = W_1$

i.e. $$\omega = \sqrt{\frac{gW_1}{wr}} \quad . \quad . \quad . \quad . \quad . \qquad \text{(XII.46)}$$

If V is the speed in miles per hour, and the diameter of the wheel is d ft., then

$$V = \frac{d\omega}{2} \times \frac{3600}{5280} = \frac{d\omega}{2{\cdot}93} \quad . \quad . \quad . \quad . \quad . \qquad \text{(XII.47)}$$

---

* The effects of " overbalancing " in this way, and the degree of overbalance required, are discussed by E. S. Cox, *A.M.I.Mech.E.*, in *Proc. I. Mech.E.*, *Dec.* 1941.

∴ Speed in miles per hour at which the wheel is about to lift is

$$V_l = \frac{d}{2{\cdot}93}\sqrt{\frac{gW_1}{wr}} \qquad . \quad . \quad . \quad . \quad . \qquad \text{(XII.48)}$$

Let W = *total* load on the coupled axles (lb.)

F = maximum vertical component of the resultant force due to the balance weights for *reciprocating parts only* (lb.)

T = useful torque exerted by engine when F is a maximum (lb.-ft.)

μ = co-efficient of friction between wheels and rails,

then slipping is about to occur when

$$\frac{2T}{d} = \mu(W - F)$$

i.e. when

$$F = W - \frac{2T}{\mu \,.\, d} \qquad . \quad . \quad . \quad . \quad . \qquad \text{(XII.49)}$$

The resultant forces F may be found by one of the preceding methods; the couples have no effect on the *total* loading.

If $w_o r_o$ is the resultant weight-arm product producing the force F, then

$$F = \frac{w_o r_o}{g} \,.\, \omega^2 = \frac{w_o \,.\, r_o}{g} \,.\, \left(\frac{2{\cdot}93V}{d}\right)^2$$

∴ From (XII.49) the speed in miles per hour at which slipping is about to occur is given by

$$V = \frac{d}{2{\cdot}93}\sqrt{\frac{g}{w_o \,.\, r_o}\left\{W - \frac{2T}{\mu \,.\, d}\right\}} \qquad . \quad . \quad . \qquad \text{(XII.50)}$$

If P is the draw-bar pull in pounds, at V miles per hour, then.

since $P = \frac{2T}{d}$, slipping is about to occur when

$$V = \frac{d}{2{\cdot}93}\sqrt{\frac{g}{w_o r_o}\left(W - \frac{P}{\mu}\right)}$$

i.e. $W - \dfrac{P}{\mu} = \dfrac{8{\cdot}57V^2}{d^2} \,.\, \dfrac{w_o r_o}{g} = \dfrac{{\cdot}255V^2}{d^2} \,.\, w_o r_o$

$$\therefore P = \mu\left\{W - {\cdot}255 \,.\, \frac{V^2}{d^2} \,.\, w_o \,.\, r_o\right\} \qquad . \quad . \quad . \qquad \text{(XII.51)}$$

Taking some figures for a modern locomotive, if $W = 40$ tons $= 89{,}600$ lb.; $V = 65$ m.p.h.; $d = 6{\cdot}67$ ft.; $\mu = {\cdot}25$; $w_o r_o = 470$ lb.-ft.; $w \,.\, r = 175$ lb.-ft. (four wheels coupled), slipping is about to occur when

$$\begin{aligned} P &= {\cdot}25\{89{,}600 - {\cdot}255 \times 96 \times 470\} \\ &= 19{,}550 \text{ lb.} \\ &= \underline{8{\cdot}7 \text{ tons.}} \end{aligned}$$

Except possibly during the initial accelerating period, P is considerably less in practice.

Lifting of the wheel will occur when

$$V_l = \frac{6{\cdot}69}{2{\cdot}93}\sqrt{\frac{32 \times 22{,}400}{175}} = 146 \text{ m.p.h.}$$

In the case of a three-cylinder locomotive with cranks at 120°, the forces due to reciprocating parts are completely balanced, and it is only necessary to balance for couples.

[For further particulars and a series of calculations relating to locomotives, a paper by Professor Dalby (*Proc.I.Mech.E.*, November, 1901) may be consulted.]

Examples 12.

1. Three masses A, B, and C, weighing 2 lb., 6 lb., and 5 lb. respectively, are rigidly attached to a disc at radii of 10 in., 5 in., and 9 in. respectively. The angles between A and B, B and C, and C and A are 60°, 165°, and 135° respectively. A fourth mass D is attached to the disc at a radius of 4 in. Determine the magnitude and angular position of D so that the four masses may completely balance one another when the disc is rotated.

Explain why, for complete balance, mass D must be placed in the same plane as the masses A, B, and C. (I.C.E.)

2. The following is the schedule of a number of rotating weights in the same plane, revolving with the same angular velocity about an axis perpendicular to the plane of the weights. Find the magnitude and angular position of the weight required at a radius of 1·3 ft. for complete balance. θ is measured in a clockwise direction from the vertical.

Weight (lb.)	2	4	3	2·5	7
Radius (ft.)	1	2	1·5	1·8	0·8
θ (degrees)	12	45	120	200	250

3. A number of rotating masses (Fig. 255), in positions 1, 2, 3, 4, 5, as shown, are to be balanced by two weights in planes XX and YY respectively, each at a radius of 1·3 ft. The distance between XX and YY is 5·5 ft. Find the magnitudes and angular positions of the balance weights.

Weights (lb.)	2	4	3	2·5	7
Radius (ft.)	1	2	1·5	1·8	0·8
θ (degrees)	12	45	120	200	250
Distance from XX (ft.)	1	2	3·2	4	4·5

4. A shaft carries four revolving masses, *A*, *B*, *C*, and *D*, in this order along its axis. The mass *A* may be assumed concentrated at a radius of 12 in., *B* at 15 in., *C* at 14 in., *D* at 18 in. The weights at *A*, *C*, and *D* are 15 lb., 10 lb., and 8 lb. respectively. The planes of revolution of *A* and *B* are 15 in. apart, and of *B* and *C* 18 in. apart. The angle between the masses of *A* and *C* is 90°. Determine (1) the angles between the masses *A*, *B*, and *D*, (2) the distance between the planes of revolution of *C* and *D*, and (3) the weight of the mass *B*, so that the shaft may be in perfect balance. (I.C.E.)

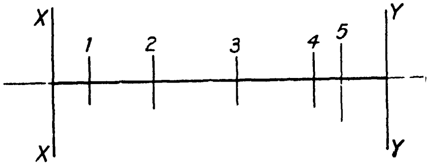


FIG. 255

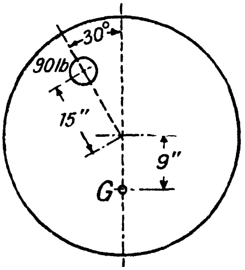


FIG. 256

5. A casting under process of machining weighs 450 lb. and is bolted to the face plate of a lathe. Its centre of gravity *G* is 9 in. from the lathe axis and 14 in. from the face plate

The workman obtains static balance by means of two weights bolted to the face plate; one of these weighs 90 lb. and is fixed in the position shown in the figure (Fig. 256). Its centre of gravity is 5 in. from the face plate. The other weighs 80 lb. and has its centre of gravity 4 in. from the face plate.

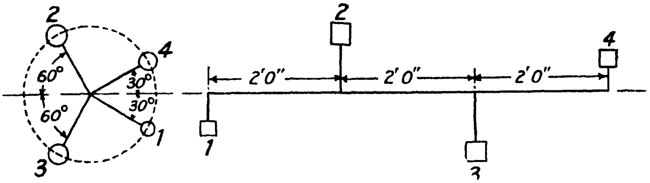


FIG. 257

Find the position of the centre of gravity of the second balance weight. Find also the magnitude of the rocking couple when the lathe is run at 60 r.p.m. (U L.)

6. A rotating drum, 5 ft. long and 4 ft. in diameter, is out of balance by reason of two weights, one of 10 lb. at one end of the drum acting at a radius of 2 ft., and the other of 5 lb. at the other end of the drum acting at a radius of 1·5 ft., displaced 90° clockwise as seen in end elevation. Find the magnitude and position of two weights placed on the surface of the drum which will produce correct dynamical balance; one of these weights is to be placed in the same cross-sectional plane as the 10 lb. weight and the other midway between the two ends of the drum. (I.C.E.)

7. A single-cylinder vertical engine with a stroke of 18 in. runs at 250 r.p.m. The reciprocating parts weigh 120 lb., and may be assumed to have simple harmonic motion; the revolving parts are equivalent to a mass of 70 lb. at a radius of 9 in. Determine the balancing mass, to be placed opposite the

crank and at a radius of 15 in., which is equivalent to all the revolving mass and two-thirds of the reciprocating mass.

Find also the magnitude of the remaining unbalanced force when the crank has turned through an angle of 30° from a dead centre. (U.L.)

8. A four-cylinder engine is arranged as shown in Fig. 257. The reciprocating masses in planes 1 and 4 are each 200 lb., and in planes 2 and 3 they are each 346 lb. If the crank radius is 1 ft., length of connecting-rod 4 ft., and the speed 120 r.p.m., determine the magnitude of the primary and secondary forces and couples. (U.L.A.)

9. The figure (Fig. 258) shows the positions of the centre lines of the cylinders and valve gears of a four-cylinder engine. The rotating masses to be balanced and the angular positions of the cranks are as below—

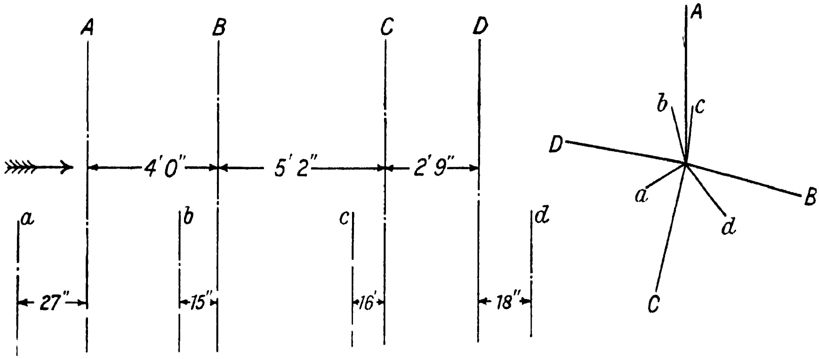


FIG. 258

Crank	A	B	C	D	a	b	c	d
Weight (lb.)	450	620	705	390	170	140	125	130
Radius (in.)	12	12	12	12	4·5	5	5	6
Angle (degrees)	0	105	195	280	240	345	5	140

(a) Find the magnitudes and angular positions of the balance weights required at 9 in radius in planes A and D.

(b) Assuming that the balance weights have been applied, and that the effect of gravity is negligible, find the couple tending to bend the shaft midway between B and C. R.p.m. = 180.

10. In a four-cylinder petrol engine, the cranks, numbered from the front end, are 1, 2, 3, and 4. Cranks 1 and 4 are in phase and 180° ahead of cranks 2 and 3. The reciprocating masses of each cylinder weigh 2 lb. The cranks are 2 in. radius and the connecting-rods 8 in. long.

Investigate the conditions as to primary and secondary balance. What are the resultant unbalanced forces and couples when cranks 1 and 4 are on top dead centre and also at 30° to that position? The engine is rotating at 1,500 r.p.m. in a clockwise direction when viewed from the front. (U.L.)

11. The cylinders of a four-cylinder vertical marine engine are pitched at equal distances apart. The reciprocating masses for each of the two middle lines of parts weigh 1 ton, and the cranks for these are set at an angle of 100°. Find the angle between each outside crank and its neighbour, and the weight of each set of reciprocating parts driven by the outer cranks so that the engine shall be in balance for primary forces and couples. (I.C.E.)

12. A simple inside cylinder locomotive has its cranks at right-angles. The cylinder centre lines are 2 ft. apart, and the distance between the planes of the balance weights is 5 ft. The masses to be balanced for each cylinder are 650 lb. rotating and 525 lb. reciprocating, at 12 in. radius. Find the necessary balance weights at a radius of 32 in. Balance all the rotating and two-thirds of the reciprocating masses. Give the positions of the centres of gravity of the balance weights relative to the cranks.

13. Taking the particulars of the last example, find the greatest vertical pressure between a driving wheel and the rail when the engine has driving wheels 7 ft. in diameter, and is travelling at 60 miles per hour, the vertical pressure when the engine is at rest being 8 tons.

14. The following data refer to a locomotive with cranks at 90° and wheels uncoupled: wheel centres, 5 ft.; cylinder centres, 2 ft.; diameter of driving wheels, 6 ft.; radius of C.G. of balance weights, 2 ft. 2 in.; crank radius, 13 in.; reciprocating parts per cylinder, 660 lb.; revolving parts per cylinder, 600 lb.

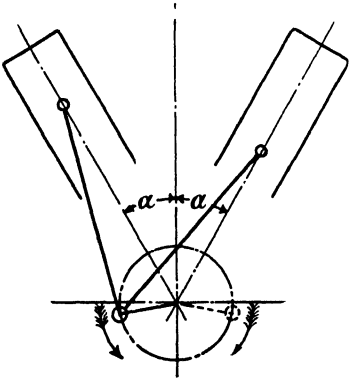


FIG. 259

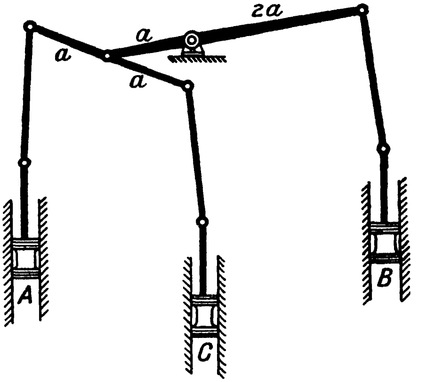


FIG. 260

(a) Find the balancing masses in the driving wheels required to balance all the revolving parts and two-thirds of the reciprocating parts.

(b) Find the speed at which the wheel just begins to lift, assuming that the load on each driving wheel is 7·5 tons and that the balance weights are placed only on the driving wheels; find also the maximum swaying couple at this speed.

(c) If the leading, driving, and trailing axles are coupled together, show how to find the revised balancing masses and the variation of the rail load per wheel. What further data would you require? (U.L.A.)

15. How far is it possible to secure balance of the primary and secondary forces and couples in a four-crank engine? Deduce the equations of equilibrium for the primary forces and couples in such an engine, and show by sketches the arrangement of the cranks which will satisfy the equations. (U.L A.)

16. A twin cylinder Vee engine has cylinders at 90°, which drive on to a single crank. The weight of the reciprocating parts of each cylinder, including one-third of the weight of the connecting-rod, is 15 lb. The weight of the crank-pin and webs plus two-thirds of the weight of each connecting-rod is equivalent to 30 lb. at the crank-pin. The stroke is 6 in. Determine the primary unbalanced force at 1,000 r.p.m. (U.L.)

17. Two cylinders, whose axes are perpendicular, drive the same crank. If the weight of the reciprocating parts of each piston-line is 100 lb., and if that of the rotating parts is 80 lb., and the speed is 300 r.p.m., find the magnitude and position of the balance weight which, fixed at the same radius as the crank-pin, will give complete primary force balance. (U.L.)

18. A Vee type twin-cylinder petrol motor (Fig. 259) has its cylinder centre lines in the same plane and inclined to each other at an angle 2α. The pistons and connecting-rods are exactly similar and are connected to a single crank-pin.

If M is the mass of the reciprocating parts of each cylinder, show that the "primary" out-of-balance effect due to the reciprocating masses when the crank revolves at constant speed is equivalent to that produced by a mass M revolving with the crank-pin, together with a mass $M \cdot \cos 2a$ attached to an imaginary crank of the same radius and revolving at the same speed as the actual crank, but in the opposite direction, the two cranks being always symmetrically disposed with respect to the cylinders. (U.L.)

19. Obtain expressions for the unbalanced forces in a radial engine with k equal fixed cylinders spaced at equal angular intervals, and show that the primary force is unbalanced and has a value $F = \frac{1}{2}m \cdot r \cdot k \cdot \omega^2$, where m is the mass of the reciprocating parts of one cylinder, r is the radius of the crank, and ω its angular velocity.

Show that if k is even, there are no other unbalanced forces, and find expressions for the magnitudes of the unbalanced forces when there is an odd number of cylinders. (U.L.A.)

20. In a nine-cylinder rotary engine the fixed crank is 3·35 in. and the connecting-rod is 11 in. long. Taking the fixed crank as vertical, obtain an expression for the turning moment per cylinder in terms of the total pressure P lb. on the piston.

Find also expressions for the radial and transverse velocities and accelerations of a piston when the cylinders are rotating at a constant speed of 1,200 r.p.m. (U.L A.)

21. Taking the data of question 10, and using the more exact expression for inertia force (XII.26), find the maximum unbalanced force corresponding to the 4θ term when the crank is revolving at 3,000 r.p.m.

22. Taking the data of question 11, find expressions for the unbalanced secondary forces and couples when the engine is balanced for primary forces and couples.

If the rotating parts are each equivalent to 1,500 lb. at crank radius, find the magnitudes and positions of the balance weights at crank radius on the outer cranks for balancing the rotating parts. Take $n = 4$, distance between centre lines 5 ft., $r = 1{\cdot}5$ ft., and r.p.m. = 120.

23. A locomotive is driven by an engine having three cylinders with cranks at 120°. The slide valves A and B of the outer cylinders (Fig. 260) are driven direct, but the valve C of the inside cylinder is driven by the linkwork shown. Show that the motion of C is approximately the same as if it were driven by a similar gear to those actuating A and B, having the same relation to the inner crank.

24. In a four-crank "symmetrical" engine the angular positions of the consecutive cranks A, B, C, D are in the order A, D, B, C, when looking along the crankshaft from A towards D. The centre lines of cylinders B and C are each a distance a from the middle cross section of the engine, while those of cylinders A and D are each a distance b from this section. The crank angle between A and C is equal to that between B and $D = \alpha$; the crank angle between A and $D = 2\beta$, and the remaining crank angle between B and C $(= 2\pi - 2\alpha - 2\beta)$ is 2γ. The recipracating masses of A and D are each W_1, and those of B and C each W_2.

Prove that, for complete balance of primary forces and couples,

$$\text{(i)}\ W_1 \cos \beta = W_2 \cos \gamma;\ \text{(ii)}\ \frac{b}{a} = \frac{\tan \gamma}{\tan \beta} \qquad \text{(U.L.)}$$

25. Referring to the data of Q. 24, show that, if the engine is completely balanced for primary forces and couples, the maximum unbalanced secondary force is

$$\frac{W_1}{n} \left\{ \cos 2\beta - \frac{\cos \beta}{\cos \gamma} \,.\, \cos 2\gamma \right\}$$

and the maximum unbalanced secondary couple is

$$\frac{W_1 b}{n} \left\{ \sin 2\beta - \frac{\sin \beta}{\sin \gamma} \,.\, \sin 2\gamma \right\}$$

Show also that these maxima occur at crank positions which are 45° apart.

CHAPTER XIII

STEERING GEARS—PRECESSION

201. **Motor-car Steering Gears.** The steering gear of a motor-car is a mechanism for varying the direction of two or more of the wheel axles with reference to the chassis, so as to cause the car to move in any desired path. Usually the two back wheels have a common axis, which is fixed in direction with reference to the chassis, and the steering is done by means of the front wheels. For correct steering the wheels should be in planes which are tangential to their respective paths.

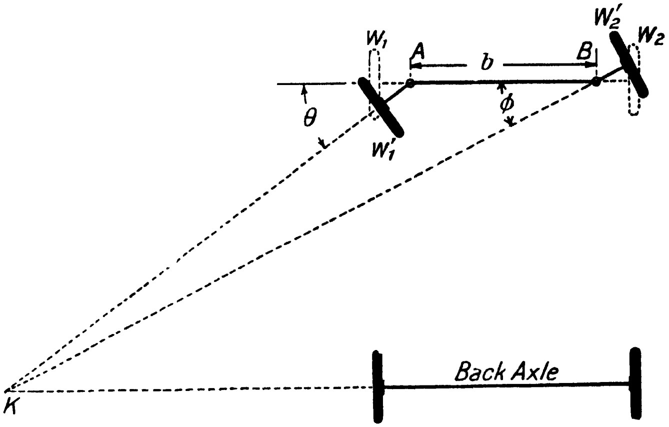


FIG. 261. POSITION OF AXLES FOR CORRECT STEERING

This disposition of the wheels is obtained if, and only if, the axes of the two front wheels intersect at a point on the common axis of the back wheels. The point of intersection is then the instantaneous centre of rotation of the chassis, and the wheels roll without sidewise slipping.

In Fig. 261, W_1 and W_2 are the front wheels in the position for steering a straight course. The axles of W_1 and W_2 are pivoted at the points A and B respectively, these points being fixed to the car chassis. If, by means of a steering gear, the wheels are moved relative to the chassis into the positions shown in full at W_1' and W_2', the car will move in a circular course. The steering will be correct if, and only if, K, the point of intersection of the axes of W_1' and W_2', lies on the axis of the back wheels.

Let $\overline{AB} = b$, H = distance between AB and the back axis, and let θ and ϕ be the acute angles between AK and BK respectively and the line AB or AB produced. We have by a well-known trigonometrical formula

$$\cot \phi - \cot \theta = \frac{b}{H} \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII.1)}$$

For all steering positions the point K should lie on the back

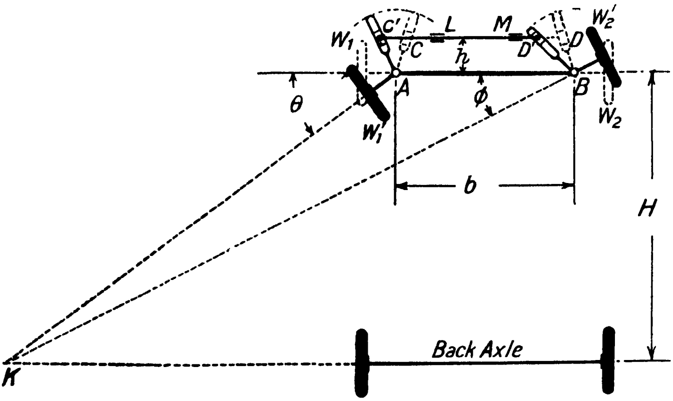


FIG. 262. DAVIS'S STEERING GEAR

axis, i.e. the values of the angles θ and ϕ should satisfy (XIII.1). Steering gears have been invented which give correct steering for all values of θ and ϕ but, owing to the necessity for sliding constraints in such gears, wear soon destroys the accuracy of the steering. We shall describe a gear of this type and shall then proceed to describe a common type of gear in which K only lies on the back axis in three positions of the steering mechanism.

202. **Davis's Steering Gear** is an exact steering mechanism. The slotted links AC and BD (Fig. 262) are each attached to one of the front wheel axles and are arranged to turn on pivots at A and B respectively. The rod CD is constrained to move in the direction of its length by sliding constraints at L and M, and is connected to the slotted levers AC and BD by a sliding and a turning pair at each end. Steering is effected by

moving CD to the right or left of its normal position. $C'D'$ shows a position of CD for turning to the left.

Let $H =$ distance between AB and the back axis

$h =$ distance between AB and CD

$b = \overline{AB}$

$\alpha = \widehat{CAB} = \widehat{DBA}$

θ and ϕ have the same meanings as in Art. 201.

Let $x =$ displacement of CD

$= \overline{D'D}$

$$\text{or} \quad x = h \cot (\alpha - \phi) - h \cot \alpha \quad . \quad . \quad . \quad \text{(XIII.2)}$$

$$\therefore \frac{x}{h} = \frac{\cos (\alpha - \phi)}{\sin (\alpha - \phi)} - \frac{\cos \alpha}{\sin \alpha}$$

$$= \frac{\sin \alpha \cos (\alpha - \phi) - \cos \alpha \sin (\alpha - \phi)}{\sin \alpha (\sin \alpha \cos \phi - \cos \alpha \sin \phi)}$$

$$= \frac{\sin \phi}{\sin^2 \alpha \cos \phi - \sin \alpha \cos \alpha \sin \phi}$$

$$\therefore \frac{h}{x} = \sin^2 \alpha \cot \phi - \sin \alpha \cos \alpha$$

$$\text{or} \quad \cot \phi = \frac{1}{\sin^2\alpha} \left(\frac{h}{x} + \sin \alpha \cos \alpha\right) \quad . \quad . \quad . \quad \text{(XIII.3)}$$

Again, $x = \overline{C'C}$

$$\text{or} \quad x = h \cot \alpha - h \cot (\alpha + \theta) \quad . \quad . \quad . \quad \text{(XIII.4)}$$

This corresponds with (XIII.2) and becomes the same equation if $-\theta$ is put for ϕ and the sign of x is changed. Making these changes in XIII.3 we have

$$\cot (-\theta) = \frac{1}{\sin^2\alpha} \left(-\frac{h}{x} + \sin \alpha \cos \alpha\right)$$

$$\text{or} \quad \cot \theta = \frac{1}{\sin^2\alpha} \left(\frac{h}{x} - \sin \alpha \cos \alpha\right)$$

Substituting the above values of $\cot \phi$ and $\cot \theta$ in the left-hand side of (XIII.1) we have

$$\cot \phi - \cot \theta = \frac{1}{\sin^2\alpha} \times 2 \sin \alpha \cos \alpha$$

$$= 2 \cot \alpha$$

which agrees with (XIII.1) if $\cot \alpha = \frac{b}{2H}$. If, therefore, the two arms AC and BD intersect at a distance H in front of AB, the steering is correct for all positions of the steering gear. Though the gear is theoretically correct, the presence of sliding constraints renders it practically incorrect as the consequent wear would soon produce slackness between the sliding surfaces, thus eliminating the original accuracy.

203. **The Ackerman Steering Gear** gives correct steering in three steering positions only. The gear is of a practical form and consists only of turning pairs. The departure from accuracy in a well-designed gear of this type is so small as to be practically negligible.

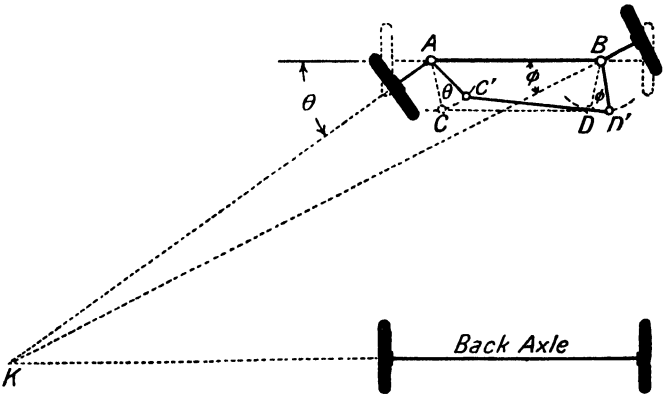


FIG. 263. THE ACKERMAN STEERING GEAR

In this gear, which is shown in Fig. 263, the mechanism $ABCD$ is a four-bar crank chain. AC and BD are connected with the front wheel axles as in the Davis gear. $\overline{AC}$ and $\overline{BD}$ are equal and CD is parallel to AB when the front wheel axles are parallel to the back axle. The thick lines show the position of the gear when steering to the left and as, in this position, the lines of the front wheel axes intersect on the back axle at K, the steering is correct. There are only two other positions for correct steering, i.e. when K is in the corresponding position for steering to the right, and when the car is being steered on a straight path. The reader should convince himself by drawing the mechanism in some other position that the steering is not correct, i.e. that K does not lie on the back axle. In the extreme position of the steering mechanism the inclination of $C'D'$ to AB will not be more than a few degrees, and therefore the projections on AB of $\overline{CD}$ and $\overline{C'D'}$ will be practically equal. If the maximum inclination is 7° and $\overline{CD} = l$, the projections are l and $l \cos 7° = \cdot 9925l$ respectively. If we assume these two projections to be equal, we have as a consequence

Projection of arc CC' on AB = projection of arc DD' on AB.

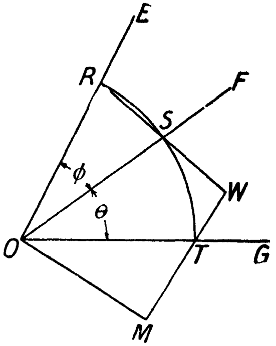


Fig. 264. Determination of Steering Angles

If a represents the magnitude of each of the angles CAB, DBA, we have to determine the value of α so that the angles ϕ and θ measured from Fig. 263 shall satisfy (XIII.1).

Draw from O (Fig. 264) any line OF, and on opposite sides of OF set off angles $\widehat{FOE} = \phi$, and $\widehat{FOG} = \theta$ as measured from Fig. 263, the given position of the mechanism being the one for which correct steering is required. With O (Fig. 264) as centre, draw a circle radius = AC cutting OE, OF, and OG in the points R, S, and T respectively. Join RS and produce it to W so that $\overline{RS} = \overline{SW}$. Join WT and draw OM perpendicular to WT produced. Then the arcs RS and ST have equal projections on OM for

Projection of arc RS on OM = projection of chord RS on OM and

Projection of arc ST on OM = projection of line SW on OM

But $\overline{RS} = \overline{SW}$, and their projections on OM are therefore equal, hence the two arcs have equal projections. Thus the angles $\widehat{TOM}$ and $\widehat{ROM}$ (Fig. 264) are respectively equal to the values of the required angles $\widehat{C'AB}$, $\widehat{ABD'}$ (Fig. 263), and $\widehat{FOM}$ is the required angle $\widehat{CAB}$ or $\widehat{DBA}$. In the above construction $\overline{OR}$ may actually be any convenient radius, as altering the value of OR will have no effect on the values of the angles in the figure. The position of K on the back axis is selected so that the greatest steering error for angles greater than θ shall be equal to the greatest steering error for angles less than θ. The method of doing this is beyond the scope of this article.

204. **Vectors Representing Rotation — Gyroscopic Effects — Precession.** The angular speed of a rotating body associated with the direction of the axis of rotation is a vector (Art. 12). As there are two senses of rotation about an axis, we must connect these with the two senses of direction along the axis. We shall adopt the usual rule, which makes the sense along the axis the same as that of the displacement of a right-handed screw working in a fixed nut, when the screw is rotated in the sense of the angular motion. Thus, if on looking along the axis of rotation, the sense of the rotation is clockwise, the sense of the vector is along the axis away from the observer; if the rotation is anti-clockwise, the sense is towards the observer.

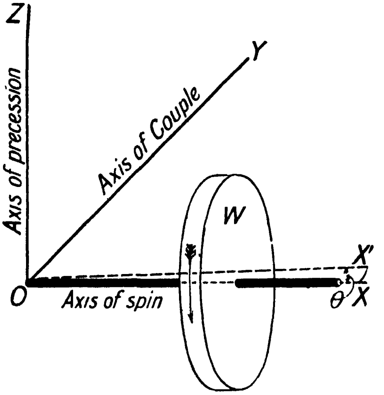


FIG. 265. SPINNING WHEEL

Angular momentum is also a vector, being the product of a scalar quantity I and a vector ω (see Art. 30). A couple is a vector whose direction is that of a line perpendicular to the plane of the forces constituting the couple, its sense being determined by the above rule. If a rigid body is spinning about its axis and a couple is applied which tends to turn the body about an axis perpendicular to the axis of spin, the body will actually rotate about a third axis perpendicular to each of the other two. This effect of spin in causing the body to rotate about an axis perpendicular to the axis about which it would rotate in the absence of spin, is known as the *gyroscopic effect*, and the rotation about the third axis is termed *precession.*

205. **Precession of a Wheel with its Axis Horizontal.** Let W (Fig. 265) be a rigid body of moment of inertia about OX of I engineers' units spinning about OX with angular velocity ω in the sense of the arrow. Let OY and OZ be perpendicular to each other and to OX. The angular momentum of W is

$I\omega$. If $\overline{OX}$ is made to represent $I\omega$ to scale, $\overrightarrow{OX}$ will represent the angular momentum. Now, suppose the axis OX to be turned in the plane XOY through a small angle θ to the position OX'. Assuming that ω remains constant, the angular momentum will now be $I\omega$ represented by $\overrightarrow{OX'}$. We have

$$\begin{aligned} \text{Change of angular momentum} &= \overrightarrow{OX'} - \overrightarrow{OX} \\ &= \overrightarrow{XX'} \\ &= \overline{OX}\ \theta \text{ in direction of } \overrightarrow{XX'} \\ &= I\omega\,.\,\theta \text{ in direction of } \overrightarrow{XX'} \end{aligned}$$

Let t secs. be the time taken for the axis to move from the first position to the second, then

$$\frac{\text{Change of angular momentum}}{t} = I\omega\frac{\theta}{t}\ . \qquad . \qquad \text{(XIII.5)}$$

As t approaches the value zero, the left-hand side of (XIII.5) tends to the value rate of change of angular momentum, and $\frac{\theta}{t}$ tends to the value ω', where ω' = speed of rotation about OZ. Thus, in the limit (XIII.5) becomes

Rate of change of angular momentum $= I\omega\ \omega'$

If C is the couple acting on the body causing precession

$C =$ rate of change of angular momentum

$$\therefore C = I\omega\omega' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII.6)}$$

C is a vector whose sense and direction are the same as those of $\overrightarrow{XX'}$ in the limiting position when $\theta = 0$, i.e. the same as those of OY. $\overrightarrow{OY}$ will, therefore, represent the couple C if $\overline{OY} = I\omega\omega'$. If $\overline{OZ} = \omega'$, then $\overrightarrow{OZ}$ will represent the vector ω', and we see that the axes of spin, couple and precession are mutually perpendicular. It is important that the reader should notice that when the senses of the three vectors are all away from O, the axes of the couple and the spin rotate about the axis of precession, so that the couple vector leads the spin vector by a right-angle. Whenever a vehicle, having wheels or other rotating masses, deviates from a straight path, there is a change in the direction of one or more of the axes of the rotating masses, and precession effects occur. The effects are only appreciable for large values of $I\omega\omega'$. In motor cars the effects of precession may be eliminated, or reduced, by arranging the flywheel to rotate in the opposite sense to the driving wheels. In aeroplanes the effects are reduced if there are two propellers on the same axis rotating in opposite senses.

EXAMPLE

A machine weighing 12 cwt. in all, has a flywheel weighing 500 lb. which is rotating about a horizontal axis at 100 revs. per min. The radius of gyration of the flywheel is 2 ft. The machine rests on a turntable, to which it is not attached, the base of the machine being a square of side 3 ft. Find the angular speed of rotation of the turntable at which the machine would just begin to overbalance owing to the precession. Centre of gravity over middle of base.

$$C = \text{couple which will just tilt the machine}$$
$$= 12 \times 112 \times \frac{3}{2} = 2016 \text{ lb.-ft. units}$$

But $C = I\,\omega\omega'$, and $\omega = \frac{100}{60} \times 2\pi = \frac{10}{3}\pi$ radians per sec.

$$\therefore \omega' = \frac{C}{I\,\omega}$$
$$= \frac{2016}{\frac{500}{32\cdot2} \times 4 \times \frac{10\pi}{3}} \text{ radians per sec.}$$
$$\therefore \omega' = \frac{2016 \times 96\cdot6}{20{,}000\pi}$$
$$= 3\cdot198 \text{ radians per sec.}$$

which corresponds to a speed of 30·5 revs. per min.

206. **Effects of Precession on the Stability of a Motor Vehicle Moving in a Curved Path.** The curves in Fig. 266 represent the four wheels of the vehicle which is moving on a curved horizontal track, the radius of curvature of the track being R ft. The wheels A and B are on the inside of the track, and the wheels D and C on the outside. The radius R is supposed to be large compared with the dimensions of the car. G is the centre of gravity of the car, and h is its height in feet above the track. Let $\overline{AD} = \overline{BC} = a$ ft. We shall assume that G is vertically over the centre of the rectangle $ABCD$, so that the weight of the car W lb. is distributed equally over the four wheels. The upward force between the road and each wheel due to the weight of the car is therefore $\frac{W}{4}$ lb. Owing to the

precession of the wheels there is a couple acting of magnitude

$$C = \Sigma I\omega\omega'$$

where I is the moment of inertia of any one of the wheels, ω is its angular velocity about its own axis, and ω' is the angular velocity of the precession. Σ denotes that the summation extends to the four wheels and to any other rotating masses in or on the car, whose planes of rotation are parallel to those of the wheels. If any masses rotate in the opposite sense to that in which the wheels A, B, C and D are rotating, the values of ω for these masses will be negative. If v is the speed of the car in feet per second, and r the radius of each of the four wheels, A, B, C, and D, then $v = \omega r$ and

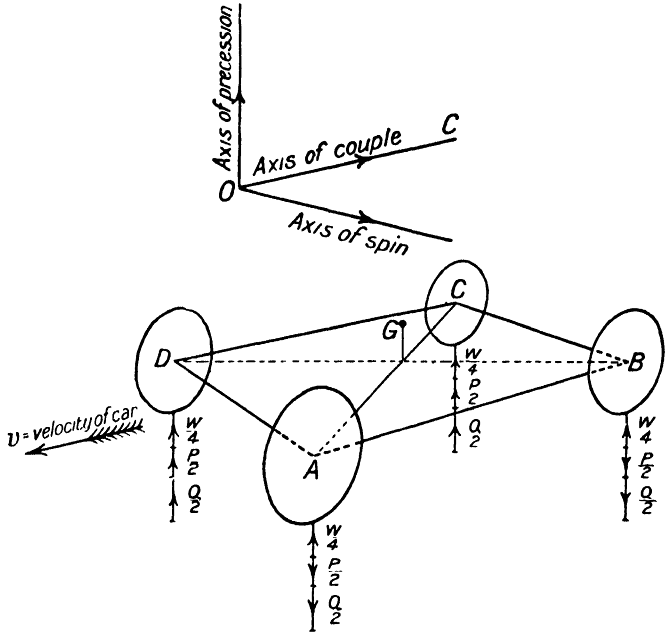


FIG. 266. EFFECTS OF PRECESSION ON STABILITY OF A MOTOR VEHICLE

$$\omega' = \frac{v}{R}$$

and

$$C = \Sigma I\omega\omega'$$

$$\therefore C = \frac{v}{R}\,\Sigma I\omega \qquad \text{(XIII 7)}$$

This couple acts about an axis in the direction of AB and DC, and since the spin and precession vectors have the directions and senses shown in the figure, the couple vector will have the direction and sense of the vector $\overrightarrow{OC}$. Such a couple would be produced by upward forces $\frac{P}{2}$ lb. acting on each of the outer wheels D and C, and downward forces $\frac{P}{2}$ lb. each on the inner wheels A and B where

$$Pa = C$$

i.e. $$Pa = \frac{v}{R}\Sigma I\omega$$

or $$\frac{P}{2} = \frac{v}{2aR}\Sigma I\omega \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII.8)}$$

The centripetal force necessary to keep the car moving in its circular track is one of $\frac{Wv^2}{gR}$ lb. This force is supplied by the radial force of friction of the ground on the tyres. This latter force is equivalent to a force of magnitude $\frac{Wv^2}{gR}$ lb. through the centre of gravity, together with a couple of magnitude $\frac{Wv^2}{gR}h$ lb.-ft., tending to overturn the car, so as to make the inner wheels lift from the road surface. In order to balance this couple, vertical forces, each of magnitude $\frac{1}{2}Q$ lb., must act on the four wheels, those on the inner wheels acting vertically downward, and the other two acting in the upward direction. The magnitude of Q is given by $Qa = \frac{Wv^2}{gR}h$

or $$\frac{Q}{2} = \frac{Wv^2h}{2agR} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIII.9)}$$

The various forces $\frac{P}{2}$, $\frac{Q}{2}$ and $\frac{W}{4}$ are shown in Fig. 266.

Let P_o = pressure of the ground on each outer wheel in lb.

then $$P_o = \frac{W}{4} + \frac{P}{2} + \frac{Q}{2}$$

$$= \frac{W}{4} + \frac{v}{2aR}\Sigma I\omega + \frac{Wv^2h}{2agR}$$

and P_i = pressure on inner wheels in lb.

$$= \frac{W}{4} - \frac{P}{2} - \frac{Q}{2}$$

$$= \frac{W}{4} - \frac{v}{2aR}\Sigma I\omega - \frac{Wv^2h}{2agR}$$

P_i is less than P_o, and if the speed is sufficiently high, P_i will become zero and the tyres of the inner wheels will leave the ground. It should be noticed that precession assists centripetal force in tending to upset the car. If, however, $\Sigma I\omega$ is negative, a condition which may be satisfied by having a heavy flywheel rotating at high speed in the opposite sense to that of rotation of the driving wheels, the precession effect will oppose the centripetal force couple and will tend to prevent upsetting of the car. For wheels whose axes are horizontal and perpendicular to the driving axle (XIII.7) gives the necessary couple due to precession. In this case the couple acts about an axis parallel to the driving axle, and so alters the distribution of pressure on the ground between the front and back wheels.

When a ship steers a circular course, precession of the turbine rotors causes the ship to incline slightly to the horizontal in the direction of motion. The bow of the ship is lowered relative to the stern for turning on one side of a straight course, and raised for turning to the other side. Pitching of a ship causes forces on the bearings in a horizontal direction perpendicular to that of the motion.

Gyrostatic Torque on a Motor Cycle.

Let I_E = M.I. of rotating parts of engine
I_W = M.I. of rotating parts of wheels
r_W = radius of wheels
R = radius of curve
G = gear ratio = $\frac{\omega_E}{\omega_W}$
h = distance of C.G. of machine and rider from ground when in vertical position
v = speed of cycle
W = total wt. of cycle and rider
a = inclination of machine to vertical for equilibrium.

(Feet and seconds units throughout.)

$$\text{Then } \omega_W = \frac{v}{r_W}\,;\quad \omega_E = \frac{G\,.\,v}{r_W}$$

$$\text{Total } I\omega = v\left(\frac{I_W}{r_W} + \frac{G\,.\,I_E}{r_W}\right) = \frac{v}{r_W}\,(I_W + G\,.\,I_E)$$

$$\text{Speed of precession } \omega' = \frac{v}{R}$$

$$\therefore\ \text{Gyrostatic torque} = \frac{v^2}{R\,.\,r_W}\,(I_W + G\,.\,I_E) \qquad . \quad . \quad (1)$$

$$\text{Overturning torque due to C.F.} = \frac{Wv^2\,h\cos a}{gR} \qquad . \quad . \quad (2)$$

$$\frac{\text{Gyrostatic torque}}{\text{Centrifugal torque}} = \frac{(1)}{(2)} = \frac{g(I_W + G\,.\,I_E)}{Wr_W\,h\,\cos a} \qquad . \quad . \quad (3)$$

Total overturning torque = (1) + (2)

$$= \frac{v^2}{R}\left(\frac{I_W + G\,.\,I_E}{r_W} + \frac{W}{g}\,.\,h\cos a\right) \quad (4)$$

For equilibrium,

$$\frac{v^2}{R}\left(\frac{I_W + G\,.\,I_E}{r_W} + \frac{W}{g}\,.\,h\,.\,\cos a\right) = W\,.\,h\,.\,\sin a$$

$$\therefore\ v^2 = \frac{Wh\sin a}{\dfrac{I_W + G\,.\,I_E}{r_W} + \dfrac{W}{g}\,.\,h\cos a}\,.\,R \qquad . \quad . \quad . \quad (5)$$

EXAMPLE.

W = 350 lb.; wt. of wheels = 2 × 25 = 50 lb.; r_W = 1·1 ft.; h = 2 ft.; wt. of rotating parts of engine = 16 lb. (k = 0·42 ft.); G = 4·5. If a is not to exceed 40°, (*a*) what is the maximum permissible speed on a curve of 45 ft. radius?; (*b*) what is the minimum value of R for a speed of 88 ft. per sec?

$$I_W = \frac{50 \times 1{\cdot}21}{32{\cdot}2} = 1{\cdot}875 \text{ (approx.)}$$

$$I_E = \frac{16 \times 0{\cdot}176}{32{\cdot}2} = 0{\cdot}0875;\quad G\,.\,I_E = 0{\cdot}393$$

∴ From (5)

$$v^2 = \frac{350 \times 2 \times 0{\cdot}643}{\dfrac{2{\cdot}268}{1{\cdot}1} + \dfrac{350 \times 2 \times 0{\cdot}766}{32{\cdot}2}} \times 45 = 24{\cdot}1 \times 45 = 1086$$

(*a*) v = 33 ft. per sec. (22·5 m.p.h.)

Since R is proportional to v^2, then for $v = 88$ ft. per sec.,

$$(b) \qquad R = \left(\frac{88}{33}\right)^2 \times 45 = 320 \text{ ft.}$$

$$\text{From (3), } \frac{\text{Gyrostatic torque}}{\text{Centrifugal torque}} = 0{\cdot}124 \text{ (independent of speed).}$$

In this case the effect of the gyroscopic couple is to reduce the permissible speed by about 6 per cent. The effect of this couple will also be to reduce the tendency to "skid."

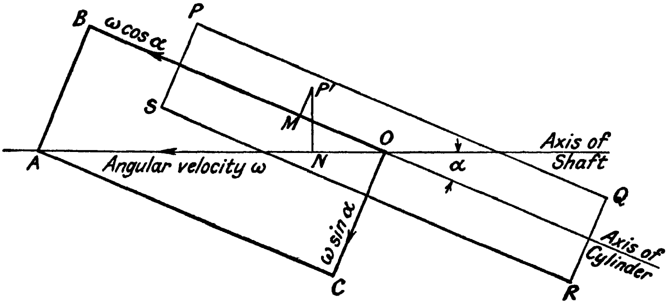


FIG. 267. COUPLE CAUSED BY ROTATION

207. **Couple Exerted on a Shaft by a Body Rotating with It.** We saw in Art. 39 that when a solid body in the form of a plane lamina rotates about an axis perpendicular to its plane, the centripetal force on the body acts along the line joining the centre of gravity to the centre of rotation, and its magnitude in pounds is $M\omega^2$ times that distance in feet; M being the mass in engineers' units, and ω the angular velocity of rotation in radians per second. If, however, the rotating body is not such a lamina, there is, in general, a couple acting on the body about an axis perpendicular to a plane containing the axis of rotation.

We shall consider the case of a circular cylinder $PQRS$ of uniform density, Fig. 267, with its axis in the same plane as the axis of rotation of the shaft, the angle between the axes being α, and with its centre of gravity O on the axis of rotation. The angular velocity of the shaft is ω radians per second in the clockwise sense, looking in the direction O to A. $\overrightarrow{OA}$ represents ω, and $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are the respective components of $\overrightarrow{OA}$ along the axis and perpendicular to it. Then $\overline{OB} = \omega \cos a$ and $\overline{OC} = \omega \sin a$. First consider the vector $\overrightarrow{OB}$. Owing to the rotation, B moves down through the plane of the figure, i.e. precession takes place, and the relation (XIII.6) enables us to calculate the couple C_B needed to produce this precession. The angular velocity of rotation about OB is $\omega \cos a$. The speed at which B is moving is $\omega \, . \, \overline{OB} \sin a$. If ω' is the angular speed of rotation of OB about OC, the speed of B is also $\omega' \, . \, \overline{OB}$.

Hence $\quad \omega' \, . \, \overline{OB} = \omega \, . \, \overline{OB} \sin a$

or, $\quad \omega' = \omega \sin a$

Substituting in (XIII.6) we have, therefore,

$$C_B = I_P \omega \cos a \, . \, \omega \sin a$$

$$\therefore \quad C_B = I_P \omega^2 \sin a \, . \cos a, \text{ lb.-ft.} \qquad \text{(XIII.10)}$$

where I_P is the polar moment of inertia of the cylinder about its axis OB. This couple tends to turn the body about an axis through O, perpendicular to the plane of the figure, in the clockwise sense.

Let C_C be the couple needed to produce precession owing to the rotation of the vector $\overrightarrow{OC}$. If ω'' is the angular speed of precession, the two expressions for the speed of C are $\omega \, . \, \overline{OC} \cos a$, and $\omega'' \, . \, \overline{OC}$

hence $\quad \omega'' = \omega \cos a$

Substituting in (XIII.6), we have

$$C_C = I_D \omega \sin a \, . \, \omega \cos a$$

i.e. $$C_C = I_D \omega^2 \sin a \, . \cos a, \text{ lb.-ft.} \qquad \text{(XIII.11)}$$

where I_D is the moment of inertia about the diameter through O in the plane of the figure. This couple tends to turn the body about an axis through O, perpendicular to the plane of the figure in the anti-clockwise sense. If C is the total couple due to both effects, then

$$C = C_C - C_B$$

$$= (I_D - I_P) \, \omega^2 \sin a \, . \cos a$$

or $$C = \tfrac{1}{2}\omega^2 \sin 2a \, (I_D - I_P) \text{ lb.-ft.} \qquad \text{(XIII.12)}$$

in the anti-clockwise sense. This is the couple exerted by the shaft on the body. The couple exerted on the shaft is equal and opposite to this, i.e. in the clockwise sense. If the shaft is short enough, I_D may be less than I_{P1}, in which case the couple on the shaft acts in the anti-clockwise sense. If $I_D = I_P$, the couple is zero in all cases.

If the cylinder is solid, of length L and radius R, we have

$$C = \tfrac{1}{2}\omega^2 \sin 2a \left[\frac{W}{g}\left(\frac{L^2}{12}+\frac{R^2}{4}\right)-\frac{W}{g}\frac{R^2}{2}\right]$$

or

$$C = \frac{W}{24g}\,\omega^2 \sin 2a\,(L^2 - 3R^2)$$

The above investigation assumes the body to be a cylinder, but it is clear that the same proof would apply in the case of any uniform body of which OB is an axis of symmetry and $PQRS$ a plane of symmetry.

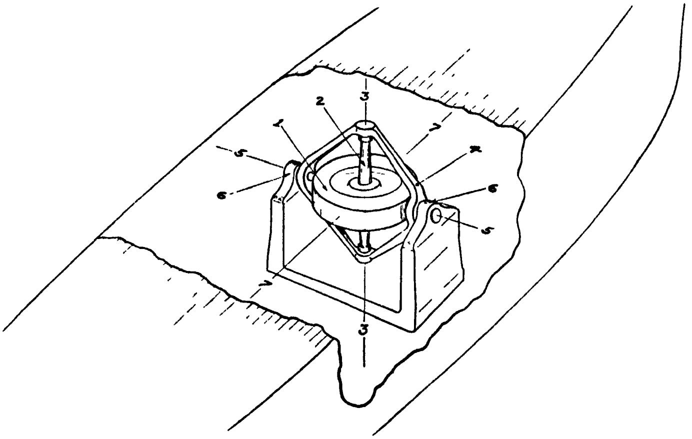


FIG. 268A. GYRO-STABILIZER ON A SHIP

If the centre of gravity is not on the axis of rotation, but is at a distance h feet from it, there is, in addition to the couple C exerted on the shaft, a force $\omega^2 h\, w/g$ lb. acting outwards through the centre of gravity. The effect of the couple and force are to produce bending moments on the shaft, and forces on its supports. Owing to the alternating nature of these disturbances, they may produce forced oscillations in the shaft on its supports. A second method of proof of (XIII.12) is indicated in examples 15 and 16 below.

208. **The Gyroscope as a Stabilizer. The Sperry Gyrostabilizer.** The gyroscopic properties of spinning rotors have been put to many practical uses. The property of keeping its direction fixed when mounted freely obviously suggests its use instead of a magnetic compass but, as that direction is fixed relative to outer space and not relative to the earth, ingenuity had to be used in order to produce the gyro compass now in such wide use. The above property, combined with the property of precession have also been used in the automatic pilot for ships and aircraft in steering torpedoes, in stabilizing rifle bullets and in many other ways, including the stabilization of land, air, and sea vehicles. Before 1939 many ships were fitted with gyro-stabilizers; the largest of these was sunk during the war, and since then the fitting of ships with these contrivances appears to have been discontinued. Fig. 268A is a diagram showing a stabilizing gyro mounted on a ship. (1) is the rotor electrically driven at constant speed and mounted on bearings in a frame (4), which is also mounted on bearings (5), whose axis is in the athwart-ship direction. These bearings are rigidly attached through the frame supporting them to the ship's structure. Thus the rotor has: (i) freedom to rotate on its own axis, (ii) freedom to precess about the axis 55, and (iii) partial freedom to rotate with the ship about a fore and aft axis. Let us suppose that the ship is fixed in position with the rotor, of moment of inertia I engineer's units, spinning at ω radians per second in the clockwise direction as seen from below. If the rotor is now caused to precess about the axis 55 with angular speed of ω' radians per second, the necessary couple must be applied by the support 55, and this in turn will produce an equal and opposite couple on the ship. In the diagram, Fig. 268B, $\overrightarrow{OV}$ represents the angular momentum $I\omega$ of the rotor in the vertical position. When the rotor has precessed through an angle θ from the vertical the angular momentum is represented by $\overrightarrow{OV'}$ and is still of the same magnitude, since ω is constant. The speed of V' as it turns about O with angular speed $\dot{\theta} = \omega'$ is $I\omega\omega'$; this is the rate at which the angular momentum OV' is

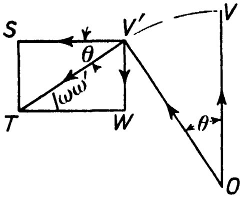


FIG. 268B. SIDE VIEW OF PRECESSION

changing and is represented by $\overrightarrow{V'T}$ of length $I\omega\omega'$. But the rate of change of angular momentum is equal to the couple causing it, so that the bearings exert on the frame (4) a couple $I\omega\omega'$ in the direction and sense of $\overrightarrow{V'T}$. This couple splits up into a horizontal couple $I\omega\omega' \cos \theta$ along and in the sense of $\overrightarrow{V'S}$, and a vertical couple $I\omega\omega' \cos \theta$ along and in the sense of $\overrightarrow{V'W}$. The horizontal couple is $C_H = I\omega\omega' \cos \omega' t$ if we assume the vertical to be the position at time $t = 0$. Thus the uniform rate of precession needs the action on the frame 4 of the two component couples given above. Couples equal and opposite to these will act on the ship. The vertical component will tend to produce angular vibrations of the ship about a vertical axis through its centre of gravity but, owing to the high moment of inertia of the mass of the ship about this axis, and the large moment of the water resistance, with the ship afloat, this vibration or "yawing" can be neglected. The horizontal component will tend to produce angular vibrations about a fore and aft axis about which the moment of inertia is least. Thus we have a rolling couple $I\omega\omega' \cos \omega' t$ acting on the ship in the sense opposite to that of $\overrightarrow{V'S}$. Now suppose the ship is in the water and that owing to the waves acting on the ship there is a rolling couple, i.e., one about a fore and aft axis, of magnitude $M_0 \cos (nt + a)$ in the direction and sense of $\overrightarrow{V'S}$. It is clear that if we can make this couple the one necessary to maintain precession of the gyroscope there will be no rolling on the ship. Thus we must have identical equality between the two couples, i.e., $I\omega\omega' \cos \omega' t \equiv M_0 \cos (nt + a)$, or, $a = 0$, $\omega' = n$, and $I\omega\omega' = M_0$; a can be made equal to 0 by fixing the time of starting the precession, ω' can be made equal to n, and the value of ω can be selected so that $\omega = M_0/In$ by variations in the speed of the driving motor. The sense of the couple could also be adjusted. Rolling could be prevented in this way if the conditions were as we have assumed. The heeling couple produced by the waves is neither harmonic nor constant; any change of course would vary the frequency of the couple as well as its magnitude. The speed of precession would have to be synchronized with the wave frequency, and the speed of spin would have to be adjusted to the magnitude of the wave effects. Such an arrangement is not practical.

Practical stabilizers have been used in which the rotor was mounted as in the figure, but instead of the precession being uniform rotation through complete revolutions, it was limited to oscillation through an acute angle on each side of the vertical. Fig. 269 is a diagram illustrating the Sperry Stabilizer. The rotor is motor driven and of considerable size and mass up to 100 tons weight or more, with speeds up to 1000 r.p.m. Three such units on a large Italian liner, sunk during the war, produced a stabilizing torque of nearly 8000 ton-ft. and reduced an average roll amplitude of 7½ degrees to one of 1¼ degrees. The unit or units may be placed in any convenient part of the ship. In addition to the stabilizing gyro there is a control gyro. The main rotor is in an enclosed case in which a vacuum is maintained equivalent to 24 in. of mercury in order to reduce skin friction. The rotor does not precess by itself, but is made to do so by a motor operating through gearing on a sector of a toothed wheel carried by the rotor casing; there is a powerful set of brakes on the motor shaft. When the ship begins to roll to the left, the rotor of the control gyro rolls with it about the fore-aft direction. This causes precession about the vertical axis of its supporting frame and brings a projection on the casing into contact with one of the two pins shown; this completes an electrical circuit which operates the motor and brake. When no current passes through

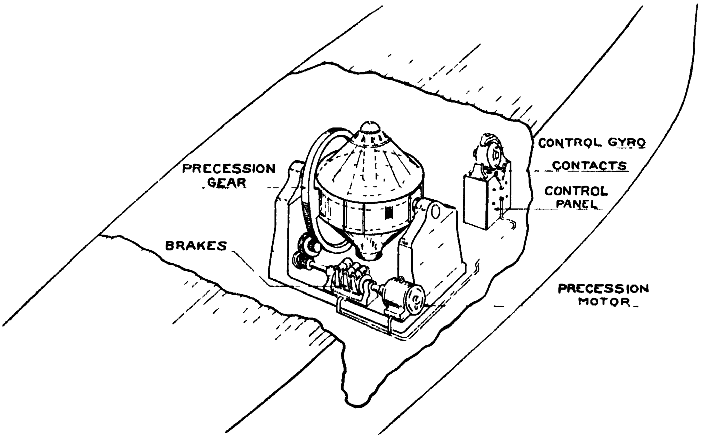


FIG. 269. GENERAL ARRANGEMENT OF THE SPERRY GYRO-STABILIZER

this circuit the brakes are kept "on" by heavy helical springs. When current flows the electric brake coils overcome the spring pressure and the brakes are "off." At the same time the motor starts up and precession takes place in the correct sense for promoting a couple to oppose the heeling. The precession builds up quickly because of the surge of current at starting. As this surge dies down there is a slight braking effect which prevents the precession from overrunning the motor. Such tendency to overrun reduces the current flowing in the brake and motor coils and causes the brakes to clamp down on the motor shaft. The motor is used to start up the precession promptly and the brakes to keep precession under control once it has started; in practice the brakes float on the shaft during the precession cycle. When the roll to the left stops and reverses, the control gyro, which has been returned to its neutral position by springs, begins to precess in the opposite sense and at once comes into contact with the other stop. Contact with this second stop causes the brake to come "off" and the motor to reverse through reversed connections to the motor only. Thus the precession is reversed and a couple produced to oppose the rolling. In this way damping is produced. A special device ensures that the mean position of the main rotor is maintained in its vertical position by making the control gyro less sensitive to the next half roll which would precess the main gyro still further in the sense of the displaced mean position. The rate of precession does not vary harmonically, but is kept approximately constant by the combined effect of the motor and brakes. The maximum tilt of the rotor axis due to precession is limited to about 35° on each side of the vertical. Beyond this the damping couple would be very small. In the Schlick Stabilizer there is no motor drive, and the rate of precession is controlled by friction on a drum which is mounted on the frame carrying the rotor and rotates with it about the athwart-ship axis.

EXAMPLE.

The weight of a ship is 2000 tons and its radius of gyration about a longitudinal axis is 12 ft. In a seaway its uncontrolled angle of roll is ± 25°, the time of a complete oscillation being 20 sec. A gyroscopic control is to be fitted which is to reduce the angle of roll to ± 5° under the same conditions. If the gyro wheel weighs 20 tons, its radius of gyration is 4·5 ft., and the amplitude of its forced precession is ± 30°; at what speed should it rotate?

Assume that the torque required for angular acceleration of the gyroscope frame is negligible, and the rate of precession is simple harmonic.

External torque applied to ship $= T = T_0 \,.\, \sin at = \dfrac{d^2\theta}{dt^2} \,.\, I_s$
where I_s = moment of inertia of ship about the longitudinal axis, and $\theta = \theta_r \sin at$ in magnitude.

$$\therefore\ \theta_r = \frac{T_0 \,.\, \sin at}{a^2 \,.\, I_s}$$

If t_0 = time of one complete oscillation, $a = \dfrac{2\pi.}{t_0}$

If θ_r is to be reduced to $\dfrac{\theta_r}{k}$, the maximum gyroscopic torque to be applied must be $T_0\left(1 - \dfrac{1}{k}\right) = \dfrac{k-1}{k} \,.\, T_0$, with a frequency of $\dfrac{2\pi.}{t_0}$

For the particulars given,

$$\theta_r = \frac{25}{57{\cdot}3} = 0{\cdot}435 \text{ rad.}$$

$$a = \frac{6{\cdot}28}{20} = 0{\cdot}314;\ a^2 = 0{\cdot}0986.$$

$$I_s = \frac{2000}{32{\cdot}2} \times 144 = 8940 \text{ tons-ft.}^2/\text{g.}$$

$$\therefore\ T_0 = 0{\cdot}435 \times 0{\cdot}0986 \times 8940 = 383 \text{ tons-ft.}$$

$\therefore$ Maximum gyroscopic torque $= 0{\cdot}8T_0 = 307$ tons-ft.

M.I. of wheel $= \dfrac{20 \times 20{\cdot}25}{32{\cdot}2} = 12{\cdot}75$ tons-ft.²/g.

If θ' is the angular amplitude of precession,

$$\theta' = \theta_0' \,.\, \sin at$$

$$\omega' = \frac{d\theta'}{dt} = a \,.\, \theta_0' \,.\, \cos at$$

$$\therefore\ \omega' \text{ (max.)} = a \,.\, \theta_0' = 0{\cdot}314 \times 0{\cdot}524 = 0{\cdot}165$$

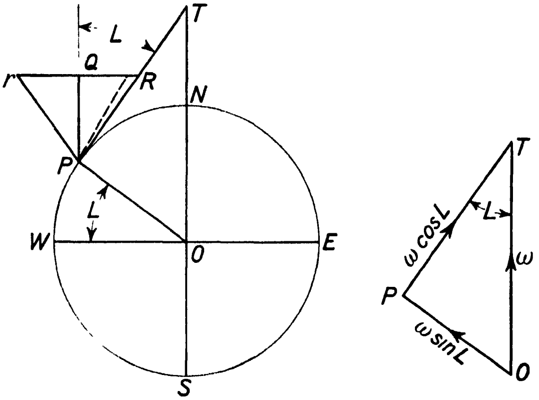
$$\therefore\ 12{\cdot}75 \times \omega \times 0{\cdot}165 = 307$$

$$\omega = 150 \text{ rad.-sec.} = 1430 \text{ r.p.m.}$$

The transverse torque will be accompanied by a "yawing" torque and a small pitching torque, the latter due to transverse precession of the gyroscope frame. Owing to the large moment of inertia of the ship about the vertical and transverse axes, the effects of these will be small.

209. **The Gyro Compass.** A freely supported spinning rotor keeps the direction of its axis fixed in space. Consider such a rotor at the point P on the earth's surface in latitude L, Fig. 270.

The circle is the meridian through P. PT is the tangent at P, and is therefore directed towards the north, NS is the polar axis, WE the equator and $\widehat{POW} = L$. Suppose the rotor to be set spinning at P with its axis PR directed along PT, i.e. towards the north. As the earth rotates about its axis carrying P with it, two lines through P remain fixed in direction—(1) PQ parallel to the earth axis, and (2) PR the axis of the rotor. As the angle between these directions is equal to L it will remain so, and the motion of PR relative to the earth is that of moving over the surface of a circular cone with vertex at P and having a semi-vertical angle L. In this motion P remains fixed and R moves at a uniform rate round the circumference of the base rR of the cone. At the start of this motion R is moving along a perpendicular to the meridian and PR is tilting out of the meridian but not out of the horizontal plane. After a small displacement of PR to the position shown dotted, PR will be inclined to the horizontal plane through P, PT being an edge view of that plane. Thus the initial tilting from the vertical is accompanied by tilting from the horizontal at a much slower rate. During 24 hours, tilt from the meridian will vary from 0 to L to $-L$ and back to 0, whilst tilt to the horizontal will vary from 0 to $2L$ and back to 0. Thus, whilst the rotor axis would be an excellent direction pointer in outer space, it would be of little use for direction finding on the



Tilt of Gyro Axis *Component Angular Velocities*

FIG. 270. TILT OF GYRO AXIS

earth's surface. In order to adapt the rotor for use as a substitute for the magnetic compass, the rotor axis must be given the north pole-seeking properties. Fig. 271 is a diagram of a gyro compass and shows the manner in which this is done in the Sperry Gyro Compass. The rotor is seen spinning anticlockwise, part of the rotor case being removed to show the rotor and its suspension. It will be seen that the rotor has freedom to spin on its own axis, to turn about a horizontal axis, and to turn about a vertical axis. It looks as if the suspension would cause a torque about the vertical axis but, in the actual instrument, the top end of the suspension by means of an electric motor, is made to follow the lower end in its movements, thus eliminating friction as well as torque. The rotor case moves with the rotor except that it doesn't spin. As the rotor axis tilts from the meridian and, therefore, from the horizontal, the case tilts with it, taking the point *B* down through the figure. This, through the connection shown, takes with it the lower part of the half ring to which is rigidly attached the U-shaped mercury tube. The tilting causes mercury to flow from the limb of the U tube behind the case into the one shown, Fig. 272. The unbalanced weight produces a couple about the horizontal axis of the half ring which causes the latter to push at *B*. This produces a couple about the horizontal axis of the rotor case and causes precession round the vertical axis, thus bringing the rotor axis back to the meridian. This precession does not bring the rotor axis back to the horizontal, as that requires a couple about a vertical axis. This couple is produced because of the eccentric connection at *B*. The pressure of the half ring on *B* has a small moment about the vertical axis and the point *B* is so located that the two angles

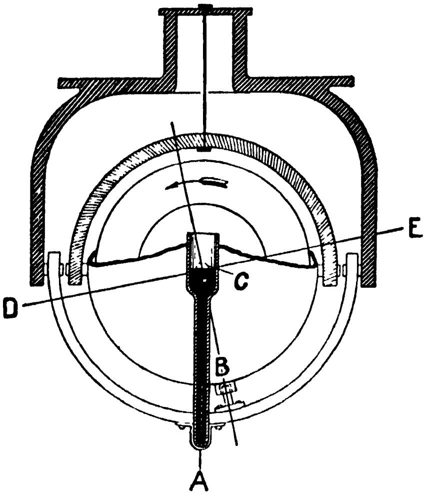


Fig. 271. Diagram of the Gyro Compass

of tilt are simultaneously corrected. This arrangement is known as the Mercury Ballistic Controlled Gyro (Damped). Fig. 272 shows diagrams of such a gyro moving at the equator with axis originally east-west at (*a*). After a short time, owing to the rotation, the axis will have tilted from the horizontal (*b*). The mercury levels will both be the same height above the sea level and so the south reservoir will contain more mercury than the other. This produces a couple about the horizontal axis carrying the rotor case and so precession takes place about the vertical axis, and this brings the rotor axis round towards the meridian; successive stages of this precession are shown at (*c*), (*d*), and (*e*). The reader should verify that the sense of precession is that shown. The gyro compass is thus seen to be north-seeking. Why is the mercury ballistic used when a single weight appropriately placed would produce the desired result? A single weight placed at *B* in place of the ballistic would produce the couples about the two axes, but in the opposite sense. The sense could be made right by reversing the sense of spin of the rotor and the compass would be north-seeking, as before. Any changes of speed or course, or both, or any vibration would tend to set the weight swinging like a pendulum, and this would produce unwanted precessions of the rotor. The inertia of the mercury in the tube, the damping effect due to change of section and the smaller bore of the lower portions, the partial balancing effect of inertia effects on the two side tubes all reduce the effects of inertia forces. The instrument is an effective vibration damper and the position

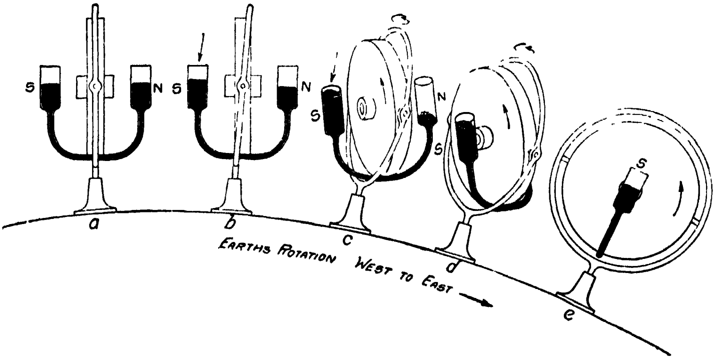


FIG. 272. ACTION OF THE MERCURY BALLISTIC IN THE SPERRY GYRO COMPASS

of P, Fig. 271, is so adjusted that if the rotor axis is set vibrating about the meridian the ratio of the amplitude of any swing on one side of the meridian to that of the preceding swing on the other side is about one-third, so that after a few swings the compass settles down in the meridian. The motion of the ship carrying the compass over the earth's surface can be split into (1) an easterly or westerly component and (2) a northerly or southerly component. The effect of (1) on the compass is negligible, as it only adds or subtracts the speed of the ship to or from the speed of rotation of the earth. As to (2), all the above reasoning has been on the assumption of a gyroscope carried in an easterly direction by the earth round a circle of latitude. If the ship's speed has a northerly (or a southerly) component V_N, and V_E is the earth's speed in that latitude, the course of the ship will make an angle $\tan^{-1} V_N/V_E$ with the circle of latitude. In all the above this course should take the place of the circle of latitude and a course perpendicular to it would be the apparent meridian. The value of ω would only be slightly altered and with $V_N = 25$ knots and $V_E = 900 \cos L$ knots the apparent angular deflection of the meridian would be $\tan^{-1} \frac{25}{900} \sec L$ west of north. At the equator the deflection would be $\tan^{-1}\frac{1}{36}$ about 1·6°, in latitude 60° about twice as much, but in high latitudes the error would be considerable. The Sperry gyro compass is fitted with a speed and latitude corrector which is manually operated. If set within 5° of the ship's latitude and ± 5 knots of the speed, the instrument applies the appropriate correction for any course on which the ship may be set. Small vibrations of the compass about the meridian are discussed in Art. 234.

EXAMPLES 13.

1. What are the conditions for correct steering of motor cars? Show that if these conditions are not satisfied, one at least of the wheels must skid.

2. Describe any form of steering gear which ensures correct steering. If the mechanism has any disadvantages over the ordinary front wheel steering gear, explain them.

3. A motor-car wheel has its plane inclined at 2 degrees to the direction of motion, the wheel being vertical; if the velocity of the car is v miles per hour, find the velocity of skid.

4. Explain how the four-bar mechanism is used as a steering gear. In which positions is there correct steering? Design such a gear so as to give correct steering when rounding a corner in a circle of 16 ft. radius. Take the length of the car between axles as 7 ft. 6 in., and the distance between the steering pivots as 4 ft. 3 in.

5. Sketch and describe a motor steering mechanism which enables the axes of all four wheels to meet at a common point when the car is rounding a corner. Show that the mechanism enables this to hold good at all times. (U.L.)

6. A wheel of mass 50 lb. and radius of gyration 4 ft., is spinning about its axis with an angular speed of 20 radians per sec. It is suspended with its axis horizontal by means of a vertical string attached to the axle at a point 4 in. from the centre of gravity. Find the rate of precession of the wheel.

7. A machine, weighing 2 tons, is carried on a rotating turntable. The machine stands on a circular base 1 ft. diameter, and has a flywheel rotating about a horizontal axis at 100 revs. per min. If the flywheel has a moment of inertia of 150 engineers' units, at what speed of rotation would the machine overturn ? The centre of gravity of the machine is on the axis of rotation of the turn-table.

8. Explain how the weight is distributed on the four wheels of a car, (1) when moving uniformly on a straight course, (2) when rounding a corner. Explain the effects of both centrifugal force and precession of the wheels.

9. Explain the effect on the distribution of pressure on the wheels of a car of the precession of a flywheel with a horizontal axis, (1) when the axis is parallel to the back axle, and (2) when the axis is at right-angles to this position.

10. A steam turbine, mounted in a boat, makes 3,000 revs. per min. The effective moment of inertia of the rotor, shaft, and propeller, is 0·5 ft.-ton units. Find the magnitude and direction of the couple acting on the hull when the boat describes a circular path, making a complete turn in 12 secs. Prove any formula you use. (U.L.)

11. Discuss the effects of precession on the pressure on the bearings of a turbine rotor carried by a ship due to the ship (*a*) rolling, (*b*) pitching.

12. The rotating parts of a motor-car engine are equivalent to a flywheel of 120 lb. at a radius of gyration of 8 in. The car takes a corner of 60 ft. radius at 20 miles per hour. Determine the magnitude of the gyroscopic couple and its effect on the car for the following dispositions of the engine: (*a*) engine shaft in longitudinal axis of the car, and rotation clockwise when viewed from front to rear. Engine speed 800 revs. per min.

(*b*) Engine shaft transversely across the car, and rotation clockwise when viewed from left-hand side to right-hand side of car. (U.L.A.)

13 The turbine rotors in a steamship have a weight W tons, radius of gyration k ft, and rotate at n revs. per min. Show what effect turning the ship and the pitching of the ship will have upon the forces acting on the turbine bearings and in producing bending moments on the shaft.

If there are a large number of rotors in a multi-stage turbine between two bearings A and B, show how to find the pressure on the bearings when the ship turns, and also the moment on the shaft between the bearings.

A turbine rotor rotating in a ship about an axis along the ship has a moment of inertia of 200 ft. ton units, and it rotates at 300 revs. per min. The ship pitches 14 degrees and has a period of oscillation of 10 secs. Find the gyrostatic couple acting on the turbine. (U.L.A)

14. Show by a sketch the positions of the wheels of a motor car, with front-wheel steering, when steering "correctly" on a curve, that is, when the wheels have pure rolling. Prove that the angles α, β, of the axes of the near and off wheels, respectively, with the back axle are connected by the equation: $\cot\beta - \cot\alpha = \frac{A}{B}$, where A is the distance between the pivots of the front wheel and B is the wheel-base of the car.

A car is turning on a radius of 20 ft. at 10 ft. per sec. The car weighs 1 ton; $A = 4$ ft, $B = 10$ ft. The centre of gravity is on the axis and equidistant from both axles. Find the side-thrust on each front wheel.

15. Suppose the body described in Art. 207 with its double symmetry to be divided up into an infinite number of thin rods perpendicular to the plane of symmetry. One of these is shown at P' (Fig. 267). The resultant force on this element acts along $P'N$. Let $P'M$ be perpendicular to OB. Let $\overline{OM} = x$ ft., $\overline{P'M} = y$ ft. Find the moment about O of the force on the element and show that for the whole body the total moment $C = \omega^2 \sin 2\alpha[\Sigma mx^2 - \Sigma my^2]$ lb.-ft., where m is the mass in engineers' units of the thin rod through P'. Show that this is the same result as (XII.12).

16. Take the same body as in the previous example, but with its centre of gravity at a distance h ft. from the axis of the shaft, and prove that the couple is still given by the same expression, but that there is in addition to the couple a force equal to that produced by the rotation of the whole mass assumed collected at O.

17. A swashplate consisting of a circular disc 12 in. in diameter and 3 in. thick is mounted on a shaft rotating at 500 r.p.m. The plane of the disc makes an angle of $\theta = 45°$ with the axis of the shaft, which passes through the centre of gravity of the disc. The material of the disc weighs 490 lb. per cubic foot. Show that the magnitude of the bending moment set up in the shaft is given by $\frac{\omega^2}{2}(I_P - I_D) \sin 2\theta$, and calculate its value where ω is the angular velocity, $I_P{}^1$ is the polar moment of inertia of the disc, and I_D the moment of inertia about the diameter. (U.L.A.)

18. Find the ratio of the length to the diameter of a cylinder on which there would be no couple produced by rotation under the conditions shown in Fig. 267, (1) when the cylinder is solid, (2) when it is hollow with the inside diameter half the outside diameter.

19. A gyro-wheel whose moment of inertia I spins about a horizontal axis with angular velocity p radians per sec. It is subjected to a torque G, which causes it to process with angular velocity r radians per sec. about a vertical axis. Determine the equation connecting these quantities.

The rotor of a motor used for electric traction weighs 1,200 lb. and has a radius of gyration of 9 in. The rotor shaft is parallel to the axles of the truck and runs in the opposite direction to the track wheels in two bearings spaced 30 in. apart. The centre of mass of the rotor is midway between the bearings. The motor runs at 1,500 r.p.m. when the speed of the train is 40 m.p.h.

Determine the force on each bearing when the train traverses a curve of 500 ft. radius. The position of the bearings with respect to the centre of the curve must be stated. (U.L.A.)

20. A uniform shaft, 2 in. diameter, rotating at 200 r.p.m. has attached to it a rectangular bar of metal 3 ft. long and 3 ft. square in section. The axes of the bar and shaft are co-planar and meet at an angle of 30°. Two faces of the bar are parallel to the axis of the shaft, and one end of the bar just touches the shaft. Find (i) the force and (ii) the couple exerted on the shaft by the bar.

21. An instrument carried on an aeroplane for blind flying consists of a spinning rotor with its axis horizontal mounted on bearings in a horizontal ring or gimbal. This ring can rock about a horizontal axis perpendicular to the rotor axis. The axis about which the ring rocks is fixed in the plane in a fore and aft direction, and the gimbal ring is prevented from rotating by the action of two springs, one on each side, so that if the gimbal moves relative to the plane through an acute angle the springs exert a restoring couple. There is a vertical pointer attached to the gimbal in line with the plane of the rotor. What do you think the pointer movement will indicate on a graduated scale fixed to the plane? Which way should the rotor spin?

22. A freely supported rotor of moment of inertia 10 engineers' units is rotating at 12,000 r.p.m. It is processing round the vertical at the rate of 15 degrees an hour. What couple is acting on it and about what axis, if the axis is (1) horizontal, (2) inclined at 45° to the horizontal?

CHAPTER XIV

OSCILLATIONS—WHIRLING OF SHAFTS—DYNAMICS (*continued*)

210. **The Energy Equation.** In Chapter III we considered several simple cases of oscillations. In each case we obtained the equation of motion from one of the fundamental relations

Force = mass × acceleration

Couple = moment of inertia × angular acceleration

An alternative method of obtaining the equation of motion makes use of the relation

$$\text{Total energy of system} = \text{constant} \quad . \quad . \quad . \quad \text{(XIV.1)}$$

This is the law of the conservation of energy (Chapter II), and (XIV.1) is known as the *equation of energy*. The assumption underlying the use of this equation is that there is no friction, viscosity, impact, or other agency for dissipating energy. When a body, or system, has only one degree of freedom, i.e. when the position of the body, or system, can be given in terms of a single variable, the energy equation will contain only this variable and its first differential coefficient with respect to time, and the equation of motion is obtained by differentiating with respect to time both sides of (XIV.1). The total energy of the body, or system, is partly kinetic and partly potential. The potential energy may be energy due to position, as when a body has the power of doing work by falling under gravity, or it may be energy due to condition, as in the case of a stretched spring which has the capacity of doing work in contracting. The former kind of potential energy is known as potential energy due to gravity, and the latter kind is *elastic energy* or *strain energy*. We shall consider again the case of Art. 46, using the equation of energy. When the weight W lb. is moving downwards through a position distant x ft. below its equilibrium position, the total energy is made up of

(1) The potential energy due to gravity $= -Wx$ ft.-lb.

(2) The elastic energy of the spring $= (W + \frac{1}{2}Ex)x$ ft.-lb.

(3) The kinetic energy of the weight $= \dfrac{W}{2g}\left(\dfrac{dx}{dt}\right)^2$ ft.-lb.

In (1) the potential energy is calculated with reference to the static position, and, as Wx ft.-lb. of work would be necessary to raise W from its displaced position to its static position, the potential energy is $-Wx$ ft.-lb. In (2) the mean force deflecting the spring is $\frac{1}{2}(W + \overline{W + Ex})$ lb., and this, multiplied by x, is the elastic energy. Substituting the above in (XIV.1) we obtain the energy equation

$$\frac{W}{2g}\left(\frac{dx}{dt}\right)^2 + (W + \tfrac{1}{2}Ex)x - Wx = \text{constant}$$

$$\therefore \frac{W}{2g}\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}Ex^2 = \text{constant} \quad . \qquad \text{(XIV.2)}$$

Differentiating both sides of this with respect to t,

$$\frac{W}{g}\frac{dx}{dt}\frac{d^2x}{dt^2} + Ex\frac{dx}{dt} = 0$$

and dividing through by $\frac{W}{g}\frac{dx}{dt}$

$$\frac{d^2x}{dt^2} + \frac{gE}{W}x = 0 \quad . \quad . \quad . \quad . \quad . \qquad \text{(XIV.3)}$$

which is the equation of motion of Art. 46.

In forming the energy equation (XIV.2) we took account of the strain energy Wx due to the load W, and also of the potential energy $-Wx$, and we found that these terms cancelled. It is usual in problems of this type to ignore the potential energy due to gravity; the results obtained will still be correct if the corresponding term in the expression for the strain energy is also ignored.

211. **Transverse Oscillations of a Thin Horizontal Rod Supported at the Ends.** Let l in. be the length of the rod, I the moment of inertia of its cross-section about the neutral axis in (inch)^4 units. and w, which may be variable or constant, the weight in pounds per inch length of the rod. There is an infinite number of ways in which such a rod can oscillate transversely. In some of the ways, all parts of the rod oscillate with simple harmonic motion, in the same period and in the same or opposite phases. These ways are known as the *normal modes* of motion. Figs. 273–275 show three of the normal modes for a uniform rod freely supported at the ends. The positions shown are stationary positions, each point being at the end of its swing. The full curve in Fig. 273 is half of a

sine wave, that in Fig. 274 is a complete sine wave, and that in Fig. 275 is one and a half sine waves. If started off properly, the rod will oscillate in any one of these modes, or in any mode in which the curve of the shaft divides the line AB into a whole number of equal parts. The dotted lines show in each case the position of the shaft at the other extremity of its oscillation. The times of oscillation are different in the different modes, and any type of oscillation other than a normal mode may be considered as being the sum of a number of motions each of which is a normal mode. Of these normal modes, the first usually predominates, the amplitudes of the oscillations of the various modes diminishing rapidly as the number representing the mode increases. The points C in Fig. 274, and D and E in Fig. 275, do not move. They, and the points A and B, are known as *nodes*, or stationary points.

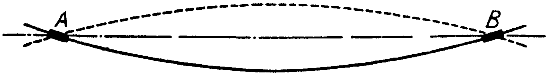


FIG. 273. FIRST NORMAL MODE

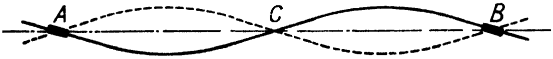


FIG. 274. SECOND NORMAL MODE

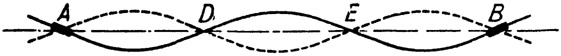


FIG. 275. THIRD NORMAL MODE

We shall study the first normal mode of the rod described above and shown in Fig. 276. Let C be the middle point of the rod AB in its statical position, and let $AC'B$ be its position at the instant of time t secs. We shall assume that all parts of the rod move along vertical lines. This is practically true, as the maximum value of z is very small compared with l. Let y in. be the deflection at time t of a point P', whose original position was P, where $\overline{AP} = x$ in. Let the central deflection at the same time be $\overline{CC'} = z$. As all points in the rod are executing simple harmonic motion, we have from (III.2) on putting $g_1 = o$ and writing ω for k_1,

$$y = a \cos \omega t \quad . \quad . \quad . \quad . \qquad \text{(XIV.4)}$$

and

$$z = c \cos \omega t \quad . \quad . \quad . \quad . \qquad \text{(XIV.5)}$$

where a and c are the maximum displacements of P' and C' respectively. Differentiating these, and dividing one by the other

$$\frac{\frac{dy}{dt}}{\frac{dz}{dt}} = \frac{a}{c} = \frac{y}{z}$$

or

$$\frac{dy}{dt} = \frac{y}{z}\,\frac{dz}{dt} \quad . \quad . \quad . \quad . \quad . \qquad \text{(XIV.6)}$$

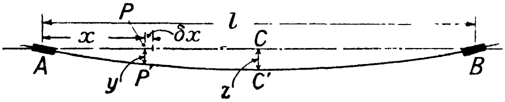


FIG. 276. OSCILLATING ROD

At any instant the kinetic energy of the rod is given by

$$KE = \text{sum of the kinetic energies of its parts}$$

$$= \text{sum of the quantities } \frac{w}{24g}\left(\frac{dy}{dt}\right)^2 \delta x \text{ inch-lb.}$$

where δx is an element of length at P, and g is the acceleration due to gravity in ft. per sec. per sec. Thus

$$KE = \int_o^l \frac{w}{24g}\left(\frac{dy}{dt}\right)^2 dx \text{ inch-lb.}$$

or from (XIV.6),

$$KE = \int_o^l \frac{w}{24g}\,\frac{y^2}{z^2}\left(\frac{dz}{dt}\right)^2 dx$$

$$= \frac{1}{24gz^2}\left(\frac{dz}{dt}\right)^2 \int_o^l wy^2 dx$$

$$= \frac{\left(\frac{dz}{dt}\right)^2}{24g}\,\frac{\int_o^l wy^2 dx}{z^2} \quad . \quad . \quad . \quad . \quad . \qquad \text{(XIV.7)}$$

The strain energy can be found if the shape of the bent rod $AC'B$ is known. When the rod deflects as a beam, under its own weight, the strain energy in the deflected position is equal to half the work done by the weights of the parts of the rod in falling to their deflected positions. We shall assume that when oscillating the rod assumes, at any instant, such a shape that the deflection at any point is k times the statical deflection at that point, due to the weight of the rod, k being a constant at any instant, but varying with time.* Let y_0 and z_0 be the statical deflections at P and C respectively, and E_e the elastic energy of the rod in the position $AC'B$. kw is the distributed load per inch of length which would cause the rod to take up statically the position $AC'B$. Hence

E_e = Half of the total work done by the weights of the parts of the rod in falling from the position ACB to $AC'B$ under a load of kw lb. per in.

$$= \tfrac{1}{2}\int_0^l kwy\,dx$$

But $$k = \frac{y}{y_0} = \frac{z}{z_0}$$

$$\therefore E_e = \frac{z}{2z_0}\int_0^l wy\,dx$$

$$= \frac{z^2}{2z_0}\,\frac{\int_0^l wy\,dx}{z}\text{ inch-lb.} \qquad \text{(XIV.8)}$$

The energy equation is

Kinetic energy + elastic energy = constant

i.e. $$\frac{1}{24g}\left(\frac{dz}{dt}\right)^2\frac{\int_0^l wy^2dx}{z^2} + \frac{z^2}{2z_0}\,\frac{\int_0^l wy\,dx}{z} = \text{constant} \qquad \text{(XIV.9)}$$

As we have assumed that in all positions $\frac{y}{z} = \frac{y_0}{z_0}$ = constant, the two quantities $\frac{\int_0^l wy^2dx}{z^2}$ and $\frac{\int_0^l wy\,dx}{z}$ are constants. Let A

---

* Lord Rayleigh showed that any assumed shape not strikingly unlike the actual curve leads to results which are very nearly correct.

and B represent their respective values. Then, from (XIV.9),

$$\frac{A}{24g}\left(\frac{dz}{dt}\right)^2 + \frac{B}{2z_0}z^2 = \text{constant}$$

Differentiating this with respect to t,

$$\frac{A}{12g}\frac{dz}{dt}\cdot\frac{d^2z}{dt^2} + \frac{B}{z_0}z\frac{dz}{dt} = 0$$

whence, dividing through by $\frac{A}{12g}\frac{dz}{dt}$, we have the equation of motion

$$\frac{d^2z}{dt^2} + \frac{12Bg}{Az_0}z = 0 \quad . \quad . \quad . \quad . \quad (\text{XIV.10})$$

From this, by comparison with (III.1) and (III.6),

t = time of oscillation in seconds

i.e. $$t = 2\pi\sqrt{\frac{Az_0}{12Bg}} \quad . \quad . \quad . \quad . \quad (\text{XIV.11})$$

The number N of oscillations per minute is given by

$$N = \frac{60}{2\pi}\sqrt{\frac{12Bg}{Az_0}} \quad . \quad . \quad . \quad . \quad (\text{XIV.12})$$

Since A and B are constants, their values may be found for any convenient position, e.g. the statical position ; in this case

$$\frac{B}{A} = \frac{\int_0^l wy_0dx}{z_0} \div \frac{\int_0^l wy_0^2dx}{z_0^2}$$

$$= \frac{z_0\int_0^l wy_0dx}{\int_0^l wy_0^2dx}$$

Hence, $$N = \frac{30}{\pi}\sqrt{\frac{12g\int_0^l wy_0dx}{\int_0^l wy_0^2dx}}$$

$$= 188\sqrt{\frac{\int_0^l wy_0dx}{\int_0^l wy_0^2dx}} \quad . \quad . \quad . \quad (\text{XIV.13})$$

If some of the loads are concentrated, or if the distributed load is not uniform, and its equation not known as a function of x, the integrals under the root are replaced by $\Sigma W y_o$, and $\Sigma W y_o^2$, respectively, where W is the concentrated load at any point, distributed loads being split up into a number of parts (say 10), and each part considered as a load at its centre of gravity. The summations are then carried out, the values of y_o being found from the deflection curve of the rod, this latter curve having been obtained previously by methods explained in textbooks on beam theory. In this case (XIV.13) becomes

$$N = 188\sqrt{\frac{\Sigma W y_o}{\Sigma W y_o^2}} \quad . \quad . \quad . \quad . \quad \text{(XIV.14)}$$

Another expression for N may be found from (XIV.12), the shape of the deflected beam being assumed to be any curve, say a sine curve, of one half wave. This curve roughly agrees with the deflection curve for a uniformly loaded beam, and Lord Rayleigh has shown that any assumed shape reasonably like the actual deflection curve will lead to results which are appreciably correct. The average height of a sine curve is $\frac{2}{\pi} \times$ (maximum height), and the average value of the square of the height is $\frac{1}{2} \times$ (the square of the maximum height), so that

$$B = \frac{\int_0^l w y_o dx}{z} = wl \times \frac{2}{\pi}$$

and

$$A = \frac{\int_0^l w y_o^2 dx}{z^2} = wl \times \tfrac{1}{2}$$

$$\therefore \frac{B}{A} = \frac{4}{\pi}, \text{ and from (XIV.12)}$$

$$N = \frac{30}{\pi}\sqrt{\frac{12g}{z_o} \cdot \frac{4}{\pi}}$$

$$= \frac{212}{\sqrt{z_o}}$$

or

$$N = \frac{212}{\sqrt{\text{maximum statical deflection in ins.}}} \quad \text{(XIV.15)}$$

If, instead of a sine curve, we had assumed the shape of the deflection curve to be a parabola, we should have obtained the formula of (XIV.15), with the constant 210 instead of 212 in the numerator. By actual integration and calculation from (XIV.13), the constant for the case of a uniform load is found to be 210, correct to three significant figures.

A general formula to suit all cases is

$$N = \frac{c}{\sqrt{\text{maximum statical deflection}}} \quad . \quad \text{(XIV.16)}$$

where c is a constant which varies from 188 for a single concentrated load at the centre, to 210 for a uniformly distributed load. In the above relations N is the number of complete oscillations per minute and the deflection is in inches. (XIV.16) gives only the frequency of the first normal mode. If there are other normal modes, as in the case of a uniformly loaded rod with free ends, the frequencies of these are found by considering only a length of shaft between two nodes, e.g. *AC* in Fig. 274, *AD* in Fig. 275, etc.

212. **Beams with Two or More Concentrated Loads.** If a beam has several attached masses it is possible to obtain an approximate expression for the time of oscillation of the system in terms of the times of oscillation of the beam when carrying the masses singly, each in its own position. Thus, let N_1, N_2, N_3, N_4, etc., be the frequencies when the beam carries single loads W_1, W_2, W_3, W_4, etc., respectively, each in its own fixed position, and let N_s be the frequency of the unloaded beam. Then the frequency N of the system is given by

$$\frac{1}{N^2} = \frac{1}{N_1^2} + \frac{1}{N_2^2} + \frac{1}{N_3^2} + \frac{1}{N_4^2} + \ . \ . \ . + \frac{1}{N_s^2} \ . \ . \ . \quad \text{(XIV.17)}$$

For a proof of the approximate truth of this, readers are referred to *Applied Elasticity*, by Dr. J. Prescott. We shall see in Art. 217 that the above relation is important in the design of rotating shafts.

213. **Whirling of Shafts.** When a shaft rotates about its axis, any portion of mass in, or attached to, the shaft has a centripetal acceleration. In order to produce this acceleration the adjacent portions of the shaft must exert a force acting inwards towards the axis. The effect on the adjacent portions

of the shaft is the same as if a force equal to the product of the mass and the acceleration were acting outwards from the axis through the centre of gravity of the mass. If, then, we introduce for each mass, or portion of mass, a centrifugal force, we may disregard the rotation of the shaft and consider it to be in equilibrium under the action of the centrifugal forces and the elastic forces in the material of the shaft. If a shaft could have a perfectly straight axis, and if the shaft and attached masses, such as pulleys, were such that the centre of gravity of every element of length parallel to the axis of the shaft were situated on the axis, then, in the absence of external disturbing forces, the axis of the shaft would remain straight at all speeds, though, as we shall see, the state of equilibrium would be unstable at certain speeds called *critical* speeds. Such perfection is not possible, and external disturbing forces are not absent. Owing to the rotation combined with the lack of straightness of the shaft, etc., centrifugal forces are set up, and at all speeds the axis of the shaft is deflected a little from its statical position. This deflection becomes dangerous at the critical speeds. When the shaft is rotating at one of these speeds, the state of affairs is known as *whirling*.

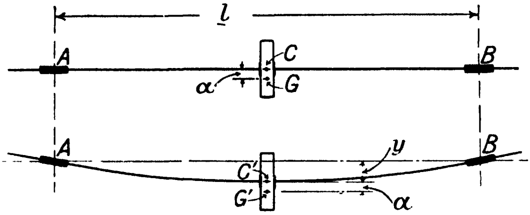


FIG. 277

214. **Whirling of a Light Flexible Shaft Carrying at the Centre of its Length a Pulley whose Centre of Gravity is not on the Axis of the Shaft.** We shall assume that the mass of the shaft is very small in comparison with that of the pulley. Let l in. be the length of the shaft, and a in. the distance $\overline{CG}$ (Fig. 277) where ACB is the shaft and G is the centre of gravity of the pulley. Let W lb. be the mass of the pulley. A and B are bearings, which fix the positions of A and B and which may, or may not, be fixed in direction.

The lower diagram shows the running conditions, $AC'B$

being the shaft and G' the centre of gravity. Owing to the rotation there is a centrifugal force of F lb. acting on the shaft, where

$$F = \frac{W}{12g}\omega^2(y + \alpha) \quad . \quad . \quad . \quad (\text{XIV.18})$$

g is the acceleration due to gravity in ft. per sec. per sec.
ω is the angular velocity in radians per sec., and
y is the radial displacement in inches of the centre point C of the axis AB.

In books on the strength of materials, it is shown that the relation between the force F lb. acting at the centre of a uniform beam and the deflection y in. at the centre, is given by

$$y = \frac{Fl^3}{cEI} \quad . \quad . \quad . \quad . \quad . \quad (\text{XIV.19})$$

where E is Young's modulus of elasticity in lb. per sq. in.; I is the moment of inertia of the section about its neutral axis in (inch)^4 units. For a circular shaft of radius r, $I = \frac{\pi r^4}{4}$ and c is a constant depending upon the nature of the end constraints at A and B.

If the ends at A and B are fixed in position but not in direction, $c = 48$, if the ends are also fixed in direction, $c = 192$.

Substituting in (XIV.19) the value of F from (XIV.18) we have

$$y = \frac{Wl^3}{12gcEI}\omega^2(y + \alpha)$$

Writing m^2 for the fraction $\frac{12gcEI}{Wl^3}$, this becomes

$$m^2y = \omega^2y + \omega^2\alpha$$

from which

$$y = \frac{\omega^2}{m^2 - \omega^2}\alpha \quad . \quad . \quad . \quad . \quad . \quad (\text{XIV.20})$$

y becomes large when ω is nearly equal to m, and, when $\omega = m$, y is infinite. The relation (XIV.19) is not true for very large values of y, as, if y is large, the elastic law breaks down, and other considerations, such as increased air resistance, internal friction, etc., are introduced. We conclude, however, that if $\omega = m$ the deflection y becomes large enough to cause damage to the material of the shaft. The speed $\omega = m$ is the only *critical speed*. At speeds greater than the critical speed, $m^2 - \omega^2$ is negative and, therefore, y is negative; thus at speeds greater than the critical speed the centre of gravity G lies between the centre of the pulley C and the straight line AB. For very large values of ω, m^2 is negligible compared with ω^2 and the value of y is very nearly given by

$$y = \frac{\omega^2}{-\omega^2}\alpha$$

i.e.

$$y = -\alpha$$

which shows that at very high speeds the centre of gravity is very nearly on the axis of rotation, the deviation becoming smaller and smaller as the speed increases. With such an arrangement as this, there is a better balance at high speeds than at low speeds. In the above investigation we have neglected the effects of gravity in deflecting the shaft. Strictly, the above results only hold true for a shaft whose axis is vertical, but, when the shaft is rotating at high speeds, the centrifugal forces are large compared with the weights, and the results may be taken to be true in all cases.

215. Uniform Shaft with no Attached Masses. Let ACB, Fig. 276, be a shaft of length l in., and let $AC'B$ be an instantaneous position of the shaft when whirling is taking place, y in. being the deflection of a point P originally at a distance x in. from A. Then, if w lb. is the weight of 1 in. length of shaft, the centrifugal force per inch length of shaft at P is $\frac{w}{12g}\omega^2 y$ lb., and the shaft will have the same shape as would a beam of the same dimensions and properties when carrying a load of $\frac{w}{12g}\omega^2 y$ lb. per inch run. In the theory of beams it is shown that

$$EI\frac{d^4y}{dx^4} = \text{load per unit length}$$

E and I having the meanings previously defined. Hence

$$EI\frac{d^4y}{dx^4} = \frac{w}{12g}\omega^2 y$$

$$\frac{d^4y}{dx^4} = m^4 y \quad . \quad . \quad . \quad . \quad \text{(XIV.21)}$$

where $$m^4 = \frac{w\omega^2}{12gEI}$$

or $$\omega = m^2 \sqrt{\frac{12gEI}{w}} \quad . \quad . \quad . \quad \text{(XIV.22)}$$

The full solution of (XIV.21) is

$$y = A \cos mx + B \sin mx + C \cosh mx + D \sinh mx \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV.23)}$$

By suitably choosing the constants m, A, B, C, and D, this relation may be made to represent any possible normal mode of motion of the shaft, the shape of the shaft when whirling being, of course, affected by the nature of the end constraints. If we assume the ends to be located in position only, there being no directional constraints, the bending moments at the ends will be zero. As the bending moment is $EI \frac{d^2y}{dx^2}$, this means that $\frac{d^2y}{dx^2} = 0$, both when $x = 0$, and when $x = l$. Thus, the end conditions give us the relations

(1) $y = 0$, when $x = 0$,

(2) $y = 0$, when $x = l$,

(3) $\frac{d^2y}{dx^2} = 0$, when $x = 0$, and

(4) $\frac{d^2y}{dx^2} = 0$, when $x = l$.

Conditions (1) and (2), when substituted in (XIV.23) give us

$$A + C = 0 \quad . \quad . \quad . \quad . \quad \text{(XIV.24)}$$

and

$$A \cos ml + B \sin ml + C \cosh ml + D \sinh ml = 0 \quad \text{(XIV.25)}$$

Also, if we find $\frac{d^2y}{dx^2}$ from (XIV.23), and substitute in turn the conditions (3) and (4), we have, on dividing through by m^2,

$$-A + C = 0 \quad . \quad . \quad . \quad . \quad \text{(XIV.26)}$$

and

$$A \cos ml - B \sin ml + C \cosh ml + D \sinh ml = 0 \quad \text{(XIV.27)}$$

From (XIV.23) and (XIV.25), $A = C = 0$, and (XIV.25) and (XIV.27) become respectively

$$B \sin ml + D \sinh ml = 0 \quad . \quad \text{(XIV.28)}$$

and
$$-B \sin ml + D \sinh ml = 0 \quad . \quad \text{(XIV.29)}$$
from which by adding,
$$2D \sinh ml = 0$$
or, since ml cannot be zero, $D = 0$. Thus (XIV.23) reduces to
$$y = B \sin mx \quad . \quad . \quad . \quad . \quad \text{(XIV.30)}$$
When y has any value in this other than zero, i.e. when the shaft is whirling, B has some value other than zero, and the axis of the shaft takes the form of a sine curve. Since $y = 0$ when $x = l$, we must have in all possible cases of whirling
$$\sin ml = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV.31)}$$
to satisfy which ml must be some multiple of π. Since m is a function of ω, whirling can only occur when ω has values which give rise to values of m satisfying (XIV.31). These values are
$$m = \frac{\pi}{l},\ m = \frac{2\pi}{l},\ m = \frac{3\pi}{l},\ m = \frac{4\pi}{l},\ \text{etc.,}$$
from which, by (XIV.22), the whirling speeds are given by
$$\omega_1 = \frac{\pi^2}{l^2}\sqrt{\frac{12gEI}{w}},\ \omega_2 = \frac{4\pi^2}{l^2}\sqrt{\frac{12gEI}{w}},\ \omega_3 = \frac{9\pi^2}{l^2}\sqrt{\frac{12gEI}{w}}$$
$$\omega_4 = \frac{16\pi^2}{l^2}\sqrt{\frac{12gEI}{w}},\ \text{etc.} \quad . \quad \text{(XIV.32)}$$

These speeds are in the ratios of the numbers, 1^2, 2^2, 3^2, 4^2, 5^2, etc., and are referred to as the first, second, third, fourth, etc., whirling speeds. Near these speeds the shaft will be in a more or less unstable condition, and will be liable to damage by overstrain. The above theory does not give us the range of speed over which it is dangerous to run the shaft.

216. **Ends Fixed in Direction.** If the ends of the shaft are fixed in the direction of the straight line AB (Fig. 276) the conditions (3) and (4) above are replaced by the conditions $\frac{dy}{dx} = 0$ when $x = 0$, and when $x = l$, from which with the other two conditions (1) and (2) the values of the constants in (XIV.23) may be determined. Thus, the conditions (1) and (2) and $\frac{dy}{dx} = 0$ when $x = 0$ and when $x = l$, give us in turn the relations
$$A + C = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV.33)}$$

$$A\cos ml + B\sin ml + C\cosh ml + D\sinh ml = 0 \qquad \text{(XIV.34)}$$

$$B + D = 0 \qquad \text{(XIV.35)}$$

$$-A\sin ml + B\cos ml + C\sinh ml + D\cosh ml = 0 \qquad \text{(XIV.36)}$$

From the first and third of these, $C = -A$, and $D = -B$ and substituting in the other two equations, we have

$$A(\cos ml - \cosh ml) + B(\sin ml - \sinh ml) = 0$$

and $$-A(\sin ml + \sinh ml) + B(\cos ml - \cosh ml) = 0$$

from which, on elimination of A and B, we have

$$-\frac{\cos ml - \cosh ml}{\sin ml + \sinh ml} = \frac{\sin ml - \sinh ml}{\cos ml - \cosh ml}$$

or $$(\cos ml - \cosh ml)^2 = \sinh^2 ml - \sin^2 ml$$

i.e. $$\cos^2 ml - 2\cos ml \cosh ml + \cosh^2 ml$$

$$= \sinh^2 ml - \sin^2 ml$$

or $$2\cos ml \cosh ml$$

$$= \cosh^2 ml - \sinh^2 ml + \cos^2 ml + \sin^2 ml$$

$$= \quad 1 \quad + \quad 1$$

$$= 2$$

$$\therefore \cos ml = \frac{1}{\cosh ml}$$

i.e. $$\cos ml = \operatorname{sech} ml \quad . \quad . \quad . \quad . \qquad \text{(XIV.37)}$$

which gives as roots the values of m for which whirling occurs.

In Fig. 278 we have sketched the graphs of $y = \cos x$, and $y = \operatorname{sech} x$, where $x = ml$. Sech x is small for values of x greater than π, its value when $x = 6$ being less than $\frac{1}{200}$. Thus, the roots of (XIV.37) are given very approximately by the points of intersection of the axis of x and the curve $y = \cos x$. The roots are therefore approximately $ml = \frac{3\pi}{2}$, $ml = \frac{5\pi}{2}$, $ml = \frac{7\pi}{2}$, $ml = \frac{9\pi}{2}$, etc., and the values of ω are given by

$$\omega = \frac{9\pi^2}{4l^2}\sqrt{\frac{12gEI}{w}},\ \omega = \frac{25\pi^2}{4l^2}\sqrt{\frac{12gEI}{w}},\ \omega = \frac{49\pi^2}{4l^2}\sqrt{\frac{12gEI}{w}}$$

etc. These speeds are in the ratios of the numbers 3^2, 5^2, 7^2, 9^2, etc.

217. Equality of the Frequency of Transverse Oscillations and the Whirling Speed. Fig. 273 shows a rod executing simple harmonic motion, the motion being of the first mode. Figs. 274 and 275 show the second and third normal modes. In each of these cases, if we assume the figures to show the shaft at one end of its oscillation, any small element of the length of the rod may be assumed to be in equilibrium under the action of two forces, (1) the resultant of the elastic forces of the other parts of the rod on the element, and (2) the reversed mass acceleration. The magnitude of (2) is $\omega_o^2 y$ times the mass of the element, where ω_o is given by $t = \frac{2\pi}{\omega_o}$, t being the time of oscillation, and y is the amplitude of the oscillation of the element. If now we consider the rod to be whirling, and not oscillating, the elastic forces on the element are the same as before, the shaft being in the same strained condition. The resultant elastic force is balanced in this case by (3) the reversed centripetal force whose magnitude is $\omega_w^2 y$ times the mass of the element, ω_w being the angular velocity of whirl. Since the same resultant elastic force is balanced in turn by the forces (2) and (3), these two forces must be equal.

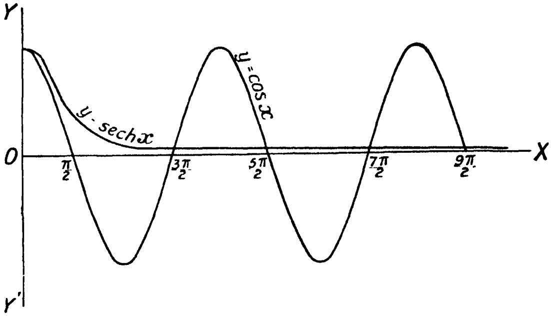


FIG. 278

Hence, $$\omega_o^2 y = \omega_w^2 y$$

or $$\omega_o = \omega_w$$

and $$\frac{2\pi}{\omega_o} = \frac{2\pi}{\omega_w}$$

that is, the frequency of oscillation and the whirling speed are equal.

Thus, the formulae of Art. 211 will be taken to apply to cases of whirling of rotating shafts having the given dimensions and loaded as stated. We have, then, the following results, in which N gives the number of revolutions per minute at the first whirling speed, the ends not being fixed in direction.

$$N = 188\sqrt{\frac{\int_o^l wy_o\,dx}{\int_o^l wy_o^2\,dx}} \text{ variable or uniform loading} \quad (\text{XIV}.38)$$

$$N = 188\sqrt{\frac{\int_o^l y_o\,dx}{\int_o^l y_o^2\,dx}} \text{ uniformly distributed load} \quad (\text{XIV}.39)$$

$$N = 188\sqrt{\frac{\Sigma Wy_o}{\Sigma Wy_o^2}} \text{ load not continuous} \quad . \quad . \quad . \quad (\text{XIV}.40)$$

$$N = \frac{c}{\sqrt{\text{maximum statical deflection}}} \quad . \quad . \quad . \quad (\text{XIV}.41)$$

where c is a constant varying from 188 for a single concentrated load to 210 for a uniformly distributed load. Also, if N_1, N_2, N_3, etc., are the numbers of revolutions per minute at which whirling occurs when a shaft is carrying single loads W_1, W_2, W_3, etc., each in its own position, and N_s is the number for the unloaded shaft then if N is the number of revolutions per minute at which whirling takes place when all the loads are on the shaft, each in its own position, then

$$\frac{1}{N^2} = \frac{1}{N_1^2} + \frac{1}{N_2^2} + \frac{1}{N_3^2} + \frac{1}{N_4^2} + \dots + \frac{1}{N_s^2} \quad (\text{XIV}.42)$$

is approximately true. The running speed should not be inside a range of about 30 per cent above or below the first critical speed.

EXAMPLE.

A weight w, suspended from a spring, has a natural frequency n per sec. The upper end of the spring is given a vertical simple harmonic motion, having the same frequency and a travel of $2a$ ft. To prevent excessive oscillation a dashpot is introduced, which produces a retarding force Kv lb. when the velocity of the weight is v feet sec. Prove that in order to limit the displacement of the weight to d, K must equal $\frac{2\pi naw}{gd}$. (U.L.A.)

Using the notation of Art. 53, the equation of motion is

$$\text{Force} = \text{mass} \times \text{acceleration}$$

$$= -\frac{w}{g} \times \frac{d^2x}{dt^2}$$

i.e. $$\frac{d^2x}{dt^2} = -\frac{g}{w} \times \text{force}$$

Let x be the displacement of the lower end of the spring measured positive downwards; the displacement of the upper end will be $a \sin 2\pi nt$, also measured positive downwards. The stretch of the spring is $(x - a \sin 2\pi nt)$ and the restoring force is $E(x - a \sin 2\pi nt)$. The frictional force is $Kv = K\frac{dx}{dt}$. Thus, the force is $E(x - a \sin 2\pi nt) + K\frac{dx}{dt}$. Substituting this in the above,

$$\frac{d^2x}{dt^2} = -\frac{g}{w}\left\{(Ex - Ea \sin 2\pi nt) + K\frac{dx}{dt}\right\}$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{gE}{w}x + \frac{gE}{w}a \sin 2\pi nt - \frac{Kg}{w}\frac{dx}{dt}$$

and putting $\frac{gE}{W} = 4\pi^2n^2$ where n is the natural frequency, (compare with Art. 53, in which n is 2π times the natural frequency),

$$\frac{d^2x}{dt^2} = -4\pi^2n^2x - \frac{Kg}{w}\frac{dx}{dt} + n^2 a \sin 2\pi nt$$

or $$\frac{d^2x}{dt^2} + \frac{Kg}{w}\frac{dx}{dt} + 4\pi^2n^2x = 4\pi^2n^2 \sin 2\pi nt$$

To find a particular integral of the above equation, put

$$x = A \sin 2\pi nt + B \cos 2\pi nt$$

Substituting this in the equation of motion,

$$- 4\pi^2n^2A \sin 2\pi nt - 4\pi^2n^2B \cos 2\pi nt$$

$$+ \frac{Kg}{w}(2\pi nA \cos 2\pi nt - 2\pi nB \sin 2\pi nt)$$

$$+ 4\pi^2n^2(A \sin 2\pi nt + B \cos 2\pi nt) = 4\pi^2n^2a \sin 2\pi nt$$

from which, by equating coefficients of sin $2\pi nt$ and cos $2\pi nt$

$$\left.\begin{aligned} -\frac{2\pi nKg}{w}B &= 4\pi n^2 a \\ \text{and} \qquad \frac{2\pi nKg}{w}A &= 0 \end{aligned}\right\}$$

from which $A = 0$, and $B = -\dfrac{4\pi n^2 wa}{2\pi nKg} = -\dfrac{2\pi nwa}{Kg}$

Thus, the particular integral is

$$x = -\frac{2\pi naw}{Kg}\cos 2\pi nt$$

To this should be added the value of x, representing the free oscillations ; this would be obtained by solving the equation

$$\frac{d^2x}{dt^2} + \frac{Kg}{w}\frac{dx}{dt} + 4\pi^2 n^2 x = 0$$

but these oscillations would ultimately be damped out by friction, and the above value of x would then give the correct displacement. As the maximum value of cos $2\,\pi nt$ is unity, the maximum amplitude is $\dfrac{2\pi naw}{Kg}$. Equating this to d, which is the greatest permissible displacement,

$$\frac{2\pi naw}{Kg} = d$$

whence

$$K = \frac{2\pi naw}{gd}$$

EXAMPLE.

A light shaft carries at the centre a pulley whose centre of gravity is on the axis of the shaft, and whose mass is W lb. Find the whirling speed, using the formula $N_w = 188\sqrt{\dfrac{\Sigma W y_o}{\Sigma W y_o^2}}$

Let l in. be the length of the shaft, which we shall assume to be fixed in position only at the ends, the directions being unconstrained. The central deflection due to a load W lb. is $\dfrac{Wl^3}{48EI}$

Hence, $$y_o = \frac{Wl^3}{48EI}$$

$$\Sigma w y_o = W y_o$$

and $$\Sigma w y_o^2 = W y_o^2$$

Hence, from the formula

$$N_w = 188\sqrt{\frac{Wy_o}{Wy_o^2}}$$

$$= \frac{188}{\sqrt{y_o}}$$

$$= 188\sqrt{\frac{48EI}{Wl^3}} = 1302\sqrt{\frac{EI}{Wl^3}}$$

This is the lowest whirling speed.

EXAMPLE.

If the shaft in the last example weighs w lb. per inch of length, find expressions for its whirling speed, (1) when unloaded, (2) when carrying a mass of W lb. weight at the centre of the span.

(1) From (XIV.32) the first critical speed is given by

$$\omega = \frac{\pi^2}{l^2}\sqrt{\frac{12gEI}{w}}$$

Hence, if N is the whirling speed in revs. per min.

$$N = \frac{2\pi}{\omega}$$

$$= \frac{60\pi}{2l^2}\sqrt{\frac{12gEI}{w}}$$

$$= 1853\sqrt{\frac{EI}{wl^4}}$$

(2) Let N_c be the required speed of whirling in revs. per min. Then by (XIV.17),

$$\frac{1}{N_c^2} = \frac{1}{N^2} + \frac{1}{N_w^2}$$

where N_w has the value $1302\sqrt{\dfrac{EI}{Wl^3}}$ found in the last example. Substituting, we have

$$\frac{1}{N_c^2} = \frac{wl^4}{1853^2EI} + \frac{Wl^3}{1302^2EI}$$

$$\therefore \frac{1}{N_c^2} = \frac{l^3}{EI}\left(\frac{wl}{1853^2} + \frac{W}{1302^2}\right)$$

$$= \frac{l^3}{EI}\,\frac{1302^2wl + 1853^2W}{2{,}413{,}000^2}$$

and
$$N_c = 2{,}413{,}000\sqrt{\frac{EI}{1302^2wl + 1853^2W}}$$

$$\therefore N_c = 1853\sqrt{\frac{EI}{wl^4 + 2{\cdot}026Wl^3}}$$

This result shows that a concentrated load at the centre is roughly equivalent to a uniformly distributed load of twice its weight.

218. **Whirling Speeds of a Shaft of Negligible Mass Carrying any Number of Attached Masses whose Centres of Gravity are Situated on the Centre-line of the Shaft.** The arrangement of the shaft and masses is shown in Fig. 279, $A, B, C, D, \ldots W, Z$, are the points at which the masses of magnitudes M_A, M_B, M_C, etc., respectively are attached to the shaft. Each mass is measured in engineers' units being the weight in pounds of the mass divided by the value of g in ft. per sec. per sec. These masses are supposed to have zero moments of inertia about axes perpendicular to the axis of the shaft. The bearings which support the shaft are not shown.

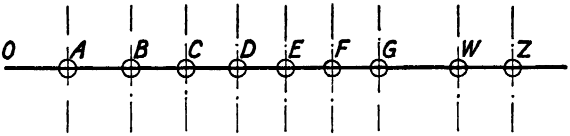


FIG. 279. WHIRLING OF LOADED SHAFT

We shall assume that when the shaft is whirling, its centre line, which is curved, is fixed relative to a plane containing it and the axis of rotation and rotating with the shaft. The shaft may therefore be treated as a beam which is deflected by a system of forces. The deflecting forces are the reversed mass accelerations of the masses, each of which moves so that its centre of gravity describes a circle. Let y_A, y_B, y_C, etc., be the respective deflections in feet, all considered positive when they are in the same sense as y_A, and negative when opposite. Let ω radians per second be the angular velocity at which whirling occurs. The deflecting forces are $\omega^2 M_A y_A$, $\omega^2 M_B y_B$ lb. respectively. Let Δ_{AA}, Δ_{AB}, Δ_{BC}, etc., represent deflections produced by unit loads. Thus Δ_{BC} represents the deflection in feet under the load C produced by a load of 1 lb. at B, Δ_{CC} that under the load C due to a load of 1 lb. at C. The method of calculating these deflections is given in textbooks on "Strength of Materials," and the deflections are known as *influence numbers*. The shaft is treated as a beam with the bearings as supports, and account is taken of all constraints, exercised by the bearings, on the slope and the deflections. A long fixed bearing will make the slope zero, whereas a very short bearing or a longer bearing mounted on a spherical seating will not fix the slope. Now consider the deflection y_A whilst the shaft is whirling. y_A is the sum of the deflections at A produced by the separate deflecting forces. The part of y_A produced by any force is the product of that force and deflection produced at A by unit load at the position of the force. Thus we have

$$y_A = \omega^2 M_A y_A \Delta_{AA} + \omega^2 M_B y_B \Delta_{BA} + \omega^2 M_C y_C \Delta_{CA} + \dots + \omega^2 M_Z y_Z \Delta_{ZA}$$

or $$y_A(\omega^2 M_A \Delta_{AA} - 1) + \omega^2 M_B y_B \Delta_{BA} + \omega^2 M_C y_C \Delta_{CA} + \dots + \omega^2 M_Z y_Z \Delta_{ZA} = 0$$

Now let $\omega^2 M_A \Delta_{AA} - 1 = P_{AA}$, $\omega^2 M_B \Delta_{BA} = R_{BA}$, etc., then

$$P_{AA} y_A + R_{BA} y_B + R_{CA} y_C + \dots + R_{ZA} y_Z = 0$$

Similarly, considering the deflections at B, C, D, etc., in turn we have

$$\begin{aligned}
R_{AB} y_A + P_{BB} y_B + R_{CB} y_C + \dots + R_{ZB} y_Z &= 0 \\
R_{AC} y_A + R_{BC} y_B + P_{CC} y_C + \dots + R_{ZC} y_Z &= 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\
R_{AZ} y_A + R_{BZ} y_B + R_{CZ} y_C + \dots + P_{ZZ} y_Z &= 0
\end{aligned}$$

The number of equations is equal to the number of unknown deflections. The number of the latter may, however, be reduced by one by dividing through in each equation by y_A, and so the deflections may be eliminated. The eliminant is most easily found by the use of a determinant (see *Practical Mathematics*, Toft and McKay, Pitman) and is

$$\begin{vmatrix} P_{AA} & R_{BA} & R_{CA} & R_{DA} & \dots & R_{ZA} \\ R_{AB} & P_{BB} & R_{CB} & R_{DB} & \dots & R_{ZB} \\ R_{AC} & R_{BC} & P_{CC} & R_{DC} & \dots & R_{ZC} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ R_{AZ} & R_{BZ} & R_{CZ} & R_{DZ} & \dots & P_{ZZ} \end{vmatrix} = 0 \quad . \quad \text{(XIV.43)}$$

Since each P and each R is a linear function of ω^2 the degree of the equation in ω^2 is equal to the number of masses. If ω_1^2, ω_2^2, ω_3^2, etc., are its roots arranged in order of increasing magnitudes, then ω_1, ω_2, ω_3, etc., are the first, second, third, etc., critical speeds. The above results also give the frequencies of the normal modes of transverse oscillations of the shaft.

EXAMPLE.

A steel shaft 3 in. diameter and 10 ft. long carries two pulleys, one weighing 1,500 lb. at a point 3 ft. from one end and the other weighing 2,400 lb. at a point 2 ft. from the other end. The centre of gravity of each pulley is on the axis of the shaft. Find the whirling speeds. Neglect the weight of the shaft and assume that the bearings, of which there are two, one at each end, do not constrain the slope.

We treat the shaft as a freely supported beam. If such a beam carries a single load of W lb. weight at a point distant L_1 and L_2 in. respectively from the left-hand and right-hand supports, the deflection under the load is given by

$$\Delta = \frac{WL_1^2L_2^2}{36EI(L_1 + L_2)} \text{ ft.} \quad . \quad . \quad . \quad (1)$$

and that at a distance x in. from the left-hand support is given by

$$\Delta = \frac{WL_2[L_1(L_1 + 2L_2)x - x^3]}{72EI(L_1 + L_2)} \quad . \quad . \quad (2)$$

We shall assume that $E = 30 \times 10^6$ lb. per sq. in. $I = \frac{\pi r^4}{4}$ where r is the radius of the shaft in inches, i.e. $r = 1{\cdot}5$. Hence $I = 3{\cdot}976$ (inch)^4 units. Let OC, Fig. 279, be the shaft and A and B the masses. Then $OA = 36$ in., $AB = 60$ in., and $BC = 24$ in.

Now suppose there is a load of 1 lb. at B. Then with the above notation, putting $L_1 = 96$ and $L_2 = 24$ in (1), we have

$$\Delta_{BB} = 1{\cdot}030 \times 10^{-5} \text{ ft.}$$

Similarly putting $L_1 = 96$, $L_2 = 24$, and $x = 36$ in (2) we find

$$\Delta_{BA} = 1{\cdot}050 \times 10^{-5} \text{ ft.}$$

Again, suppose there is a load of 1 lb. at A only. We put $L_1 = 36$ and $L_2 = 84$ in (1) to obtain

$$\Delta_{AA} = 1{\cdot}774 \times 10^{-5} \text{ ft.}$$

By Maxwell's "Reciprocal Theorem" (see Pippard's *Strain Energy Methods of Stress Analysis*) we have

$$\Delta_{AB} = \Delta_{BA}$$

$$\therefore \quad \Delta_{AB} = 1{\cdot}050 \times 10^{-5} \text{ ft.}$$

To find the critical speeds we take (XIV.43) and ignore all terms which are not in both the first two rows and the first two columns. Thus

$$\begin{vmatrix} P_{AA} & R_{BA} \\ R_{AB} & P_{BB} \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad (3)$$

or,

$$P_{AA}P_{BB} - R_{AB}R_{BA} = 0 \quad . \quad . \quad . \quad . \quad (4)$$

But

$$\begin{aligned} P_{AA} &= \omega^2 M_A \Delta_{AA} - 1 \\ &= \frac{1500}{32{\cdot}2} \times 1{\cdot}774 \times 10^{-5}\omega^2 - 1 \\ &= {\cdot}0008264\omega^2 - 1 \end{aligned}$$

Similarly,

$$\begin{aligned} P_{BB} &= \frac{2400}{32{\cdot}2} \times 1{\cdot}030 \times 10^{-5}\omega^2 - 1 \\ &= {\cdot}0007677\omega^2 - 1 \end{aligned}$$

Also,

$$\begin{aligned} R_{AB} &= \omega^2 M_A \Delta_{AB} \\ &= \frac{1500}{32.2} \times 1{\cdot}050 \times 10^{-5}\omega^2 \\ &= {\cdot}0004891\omega^2 \end{aligned}$$

and

$$\begin{aligned} R_{BA} &= \omega^2 M_B \Delta_{BA} = \frac{M_B}{M_A} R_{AB} \\ &= {\cdot}0007826\omega^2 \end{aligned}$$

Substituting in (4)

$$(\cdot 0008264\omega^2 - 1)\,(\cdot 0007677\omega^2 - 1) - \cdot 0004891 \times \cdot 0007826\omega^4 = 0$$

or $$2{\cdot}518 \times 10^{-7}\omega^4 - 1{\cdot}594 \times 10^{-3}\omega^2 + 1 = 0,$$

the roots of which are $\omega_1{}^2 = 704{\cdot}9$ and $\omega_2{}^2 = 5626$, giving $\omega_1 = 26{\cdot}55$ radians per second and $\omega_2 = 74{\cdot}51$ radians per second. In revolutions per minute the critical speeds are $N_1 = \dfrac{30\omega_1}{\pi} = 253$ and $N_2 = \dfrac{30\omega_2}{\pi} = 712$.

219. Torsional Oscillations of a Light Shaft Carrying N Masses. The shaft and the masses are shown in Fig. 280. Starting from the left-hand mass the moments of inertia of the masses

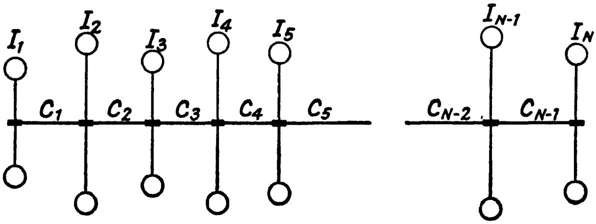


FIG. 280. TORSIONAL OSCILLATIONS

about the axis of the shaft are I_1, I_2, I_3, etc., engineers' units, respectively. Let C_1, C_2, C_3, etc., be the couples in lb.-ft. to produce unit twist in the portions of the shaft into which it is divided by the masses (Art. 50). The masses between the fifth and the $(N - 1)$th are not shown. We shall consider the normal modes of oscillation in which all the masses oscillate with simple harmonic motion of the same period and all occupy positions of maximum angular displacement at the same instant. The oscillatory motion is superposed on the steady motion of the shaft as a whole. Let the maximum angular displacements of the masses be θ_1, θ_2, θ_3, etc., radians respectively, considered positive when the sense of the displacement is the same as that of θ_1, and negative otherwise. From Art. 50 we see that the angular displacement θ radians of the first mass at time t seconds is $\theta = \theta_1 \cos (nt + \alpha)$ where n and α are constants. The time of an oscillation is $\dfrac{2\pi}{n}$ seconds (see III.26) and the frequency of oscillation is therefore $\dfrac{60n}{2\pi} = \dfrac{30n}{\pi}$ per minute. The angular acceleration is $\frac{d^2\theta}{dt^2}$ which is equal to $-n^2\theta$. When θ has its maximum value θ_1 the magnitude of the angular acceleration is $n^2\theta_1$. Now considering the system when all the masses are in position of maximum displacement, the angular accelerations of the masses are, respectively, $n^2\theta_1$, $n^2\theta_2$, $n^2\theta_3$, etc. The couples in the portions into which the shaft is divided by the masses are respectively $C_1(\theta_1 - \theta_2)$, $C_2(\theta_2 - \theta_3)$, $C_3(\theta_3 - \theta_4)$, etc. The couple in any portion of the shaft produces the algebraic sum of the angular accelerations of all the masses on one side of it. Considering the mass on the extreme left we have, therefore,

$$C_1(\theta_1 - \theta_2) = n^2 I_1 \theta_1 \quad . \quad . \quad \text{(XIV.44)}$$

or,

$$\left(\frac{n^2 I_1}{C_1} - 1\right)\theta_1 + \theta_2 = 0 \quad . \quad . \quad . \quad \text{(XIV.45)}$$

Next considering the first and second masses from the left

$$C_2(\theta_2 - \theta_3) = n^2 I_1 \theta_1 + n^2 I_2 \theta_2 \quad . \quad . \quad \text{(XIV.46)}$$

or,

$$\frac{n^2 I_1}{C_2}\theta_1 + \left(\frac{n^2 I_2}{C_2} - 1\right)\theta_2 + \theta_3 = 0 \quad . \quad \text{(XIV.47)}$$

Again, considering the first, second, and third masses from the left we have after simplification

$$\frac{n^2 I_1}{C_3}\theta_1 + \frac{n^2 I_2}{C_3}\theta_2 + \left(\frac{n^2 I_3}{C_3} - 1\right)\theta_3 + \theta_4 = 0 \quad . \quad \text{(XIV.48)}$$

Thus we proceed, obtaining an additional equation for each additional mass. As there is no couple to the right of the last mass the Nth equation is $\Sigma n^2 I\theta = 0$, the summation extending over all the masses. Dividing through by n^2

$$I_1\theta_1 + I_2\theta_2 + I_3\theta_3 + \ . \ . \ . + I_N\theta_N = 0 \quad . \quad \text{(XIV.49)}$$

In these equations there are $N + 1$ unknowns, the N values of θ and n^2. From the equations, linear in θ_1, θ_2, θ_3, etc., the N variables θ_1, θ_2, θ_3, etc., can be eliminated as, owing to all the terms on the right-hand side being zero, only $N - 1$ ratios are involved. The method of elimination by a determinant is convenient and is used in the example solved below. The eliminant obtained will form an equation of the $(N - 1)$th degree in n^2. Let n_1^2, n_2^2, n_3^2, etc., be the roots of this equation, arranged in increasing order of magnitude. Then n_1, n_2, n_3, etc., are, respectively, the first, second, third, etc., critical frequencies. In order to determine the positions of the nodes due to any particular critical frequency, the appropriate value of n^2 is substituted in all but the last of our equations. The ratios of the angular displacement are sufficient to enable us to determine the nodes and, therefore, as it is not possible to determine the absolute values of the displacements, we assume that $\theta_1 = 1$ radian. We then calculate θ_2, θ_3, etc. The last equation, $\Sigma I\theta = 0$, provides a convenient check on the correctness of the calculated values of θ_1, θ_2, etc. The values of θ_1, θ_2, θ_3, etc., are plotted vertically, against actual distances along the shaft plotted horizontally. As the shaft is assumed to be uniform and without mass the angular displacement is a linear function of the distance along the shaft. We do not assume that the portions of the shaft have all the same uniform diameter, but only that each portion is uniform. If any portion is of variable diameter it is assumed to be replaced by an equivalent uniform shaft of the same stiffness. The angular displacement graph is therefore completed by joining each pair of adjacent points by a straight line. The nodes occur at the points where the graph crosses the axis of the shaft.

EXAMPLE.

Find the critical frequencies of torsional oscillations of the following system. Three rotating masses of 1,000 lb., 1,200 lb., and 1,400 lb. respectively and radii of gyration 30 in., 24 in., and 30 in. The connecting shafts are in order 30 in. long and 3·5 in. diameter and 24 in. long and 3 in. diameter. Find the positions of the nodes. Modulus of rigidity = 12 × 10⁶ lb. per sq. in.

Using the notation of Fig. 280 we have

$$I_1 = \frac{1000}{32{\cdot}2} \times \frac{25}{4} = 194{\cdot}1 \text{ engineers' units}$$

$$I_2 = \frac{1200}{32{\cdot}2} \times 4 = 148{\cdot}4 \text{ engineers' units}$$

$$I_3 = \frac{1400}{32{\cdot}2} \times \frac{25}{4} = 271{\cdot}7 \text{ engineers' units}$$

Using (III.28) we find

$$C_1 = 4{\cdot}912 \times 10^5 \text{ lb.-ft. and } C_2 = 3{\cdot}313 \times 10^5 \text{ lb.-ft.}$$

Substituting in (XIV.45), (XIV.47), and (XIV.49) we have

$$\left(\frac{194{\cdot}1n^2}{491{,}200} - 1\right)\theta_1 + \theta_2 = 0 \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{194{\cdot}1n^2}{331{,}300}\,\theta_1 + \left(\frac{148{\cdot}4n^2}{331{,}300} - 1\right)\theta_2 + \theta_3 = 0 \quad . \quad . \quad (2)$$

$$194{\cdot}1\theta_1 + 148{\cdot}4\theta_2 + 271{\cdot}7\theta_3 = 0 \quad . \quad . \quad (3)$$

Or simplifying

$$({\cdot}0003951n^2 - 1)\theta_1 + \theta_2 = 0 \quad . \quad (4)$$

$${\cdot}0005859n^2\theta_1 + ({\cdot}0004499n^2 - 1)\theta_2 + \theta_3 = 0 \quad . \quad (5)$$

$$194{\cdot}1\theta_1 + 148{\cdot}4\theta_2 + 271{\cdot}7\theta_3 = 0 \quad . \quad (6)$$

Eliminating θ_1, θ_2, θ_3 from these we have

$$\begin{vmatrix} ({\cdot}0003951n^2 - 1) & 1 & 0 \\ {\cdot}0005859n^2 & ({\cdot}0004499n^2 - 1) & 1 \\ 194{\cdot}1 & 148{\cdot}4 & 271{\cdot}7 \end{vmatrix} = 0 \quad . \quad (7)$$

from which

$$271{\cdot}7\,({\cdot}0003951n^2 - 1)\,({\cdot}0004499n^2 - 1) - {\cdot}1592n^2$$
$$- 148{\cdot}4\,({\cdot}0003951n^2 - 1) + 194{\cdot}1 = 0$$

or, $4{\cdot}809 \times 10^{-5}n^4 - {\cdot}4461n^2 + 614{\cdot}2 = 0$

The roots of this are $n_1^2 = 1{,}677$ and $n_2^2 = 7{,}624$.

The first critical speed is $N_1 = \frac{30n_1}{\pi} = 391$ p.m. and the second critical speed is $N_2 = \frac{30}{\pi}n_2 = 834$ p.m.

In order to find the positions of the nodes for any critical speed we substitute the appropriate value of n^2 in equations (4) and (5), and, putting $\theta_1 = 1$, solve for θ_2 and θ_3 as below. Equation (6) provides a convenient check on the correctness of the results which should approximately satisfy the equation. Solving for θ_2 and θ_3 we have

From (4), $\theta_2 = 1 - {\cdot}0003952n^2$

,, (5), $\theta_3 = (1 - {\cdot}0004476n^2)\theta_2 - {\cdot}0005858n^2$

At the first critical speed $n^2 = 1677$, hence

$$\theta_2 = 1 - {\cdot}0003952 \times 1677 = 0{\cdot}3382 \text{ radians}$$

and $\theta_3 = 0{\cdot}3372\,(1 - {\cdot}0004476 \times 1677) - {\cdot}0005858 \times 1677$
$= -\,0{\cdot}897$

These calculations are better done by tabulation, thus—

(1)	(2)	(3)	(4)	(5)	(6)	(7)
I	In^2	θ	$In^2\theta$	$\Sigma In^2\theta$	C	$\theta_n - \theta_{n-1} = \Delta\theta$ $= \Sigma In^2\theta \div C$
194·1	$3{\cdot}251 \times 10^5$	1·000	$3{\cdot}251 \times 10^5$	$3{\cdot}251 \times 10^5$	$4{\cdot}912 \times 10^5$	0·6618
148·4	$2{\cdot}489 \times 10^5$	0·3382	$0{\cdot}842 \times 10^5$	$4{\cdot}093 \times 10^5$	$3{\cdot}313 \times 10^5$	1·235
271·7	$4{\cdot}556 \times 10^5$	−0·897	$-4{\cdot}087 \times 10^5$	$0{\cdot}006 \times 10^5$		

By equation (3) $\Sigma In^2\theta$ should be zero. It is $\cdot006 \times 10^5$ which is practically zero.

The values of θ for the different masses are plotted in Fig. 281.

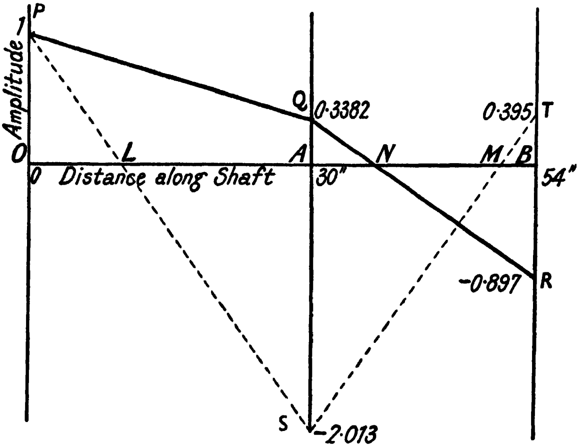


FIG. 281. AMPLITUDES OF TORSIONAL OSCILLATIONS

N is the node, and by measurement (or by calculation) $AN = 3{\cdot}28$ inches.

At the second critical frequency $n^2 = 7624$. As before, θ_1 is taken as being one radian and θ_2 and θ_3 are calculated from the equations (1) and (2). We shall use the tabular method—

I	In^2	θ	$In^2\theta$	$\Sigma In^2\theta$	C	$\Delta\theta$
194·1	$1{\cdot}480 \times 10^6$	1·000	$1{\cdot}480 \times 10^6$	$1{\cdot}480 \times 10^6$	$4{\cdot}912 \times 10^5$	3·013
148·4	$1{\cdot}132 \times 10^6$	−2·013	$-2{\cdot}278 \times 10^6$	$-0{\cdot}798 \times 10^6$	$3{\cdot}313 \times 10^5$	−2·408
271·7	$2{\cdot}071 \times 10^6$	0·395	$0{\cdot}8180 \times 10^6$	$0{\cdot}02 \times 10^6$		

Here $\Sigma In^2\theta = \cdot02 \times 10^6$ which is again negligible compared with the separate values of $In^2\theta$.

These values of θ are also plotted in Fig. 281, the graph being dotted. L and M are the nodes and by measurement (or calculation) $OL = 9{\cdot}96$ inches and $AM = 20{\cdot}13$ inches. PQR and PST are called the *elastic lines*.

When dealing with cases of four or more oscillating masses much labour is usually necessary to solve the system of equations for the values of n^2. Once n^2 is known, however, the above method of tabulation simplifies greatly the process of determining the values of θ and the positions of the nodes. In practical cases the designer will have some idea of the actual critical speeds and does not, therefore, need to solve the system of equations. He takes a probable value of n^2 and tabulates the values corresponding to those given in the above tables. If the final value in the $\Sigma In^2\theta$ column is practically zero the assumed value of n is the correct one. Otherwise he adjusts the value of n^2 and tabulates again. After two or three attempts the values of $\Sigma In^2\theta$ may be plotted against values of n^2 and the value of the latter found which makes the former zero. Using the above methods, critical speeds may be determined with an error of not more than 3 per cent. When one or more sections of the shaft are held rigidly, or prevented from oscillating, as the two ends in the example below, the above method of finding critical frequencies will apply if the shaft is assumed to carry a mass of infinite moment of inertia at each of the sections which cannot oscillate.

EXAMPLE.

The shaft of the previous example has its ends fixed. Find the critical frequency of torsional oscillations.

The masses I_1 and I_3 are assumed to be infinite and θ_1 and θ_3 to be zero. Thus in the equations (1) and (2) of the last example we put $\theta_3 = 0$ and $\theta_1 = 0$ except where θ_1 is multiplied by I_1 for $I_1\theta_1$ is not necessarily zero. Equations (1) and (2) become

$$\frac{n^2}{C_1}I_1\theta_1 + \theta_2 = 0 \quad . \quad . \quad . \quad (1)$$

$$\frac{n^2}{C_2}I_1\theta_1 + \left(\frac{n^2}{C_2}I_1 - 1\right)\theta_2 = 0 \quad . \quad . \quad . \quad (2)$$

We eliminate $I_1\theta_1$ and θ_2 from these, obtaining

$$\begin{vmatrix} \frac{n^2}{C_1} & 1 \\ \frac{n^2}{C_2} & \frac{n^2}{C_2}I_1 - 1 \end{vmatrix} = 0$$

from which dividing through by n^2 and simplifying

$$\frac{1}{C_1}\left(\frac{n^2}{C_2}I_1 - 1\right) = \frac{1}{C_2}$$

or

$$\frac{n^2 I_1}{C_2} = \frac{C_1}{C_2} + 1$$

$\therefore$

$$n^2 = \frac{C_1 + C_2}{I_1}$$

and

$$N = \frac{30}{\pi}\sqrt{\frac{C_1 + C_2}{I_2}} \quad . \quad \text{(XIV.50)}$$

This result is obvious as, if the ends are fixed, $C_1 + C_2$ is the couple acting on I_2 for an angular displacement of one radian.

Substituting the values of C_1, C_2, and I_2, we have

$$N = \frac{30}{\pi}\sqrt{\frac{491{,}200 + 331{,}300}{148{\cdot}4}}$$
$$= 710{\cdot}9$$

The critical frequency is 711 per minute.

If a mass is mounted on a shaft at the position of a node it will not affect the value of the critical frequency, and the mass itself will not oscillate. This shows the possibility of arranging some of the wheels and other rotating masses in, and connected with, a reduction gear, so that the oscillations which take place will not interfere with the gearing of the wheel teeth. Such an arrangement is known as a *nodal drive.*

EXAMPLE.

The propeller and engine flywheel of a ship have moments of inertia of 70 and 90 tons-ft.^{2} respectively and are mounted 40 ft. apart on a hollow shaft 15 in. outside and 12 in. inside diameter.

Calculate the frequency of torsional oscillations of the shaft.

In the above arrangement the flywheel is between the propeller and the engine which is 58 ft. from the propeller. In order to lessen vibration the size of the flywheel is reduced and a second flywheel is mounted on the shaft 68 ft. from the propeller. By this means one of the two nodes of the oscillation due to the three masses occurs at the engine itself, and the frequency of the oscillation is 1900 per minute.

Calculate the moments of inertia of the flywheels. Neglect the mass of the shaft and take G for the shaft as 12×10^6 lb. per sq. in. (U.L.A.)

To solve the first part of this problem we use (III.36)

$$N = \frac{30}{\pi}\sqrt{\frac{C_{AB}(I_A + I_B)}{I_A I_B}} \text{ oscillations per minute.}$$

Now $C_{AB} = \dfrac{\pi G}{24l}(R^4 - r^4)$, from (III. 28) adapting it to the case of a hollow shaft

$$\therefore \quad C_{AB} = \frac{\pi \times 12 \times 10^6}{24 \times 480}(7{\cdot}5^4 - 6^4)$$

$$= 6{,}125{,}000 \text{ lb.-ft.}$$

also

$$\frac{I_A + I_B}{I_A I_B} = \frac{160 \times 32{\cdot}2}{70 \times 90 \times 2240}$$

$$= {\cdot}0003652$$

$$\therefore \quad N = \frac{30}{\pi}\sqrt{6{,}125 \times 0{\cdot}3652}$$

$$= 452 \text{ oscillations per minute,}$$

which is the frequency of torsional oscillations.

The arrangement of part two is shown in Fig. 282.

A is the propeller of moment of inertia $I = \dfrac{70 \times 2240}{32{\cdot}2}$ engineers' units. *B* and *D* are the flywheels whose moments of inertia are I_2 and I_3 respectively. *C* is the position of the engine and is one node.

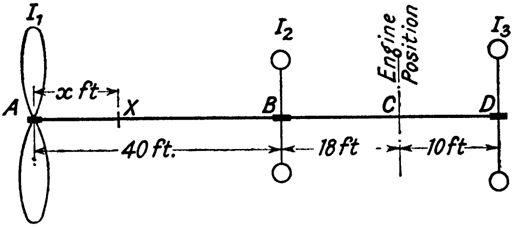


FIG. 282. CRITICAL SPEED OF PROPELLER SHAFT

Let *X* be the position of the other node where *AX* is *x* ft. long,

$C_{AB} = 6{,}125{,}000$ lb.-ft. $C_{BC} = \dfrac{40}{18} \times C_{AB} = 13{,}610{,}000$ lb.-ft.,

and $C_{CD} = 4C_{AB} = 24{,}500{,}000$ lb.-ft.

Applying (III.27) to the part *CD*, we have

$$N = \frac{30}{\pi}\sqrt{\frac{C_{CD}}{I_3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore \qquad 1900 = \frac{30}{\pi}\sqrt{\frac{24{,}500{,}000}{I_3}}$$

and
$$I_3 = \left(\frac{900}{1900\pi}\right)^2 \times 24{,}500{,}000 \text{ engineers' units}$$
$$= \frac{9 \times 24{,}500{,}000 \times 32{\cdot}2}{36{,}100\pi^2 \times 2240} \text{ tons-ft.}^2$$
$$= 8{\cdot}89 \text{ tons-ft.}^2$$

Now, considering the left-hand mass and the shaft AX, we have

$$N = \frac{30}{\pi}\sqrt{\frac{C_{AX}}{I_1}} \qquad . \quad . \quad . \quad . \quad . \qquad (2)$$

From (1) and (2)

$$\frac{C_{AX}}{I_1} = \frac{C_{CD}}{I_3}$$

Also by (III.29),

$$C_{AX} = \frac{c}{\overline{AX}}$$

and $C_{CD} = \dfrac{c}{\overline{CD}}$ the shaft being uniform,

hence, $I_1 . \overline{AX} = I_3\overline{CD}$

or, $70x = 10 \times 8{\cdot}89$

from which $x = 1{\cdot}27$ ft.

Consider next the mass at B. XC may be treated as a shaft fixed at X and C as in the previous example. The couple corresponding to an angular displacement of one radian is the sum of the couples required to produce unit twist in each of the portions of the shaft XB and BC. Let C_B be this couple, then

$$C_B = C_{XB} + C_{BC}$$
$$= \frac{n}{\overline{XB}} + \frac{n}{\overline{BC}}$$
$$= n\left(\frac{1}{38{\cdot}73} + \frac{1}{18}\right)$$
$$= \frac{n}{12{\cdot}28}$$

Now $$N = \frac{30}{\pi}\sqrt{\frac{C_B}{I_2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and from (2) and (3)

$$\frac{C_{AX}}{I_1} = \frac{C_B}{I_2}$$

or, $$I_1 . x = 12{\cdot}28 I_2$$

$$\therefore \quad I_2 = \frac{1{\cdot}27}{12{\cdot}28} \times 70$$

$$= 7{\cdot}26 \text{ tons-ft.}^2$$

220. **Equivalent Systems in Torsional Oscillation. Geared Systems.** Two torsional systems are *equivalent* when they have the same critical frequencies of oscillation. If one system has more rotors, and therefore more modes of oscillation than the other, we still consider them equivalent for the lower frequencies, even though the former system has one or more critical frequencies beyond the highest critical frequency of the latter.

In an example below we will see how this limited equivalence is made use of when reducing the system to one of three-rotors in finding the fundamental frequency.

With the notation of Art. 50, two single-rotor systems, on shafts of the same diameter and of the same material, are equivalent, if Il is the same in both; two double-rotor systems are equivalent if $\frac{I_1 I_2 l}{I_1 + I_2}$ is the same in both.

When dealing with a system connected by gearing, we replace it by an equivalent system in which there is no gearing. The upper figure (Fig. 283) shows a two-shaft system with two rotors in which the shafts are connected by gearing. AB and CD are the shafts carrying the rotors whose moments of inertia are I_1 and I_2. The ratio of the gearbox is r to 1, so that the angular speed of C is r times that of B. The lower figure shows an equivalent system consisting of a single shaft carrying two rotors I_1 and I_2'. The portion of shaft EF is considered to be of the same length and diameter as the shaft AB; the portions of shaft in the gear box are considered to be rigid, and the gears without inertia. We shall assume that C_1 is the couple for unit twist in each of the shafts AB and EF, C_2 that in

CD, and C_2' that in FG, and we shall determine the values of C_2' and I_2' so that the two systems are equivalent.

Considering the geared system, we have, if θ_1 is the maximum angular displacement of I_1, and θ_2 that of I_2, when the system is at its maximum displacement

$$\text{couple in shaft } AB = I_1\theta_1 n^2$$

and

$$\text{couple in shaft } CD = I_2\theta_2 n^2$$

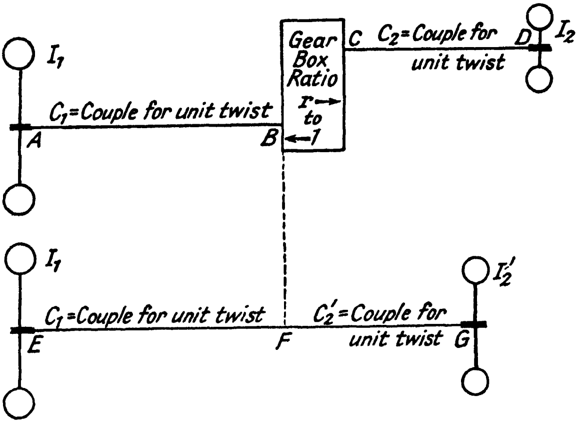

I_1
Gear Box Ratio r to 1
C
C_2=Couple for unit twist
D
I_2
C_1=Couple for unit twist
A
B
I_1
I_2'
C_1=Couple for unit twist
C_2'=Couple for unit twist
E
F
G

FIG. 283. TORSIONAL OSCILLATIONS OF GEARED SHAFTS

By the principle of work, the couple at B is r times that at C

$$\therefore\ I_1\theta_1 n^2 = rI_2\theta_2 n^2$$

or

$$I_1\theta_1 = rI_2\theta_2 \quad . \quad . \quad . \quad \text{(XIV.51)}$$

The angular displacement of the shaft AB at B is $\theta_1 - \frac{I_1\theta_1 n^2}{C_1}$, and that of C in the same sense, is $-\theta_2 + \frac{I_2\theta_2 n^2}{C_2}$, and since the gear ratio is r, the latter displacement is r times the former, hence,

$$r\left(\theta_1 - \frac{I_1\theta_1 n^2}{C_1}\right) = -\theta_2 + \frac{I_2\theta_2 n^2}{C_2} \quad . \quad \text{(XIV.52)}$$

Dividing (XIV.52) by (XIV.51) and dividing through by r, we have

$$\frac{1}{I_1} - \frac{n^2}{C_1} = -\frac{1}{I_2 r^2} + \frac{n^2}{C_2 r^2} \quad . \quad \text{(XIV.53)}$$

If we put $r = 1$, $I_2 = I_2'$, $C_2 = C_2'$, the relation applies to the system in the lower figure,

hence

$$\frac{1}{I_1} - \frac{n^2}{C_1} = -\frac{1}{I_2'} + \frac{n^2}{C_2'} \quad . \quad . \quad \text{(XIV.54)}$$

Since the two systems are equivalent, and the left-hand sides of the last two relations are identical, the right-hand sides must be equal to one another.

If we make $I_2' = I_2 r^2$ and $C_2' = C_2 r^2$, the right-hand sides are equal, and the lower system is equivalent to the upper system. We can, therefore, replace any system (e.g. the upper) by a non-geared system, if we refer it to the first shaft AB, by increasing the stiffness of the shaft CD in the ratio r^2 to 1, and the moment of inertia I_2 at the end of it in the same ratio.

If the gears in the gear box are not massless, the upper system reduces to a three-mass system with the arrangement of the lower system, but with a third rotor at F whose moment of inertia is $\Sigma I r'^2$, where I is the moment of inertia of a wheel and r' its angular speed ratio to that of AB, the summation extending over all the rotating masses in the gear box.

EXAMPLE.

The values of I_1, I_2 and C_1 and C_2 in the geared system are 100 and 120 engineers' units, and 4×10^5 and $3{\cdot}5 \times 10^5$ lb.-ft. respectively. The gear ratio is 2 : 1, the end of the left-hand shaft being the slower moving. Find the frequency of oscillation.

The system reduces to the two-mass system shown in the lower figure, in which $I_1 = 100$, $I_2' = 480$, $C_1 = 4 \times 10^5$, and $C_2' = 14 \times 10^5$. We have now to find the single couple C_e for unit twist in the combined shaft EF, FG. When C_e is applied to the shaft the difference between the angular displacement of the ends is one radian. The angle of twist of EF is $\frac{C_e}{C_1}$ and that of FG is $\frac{C_e}{C_2'}$, hence

$$\frac{C_e}{C_1} + \frac{C_e}{C_2'} = 1, \text{ or } C_e = \frac{C_1 C_2'}{C_1 + C_2'}$$

Substituting the values, we have

$$C_e = \frac{56 \times 10^{10}}{18 \times 10^5} = 3{\cdot}111 \times 10^5 \text{ lb.-ft.}$$

Hence by (111·36)

$$N = \frac{30}{\pi}\sqrt{\frac{3{\cdot}111 \times 10^5 \times 580}{48{,}000}} \text{ is the frequency}$$

$$\therefore\ N = \frac{30}{\pi}\sqrt{3759}$$

$\therefore$ Frequency = 585 per minute.

EXAMPLE.

Find the fundamental frequency of a system of five rotors which is derived from the system of Fig. 281, with the following alterations. The right-hand rotor has its moment of inertia reduced by 70 engineers' units, and two additional rotors are introduced on opposite sides of it and 6 in. from it, the right-hand shaft being extended by 6 in.; the additional rotors have each a moment of inertia of 35 engineers' units.

Replace the system by an equivalent three-mass system to obtain a first approximation to the fundamental frequency, and then find a closer approximation by the method of tabulation.

For the equivalent three-mass system we combine the three rotors at the right-hand end, thus obtaining a single rotor of moment of inertia 271·7 engineers' units in the position of the large rotor. The system is then the three-mass system of Fig. 281, and the frequency has been found to be 391 per minute, at which frequency n^2 has the value 1677. We tabulate below for this value of n^2 in the upper table.

I	$In^2 \times 10^{-5}$	θ	$In^2\theta \times 10^{-5}$	$\Sigma In^2\theta \times 10^{-5}$	$C \times 10^{-5}$	$\theta_n - \theta_{n+1} = \frac{\Sigma In^2\theta}{C}$
194·1	3·255	1·0000	3·255	3·255	4·912	0·6625
148·4	2·489	0 3375	0·8399	4·095	4·417	0·9270
35 0	0·587	− 0·5895	0·3461	3 749	13·25	0·2830
201·7	3·382	− 0·8725	− 0·2950	0·799	13·25	0·0603
35·0	0·587	− 0·9328	− 0·5475	0·251		
194·1	3·281	1 0000	3·281	3·281	4·912	0 6679
148·4	2·507	0·3321	0·8236	4·114	4·417	0·9313
35·0	0·5915	− 0·5992	− 1·354	3·760	13·25	0·2838
201·7	3·416	− 0 8830	− 3·016	0·744	13·25	0·5615
35·0	0·5919	− 1·445	− 0·8547	− 0·111		

The residue, i.e. the last entry in the fifth column, is not negligible, and since the residue is positive we see that an appropriate increase in the value of I_5 would make it zero. But increasing I_5 reduces the critical speed, hence we have assumed too low a value of n^2.

In the lower table we have taken $n^2 = 1690$, and in this case the residue is negative. The correct value of n^2 is between 1677 and 1690, and by the method of proportional parts we find as a closer approximation

$$\begin{aligned} n^2 &= 1677 + \frac{0{\cdot}251}{0{\cdot}251 + 0{\cdot}111} \times (1690 - 1677) \\ &= 1677 + 6 = 1683 \end{aligned}$$

The frequency is therefore $\frac{30}{\pi}\sqrt{1683} = 392$ per minute.

221. **Further Considerations of Oscillations.** Consider a mass of $\frac{W}{g}$ engineers' units oscillating along a straight path under an elastic constraint, and suppose that this constraint opposes the motion a force of magnitude Ex lb., where x ft. is the displacement of the mass from its equilibrium position, and E is the stiffness of the constraint, i.e. the restoring force in lb. per foot of displacement. Let t be the time in seconds.

If there is an external periodic disturbing force, $P \cos \omega t$, acting on the mass and a viscous resistance proportional to the velocity of amount $f\frac{dx}{dt}$, the equation of motion is

$$P \cos \omega t - Ex - f\frac{dx}{dt} = \frac{W}{g} \cdot \frac{d^2x}{dt^2}$$

which simplifies to

$$\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + n^2x = n^2x_0 \cos \omega t \qquad \text{(XIV.55)}$$

where $a = \frac{fg}{2W}$, $n^2 = \frac{Eg}{W}$, $x_0 = \frac{P}{E}$ = static deflection due to P.

If there is no external disturbing force $x_0 = P = 0$, and the equation of motion is

$$\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + n^2x = 0 \quad . \quad . \quad \text{(XIV.56)}$$

In this case the motion is said to be *free*, while if x_0 is not zero, the motion is said to be *forced.*

Free Oscillations. The solution of (XIV 56) depends upon the nature of the quantity $a^2 - n^2$. a^2 and n^2 are, of course, essentially positive quantities.

(i) If $a^2 > n^2$, the solution is

$$x = Ae^{\{-a + \sqrt{(a^2-n^2)}\}t} + Be^{\{-a - \sqrt{(a^2-n^2)}\}t} \quad \text{(XIV.57}a\text{)}$$

(ii) If $a^2 = n^2$, the solution is

$$x = (A + Bt)e^{-at} \quad . \quad . \quad . \quad \text{(XIV.57}b\text{)}$$

(iii) If $a^2 < n^2$, the solution is

$$x = Ae^{-at} \cos\{\sqrt{(n^2 - a^2)}t + \alpha\} \quad . \quad \text{(XIV.57}c\text{)}$$

where A, B, and α are arbitrary constants.

In cases (i) and (ii), as t increases, the value of x after passing once, at most, through the value $x = 0$, steadily settles down, continually approaching to that value. In case (iii) we see that x is the product of the decaying function Ae^{-at} and the periodic function $\cos\left\{\sqrt{(n^2 - a^2)}t + \alpha\right\}$. If $f = 0$, and consequently $a = 0$, the solution is $x = A \cos (nt + \alpha)$ and the motion is simple harmonic. If f is not zero, the motion is oscillatory with decreasing amplitude Ae^{-at}. The motion in cases (i) and (ii) is known as *dead-beat* motion; that in (iii) as *damped oscillations*. (ii) is the division or critical case between (i) and (iii) and occurs when $f = \sqrt{\dfrac{4WE}{g}}$. If f is greater than this, dead-beat motion occurs, whilst if it is less, damped oscillations occur. When f has its critical value, the damping is known as *critical damping.*

Just as simple harmonic motion can be considered as the projection of uniform circular motion (e.g. in Art. 45), so the damped oscillations given by (XIV.57c) can be looked upon as the projection of motion on a logarithmic spiral. If we imagine the radius vector of the spiral $r = Ae^{-\frac{a\theta}{\sqrt{(n^2-a^2)}}}$ to rotate with angular velocity $\sqrt{(n^2 - a^2)}$ radians per second about the origin, the projection of the outer end of the vector on to any straight line will move in accordance with (XIV.57c). Since all loops of the logarithmic spiral are similar, and are similarly situated with respect to the origin, each oscillation will differ from the preceding one only in its extent, and the time taken for any fraction of the displacement from the position of rest to be covered is independent of the amplitude. The periodic time is, therefore, most simply calculated by finding the time between two successive occasions on which $x = 0$.

From (XIV.57c) this time of oscillation is

$$t_D = \frac{2\pi}{\sqrt{(n^2 - a^2)}} \quad . \quad . \quad . \qquad \text{(XIV.58}a\text{)}$$

If the motion is undamped f and a are zero, and the time of oscillation is

$$t_0 = \frac{2\pi}{n} \quad . \quad . \quad . \quad . \quad . \qquad \text{(XIV.58}b\text{)}$$

$$\therefore \frac{t_0}{t_D} = \sqrt{1 - \left(\frac{a}{n}\right)^2} \quad . \quad . \quad . \quad . \qquad \text{(XIV.59}a\text{)}$$

or $$\frac{t_0}{t_D} = \sqrt{\left(1 - \frac{f^2 g}{4EW}\right)} \quad . \quad . \quad . \quad . \quad (XIV.59b)$$

The ratio of the frequencies is the reciprocal of this.

From this we see again that critical damping occurs when

$$f = 2\sqrt{\frac{EW}{g}} \quad . \quad . \quad . \quad . \quad . \quad (XIV.60)$$

The effect on the amplitude of increasing the time by the periodic time t_D is to change it from Ae^{-at} to $Ae^{-a(t+t_D)}$, that is, to multiply it by the constant fraction $K = e^{-\frac{2\pi a}{\sqrt{(n^2-a^2)}}}$.

Successive values of the maximum displacements, in the same sense, form a geometrical series whose common ratio is K. The hyperbolic logarithm of $\frac{1}{K}$ is $\frac{2\pi a}{\sqrt{(n^2 - a^2)}}$ and this constant is called the *logarithmic decrement.*

Forced Oscillations with Fluid Friction. When an external periodic force, $P \cos \omega t$, is applied, the conditions are given by equation (XIV.55). If X is a function of t, which satisfies this it is known as a *particular integral,* and the full solution is found by adding X to the right-hand side of the appropriate solution of (XIV.56). The value of X is given by

$$X = \frac{n^2 x_0}{D^2 + 2aD + n^2} \cos \omega t \text{ where } D = d/dt.$$

Hence putting $D^2 = -\omega^2$

$$X = \frac{n^2 x_0}{(n^2 - \omega^2) + 2aD} \cos \omega t = \frac{n^2 x_0 [(n^2 - \omega^2) - 2aD]}{(n^2 - \omega^2)^2 - 4a^2 D^2} \cos \omega t$$

$$= \frac{n^2 x_0}{(n^2 - \omega^2)^2 + 4a^2\omega^2} \{(n^2 - \omega^2) \cos \omega t + 2a\omega \sin \omega t\}$$

$$X = \frac{n^2 x_0}{\sqrt{\{(n^2 - \omega^2)^2 + 4a^2\omega^2\}}} \cos (\omega t - \phi) \quad (XIV.61)$$

where $\tan \phi = \frac{2a\omega}{n^2 - \omega^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (XIV.62)$

If $f = 0$ and, therefore, $a = 0$, these reduce to

$$X = \frac{n^2 x_0}{n^2 - \omega^2} \cos \omega t \quad . \quad . \quad . \quad . \quad . \quad (XIV.63)$$

In all cases the free oscillations will be damped out after an interval of time, and the motion will then be represented by (XIV.61), in which X is now the total displacement of the mass.

$x_0 = \frac{P}{E}$ is called the amplitude of the forced vibration. It is the displacement which the maximum disturbing force would produce at its point of application. From (XIV.61) the amplitude A of the forced vibration is

$$A = \frac{n^2 x_0}{\sqrt{\{(n^2 - \omega^2)^2 + 4a^2\omega^2\}}}$$

i.e.

$$A = \frac{x_0}{\sqrt{\left\{\left(1 - \left(\frac{\omega}{n}\right)^2\right)^2 + 4\left(\frac{\omega}{n}\right)^2\left(\frac{a}{n}\right)^2\right\}}} \quad \text{(XIV.64)}$$

ω is 2π times the frequency of the forcing vibrations. n is 2π times the frequency of the free vibrations; $a = \frac{gf}{2W}$, where f is the frictional resistance in pounds at a speed of 1 ft. per second.

This expression is not so complicated as it appears, and some useful deductions may be drawn from it. If $f = 0$, i.e. there is no friction, then $a = 0$ and $A = \frac{n^2 x_0}{n^2 - \omega^2}$, which agrees with (III.38).

If $\frac{\omega}{n} = 1$, i.e. synchronism occurs, and f is not zero, the value of $\frac{A}{x_0}$ does not tend to the value infinity as in (III.38), but is given by $\frac{A}{x_0} = \frac{n}{2a}$.

The vibration of W will not be in phase with the forcing vibration but will lag behind it, the amount of lag depending upon the relative values of ω, n, and a. This is best shown by Fig. 284, in which OP' represents the amplitude A of the vibration of W, and OP is the amplitude of the forcing vibration, there being a constant angle ϕ between OP and OP' such that $\tan \phi = \frac{2\omega a}{n^2 - \omega^2}$.

If the motion of the point N along KOX' represents the forcing vibration, then the motion of N' represents that of W.

This has an important application in the balancing of high-speed rotors. It is well known that if the rotor is rotated at high speed on flexible supports, the out-of-balance position of the centre of gravity is in advance of a mark made by a scriber point while the rotor is revolving, and, to obtain the true out-of-balance position, it is necessary to reverse the direction of rotation and halve the distance between the two marks.

Note that if $\omega = n$, $\phi = 90°$ and if ω is greater than n, ϕ is greater than 90°.

When ω^2 is very large compared with n^2 and a^2, (XIV.61) approximates to the form $X = \frac{n^2 x_0}{\omega^2} \cos(\omega t - \pi)$; the amplitude being small and the phase of the vibration opposite that of the disturbance.

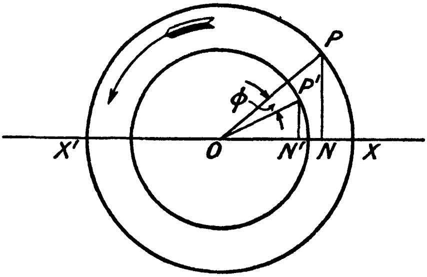


FIG. 284. ANGLE OF LAG

Equation (XIV.64) may be written

$$D = \frac{1}{\sqrt{(R^2 - 1)^2 + \left(\frac{2a}{n}\right)^2 . R^2}} \quad . \quad . \quad . \quad \text{(XIV.64}a\text{)}$$

where $D = \frac{A}{x_o}$; $R = \frac{\text{forced frequency}}{\text{synchronous frequency}}$; $\frac{2a}{n} = f\sqrt{\frac{g}{WE}}$.

Let $\frac{2a}{n} = k = \frac{1}{D_s}$, where D_s is the value of D at synchronism ($R = 1$).

Then *D is a maximum* when $(R^2 - 1)^2 + k^2 . R^2$ is a minimum,

i.e., when $A(R^2 - 1) . R + 2k^2 . R = 0$

$$R = \sqrt{1 - \frac{k^2}{2}} \quad . \quad . \quad . \quad . \quad \text{(XIV.64}b\text{)}$$

Thus with damped oscillations the maximum value of A occurs at a speed below synchronism.

Substituting for R in (XIV.64*a*), then

$$D_{max.} = \frac{1}{\sqrt{k^2\left(1 - \frac{k^2}{4}\right)}} \quad . \quad . \quad \text{(XIV.64c)}$$

EXAMPLE 1.

Given $W = 200$ lb.; $S = 48000$ lb. per ft.; at synchronism ($R = 1$), D_s is not to exceed 2. Find the necessary value of f.

Then,
$$k = \frac{1}{D_s} = 0{\cdot}5$$

$$f = k\sqrt{\frac{W \, . \, s}{g}} = 0{\cdot}5\sqrt{\frac{9{\cdot}6 \times 10^6}{32{\cdot}2}}$$

$\therefore f = 270$ lb. at 1 ft. per sec.

R for maximum value of $D = \sqrt{1 - 0{\cdot}125} = 0{\cdot}934$

$$D_{max.} = \frac{1}{\sqrt{0{\cdot}25 \times 0{\cdot}9375}} = \underline{\underline{2{\cdot}06.}}$$

If an oil dashpot is used in this case, the bye-pass valve must be so adjusted that a load of 270 lb. will cause the piston to descend at the rate of 1 ft. per sec.

EXAMPLE 2.

A vertical engine weighing 200 lb. is mounted at the mid-point of two steel floor joists which deflect $\frac{1}{20}$ in. under the load. There is an out-of-balance force on the engine which is equivalent to a mass of 0·05 lb. rotating at 0·1 ft. radius; the force acts in a vertical plane midway between the joists.

(*a*) Neglecting the mass of the joists and any damping effects, find the amplitude of vibration due to an engine speed of 800 r.p.m.

(*b*) At what speed will resonance occur?

(*c*) In order to limit the amplitude of the vibration at resonance to 0·01 in., a damping device is fitted in which the frictional force is proportional to the velocity. Calculate the damping force required in pounds per unit velocity.

$W = 200$; $P = \frac{0{\cdot}05}{g}\left(\frac{80\pi}{3}\right)^2 \times 0{\cdot}1$; $n^2x_0 = \frac{Pg}{W} = \frac{0{\cdot}05 \times 6400 \times \pi^2 \times 0{\cdot}1}{200 \times 9}$
$= 0{\cdot}1754$; $\omega = \frac{80\pi}{3} = 83{\cdot}81$; $n^2 = \frac{Eg}{W} = \frac{200 \times 240 \times 32{\cdot}2}{200} = 772$

(*a*) The equation of motion is, from (XIV.55), since $a = 0$

$$\frac{d^2x}{dt^2} + n^2x = n^2x_0 \cos \omega t$$

i.e.
$$\frac{d^2x}{dt^2} + 7728x = 0{\cdot}1754 \cos 83{\cdot}81t$$

From (XIV.63) the particular integral of this is

$$x = \frac{0{\cdot}1754}{7728 - (83{\cdot}81)^2} \cos 83{\cdot}81t$$

The amplitude of the forced oscillation is therefore

$$\frac{0{\cdot}1754}{7728 - (83{\cdot}81)^2} = 0{\cdot}000250 \text{ ft.}$$

(*b*) Resonance occurs when the denominator on the right of (XIV. 61) is zero. This is the case if $f = 0$, in which case $a = 0$ and the denominator is zero when $\omega = n$. The resonance speed is then

$$\omega = \sqrt{7728}$$

$$= 87{\cdot}9 \text{ radians per second}$$

or $N = 840$ revolutions per minute.

(*c*) At the resonance speed the disturbing force is $\frac{0{\cdot}05}{g} \times 7728 \times 0{\cdot}1 \times \cos 87{\cdot}9t$ lb., and the equation of motion is

$$\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + n^2x = n^2x_0 \cos 87{\cdot}9t$$

i.e. $$\frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + 7728x = \frac{g}{200} \times \frac{0{\cdot}05}{g} \times 772{\cdot}8 \cos 87{\cdot}9t$$

$$\therefore \frac{d^2x}{dt^2} + 2a\frac{dx}{dt} + 7728x = 0{\cdot}1932 \cos 87{\cdot}9t$$

and from (XIV.61), since $\omega = n$, the amplitude a_o of the forced oscillation is given by

$$a_o = \frac{n^2x_0}{2a\omega}$$

$$\therefore a_o = \frac{0{\cdot}1932}{2 \times 87{\cdot}9a}$$

As the amplitude is to be 0·01 in., we have

$$\frac{0{\cdot}1932}{175{\cdot}8a} = \frac{0{\cdot}01}{12}$$

from which $a = 1{\cdot}318$

But $$f = \frac{2Wa}{g}$$

$$= \frac{400 \times 1{\cdot}318}{32{\cdot}2}$$

$$= 16{\cdot}4$$

and the damping force is 16·4 lb. at a speed of 1 ft. per second.

222. Small Free Oscillations of the Masses of a Hartnell Governor. Consider the mechanism of the governor in Fig. 226 when oscillating about the position of equilibrium, assuming that all parts except the balls are massless.

Let a be increased by a small amount θ and assume that $a + \theta$ is always small, say less than 10°, in which case we may put $\sin (\theta + a) \approx \theta + a$ and $\cos (\theta + a) \approx 1$. With the notation of Art. 184 we have, from (XI.31).

$$\frac{w}{g}(m + l \,.\, a) \,.\, \omega^2 \,.\, l = \frac{p}{2}(P_0 + s \,.\, p \,.\, a).$$

When a is increased to $a + \theta$ the outward force at R is $\frac{w'}{g} \,.\, \omega^2 \,.\, \{m + l(a + \theta)\}$ and the downward force Q is $\frac{1}{2}\{P_0 + s \,.\, p(a + \theta)\}$, so that the restoring torque is

$$T = \frac{p}{2}\{P_0 + s \,.\, p(a + \theta)\} - \frac{w}{g} \,.\, \omega^2 \,.\, l\{m + l(a + \theta)\}.$$

From these two relations we have

$$T = \frac{s \,.\, p^2 \,.\, \theta}{2} - \frac{w}{g} \,.\, \omega^2 \,.\, l^2 \,.\, \theta$$

i.e.

$$T = \left(\frac{s \,.\, p^2}{2} - \frac{w}{g} \,.\, \omega^2 \,.\, l^2\right) . \, \theta$$

The moment of inertia of the mass w lb. about θ is $\frac{w}{g} \,.\, l^2$ and the equation of motion is

$$\frac{w}{g} \,.\, l^2 \,.\, \frac{d^2\theta}{dt^2} + \left(\frac{s \,.\, p^2}{2} - \frac{w}{g} \,.\, \omega^2 \,.\, l^2\right)\theta = 0,$$

the solution of which is

$$\theta = A \,.\, \sin\left(\sqrt{\frac{g \,.\, s \,.\, p^2}{2W} - \omega^2} \,.\, t + \varepsilon\right).$$

The frequency of oscillation, n per second, is given by

$$n = \frac{1}{2\pi}\sqrt{\frac{g \,.\, s}{2w} \,.\, \frac{p^2}{l^2} - \omega^2}.$$

In practice $2\pi n$ will be considerably less than ω, so that periodic torque fluctuations are unlikely to cause excessive amplitudes of oscillation. The effect of the weights of arm, sleeve, etc., will be to increase the moment of inertia and reduce n.

By the principle of conservation of angular momentum, an increase in the radius of action of w will tend to reduce ω, but since in most cases the governor is connected to revolving masses of relatively considerable moment of inertia this effect will be negligible. Other factors of considerably greater importance would be solid and fluid (dashpot) resistance and variation of driving torque with variation of sleeve movement.

A comprehensive discussion of these and other problems is contained in a paper by G. W. Higgs-Walker (*Proc. I. Mech.E.*, Vol. 149, 1942).

223. **Free and Forced Vibrations of a Multi-mass System.** The results of Art. 219 apply to axial as well as to torsional vibrations. In the former case we assume I_1, I_2, etc., in Fig. 280, to be the masses in engineers' units of the vibrating bodies, C_1, C_2, etc., the axial stiffnesses in lb. per ft. of the elastic connections, and θ_1, θ_2, etc., the amplitudes in ft. of the axial vibrations. The subject of axial vibrations of crankshafts is one of increasing importance owing to severe stresses which may occur at critical speeds of axial vibration. When the axis of the vibrating system is vertical gravity forces cause the bodies to settle into equilibrium positions about which they vibrate, so that the gravity forces can be left out of consideration. In this case, however, one of the masses must be fixed, and this corresponds to making that particular mass infinite and its amplitude of vibration zero.

The normal modes of free axial vibration are given by (XIV.44) to (XIV.49). We shall show the application of these by considering the free vibration of the two-mass system shown in Fig. 285, in which M_1 and M_2 engineers' units are the masses suspended from springs of stiffness s_1 and s_2 lb. per ft. respectively. The top end of the upper spring is fixed in space and that of the lower spring is fixed to M_1. On the right of the figure is shown the corresponding system in torsional vibration. By substituting I_1, I_2, C_1, C_2 respectively for M_1, M_2, s_1, s_2 the results we obtain will apply in the case of torsional vibration.

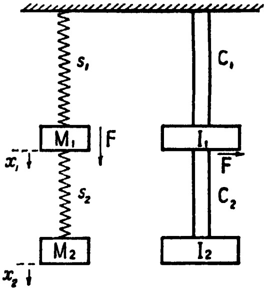


FIG. 285. TWO-MASS VIBRATING SYSTEMS

From (XIV.45) and (XIV.47) with the necessary change of notation we have, if x_1 and x_2 ft. are the amplitudes—

$$\left(\frac{n^2M_2}{s_2}-1\right)x_2+x_1=0 \quad . \quad (XIV.65)$$

$$\frac{n^2M_2}{s_1}x_2+\left(\frac{n^2M_1}{s_1}-1\right)x_1=0 \quad (XIV.66)$$

from which

$$\begin{vmatrix} \frac{n^2M_2}{s_2}-1 & 1 \\ \frac{n^2M_2}{s_1} & \frac{n^2M_1}{s_1}-1 \end{vmatrix}=0 \quad . \quad (XIV.67)$$

Let $s_1/M_1=n_1^2$, $s_2/M_2=n_2^2$ and $M_2=\mu M_1$, then n_1 and n_2 are the respective angular frequencies of free vibration of the separate vibrating systems s_1, M_1 and s_2, M_2, each with its upper end fixed.

Substituting in (XIV.67), and simplifying, we have

$$\left(\frac{n^2}{n_2^2}-1\right)\left(\frac{n^2}{n_1^2}-1\right)-\mu\frac{n^2}{n_1^2}=0 \quad . \quad . \quad (XIV.68)$$

Plotting $\left(\frac{n^2}{n_2^2}-1\right)\left(\frac{n^2}{n_2^2}-1\right)$ and $\mu\frac{n^2}{n_1^2}$ against n^2 the graphs are a parabola and a straight line through the origin. Let ω_1^2 and ω_2^2 be the roots of (XIV.68) such that $\omega_2>\omega_1$. Then ω_2 is greater than, and ω_1 less than, either n_1 or n_2. ω_1 and ω_2 are the angular frequencies of vibration of the system in its first and second normal modes respectively. The first and second normal modes of vibration of M_2 are respectively

$$x_2=A\sin(\omega_1t+\alpha) \text{ and } x_2=B\sin(\omega_2t+\beta) \quad . \quad (XIV.69)$$

where A, B, α, and β are arbitrary constants depending upon the initial conditions. From (XIV.65) the first and second normal modes of M_1 are respectively

$$x_1=A\left(1-\frac{\omega_1^2}{n_1^2}\right)\sin(\omega_1t+\alpha) \text{ and}$$

$$x_1=-B\left(\frac{\omega_2^2}{n_2^2}-1\right)\sin(\omega_2t+\beta) \quad . \quad . \quad (XIV.70)$$

In general the free vibrations will be a combination of the two modes; thus, the general motion of M_1 is given by

$$x_1=A\left(1-\frac{\omega_1^2}{n_1^2}\right)\sin(\omega_1t+\alpha)-B\left(\frac{\omega_2^2}{n_2^2}-1\right)\sin(\omega_2t+\beta) \quad (XIV.71)$$

and that of M_2 by

$$x_2 = A \sin(\omega_1 t + \alpha) + B \sin(\omega_2 t + \beta) \quad . \quad . \quad \text{(XIV.72)}$$

If $B = 0$ the system vibrates in its first normal mode only and the vibrations of M_1 and M_2 are always in phase. If $A = 0$ the motion is of the second normal mode and the vibrations are always out of phase. By suitably choosing the arbitrary constants we can make (XIV.71) and (XIV.72) represent any possible free vibratory motion of the system. Damping gradually reduces the motion, the higher frequency mode dying away the more rapidly so that, after a time, the motion may be almost entirely that of the lower mode only. When ω_1 and ω_2 are nearly equal (which can only be when n_1 is nearly equal to n_2 and μ is small) there are appreciable periods of time over which the two terms on the right of (XIV.71) and (XIV.72) are alternately nearly in phase and nearly of opposite phase. This gives rise to *beats* as the motions alternately supplement, or partly cancel, each other.

We shall now consider the forced vibrations of the above system when a periodic force of magnitude $F = F_0 \sin$ ft. acts on the mass M_1, the force being positive in the sense of x_1 positive. F acts against the tension $s_1 x_1$ in the upper spring, and its effect is to add a term F_0/s_1 to the left-hand side of (XIV.66). If we write a_1 for F_0/s_1, a_1 is the static deflection of the upper spring due to a constant force F_0, and the equations (XIV.65) and (XIV.66) become

$$x_1 = \left(1 - \frac{p^2}{n_2^2}\right) x_2$$

and

$$\mu \frac{p^2}{n_1^2} x_2 + \left(\frac{p^2}{n_1^2} - 1\right) x_1 = -a_1$$

From these we find

$$x_2 = \frac{a_1}{\left(1 - \frac{p^2}{n_1^2}\right)\left(1 - \frac{p^2}{n_2^2}\right) - \mu \frac{p^2}{n_1^2}} \quad . \quad \text{(XIV.73)}$$

and

$$x_1 = \frac{\left(1 - \frac{p^2}{n_2^2}\right) a_1}{\left(1 - \frac{p^2}{n_1^2}\right)\left(1 - \frac{p^2}{n_2^2}\right) - \mu \frac{p^2}{n_1^2}} \quad . \quad \text{(XIV.74)}$$

The ratio of the amplitude of vibration of M_1 to the static deflection produced by the force F_o is known as the *dynamic multiplier* or the *magnification factor*, that is

$$\text{Dynamic multiplier} = \frac{1 - \dfrac{p^2}{n_2^2}}{\left(1 - \dfrac{p^2}{n_1^2}\right)\left(1 - \dfrac{p^2}{n_2^2}\right) - \mu\dfrac{p^2}{n_1^2}} \quad \text{(XIV.75)}$$

The values of p^2 which make the denominator in (XIV.73) and (XIV.74) zero, and therefore make x_1 and x_2 infinite, are the roots of (XIV.68), i.e., $p^2 = \omega_1^2$ and $p^2 = \omega_2^2$. Thus ω_1 and ω_2 are the critical angular frequencies of the system.

If the frequency of the lower system is tuned to the forcing frequency, i.e., if $n_2 = p$, we have

$$x_1 = 0 \text{ and } x_2 = -\frac{n_1^2}{\mu p^2} a_1 \quad . \quad \text{(XIV.76)}$$

The upper mass remains at rest and the lower mass vibrates with amplitude $\dfrac{n_1^2 a_1}{\mu n_2^2} = \dfrac{M_1 n_1^2}{M_2 n_2^2}\dfrac{F_o}{s_1} = \dfrac{F_o}{s_2}$. The maximum tension in the lower spring is s_2 times this, i.e., F_o. Thus if the lower system is tuned to the forcing frequency the upper mass remains undisturbed by the applied force and the lower mass vibrates with an amplitude and phase which just make the extra tension in the lower spring balance the disturbing force. This property of the two-mass system is made use of by designers for neutralizing the effects of forced vibrations. Forced vibrations of a given frequency acting on a girder in a vertical direction may be neutralized by a suitably designed spring-mass system suspended from the girder. Similarly torsional vibration caused by a forcing couple may be eliminated by fitting to the system a one-mass system tuned to the forcing frequency.

In adding a vibrating system to an existing system we add a critical frequency to those already existing and at the same time change the values of those frequencies. The critical angular frequencies of a two-mass system are ω_1 and ω_2 as found above. If n_1 and n_2 are nearly equal, or equal, and μ is small, ω_1 and ω_2 are nearly equal to each other as well as to n_1 and n_2. If, further, n_2 is tuned to the forcing frequency this becomes nearly equal to a critical frequency and the slightest variation in the forcing frequency may produce resonance and consequent serious vibration.

Putting $n_1 = n_2$ in (XIV.68) we find the roots $n^2 = \omega_1^2$, $n_2 = \omega_2^2$ are given by $\omega_1^2 = \left(1 + \frac{1}{2}\mu - \sqrt{\mu + \frac{\mu^2}{4}}\right)n_2^2$ and $\omega_2^2 = \left(1 + \frac{1}{2}\mu + \sqrt{\mu + \frac{\mu^2}{4}}\right)n_2^2$. If in these we put $\mu = M_2/M_1 = 0{\cdot}02$ we find that $\omega_1 = 0{\cdot}93n_2$ and $\omega_2 = 1{\cdot}07n_2$, showing that if the forcing frequency is allowed to vary by as little as 7 per cent resonance occurs. If we put $\mu = 0{\cdot}1$ we find $\omega_1 = 0{\cdot}85n_2$ and $\omega_2 = 1{\cdot}17n_2$, the difference between the critical and forcing frequencies being now at least 15 per cent. The value chosen for μ must be large enough to provide a satisfactory gap between the two frequencies. If this is not practicable, some damping should be provided.*

The general case of forced vibrations of a multi-mass system can now be treated. We shall again consider the system of Art. 219 for the case of torsional vibrations, when harmonic torques $F_1 = E_1 \sin pt$, $F_2 = E_2 \sin pt$, $F_3 = E_3 \sin pt$, etc., lb.-ft. are applied to the successive masses counting from the left. The results obtained also apply to axial vibrations of the shaft if I_n is taken to represent the mass in engineers' units of the nth mass, C_n the axial stiffness in lb. per foot of the nth length of shaft, F_n the axial force in lb. applied to the nth mass, and θ_n its displacement in ft. Considering the system when the masses are all at maximum displacement when $\sin pt = 1$, we see that the acceleration of the nth mass is $-n^2I\theta_n$ and the torque $F_n = E_n$. The torque and the reversed mass acceleration couple $n^2I\theta_n$ have the same algebraic sign and we obtain the equations of motion for the forced vibrations from (XIV.45), (XIV.47), (XIV.48), etc., by adding to the left-hand sides of these relations a term E_n/C_m for each term $\dfrac{n^2I_n\theta_n}{C_m}$ which appears there. The equations are then, writing p^2 for n^2,

$$\left(\frac{p^2I_1}{C_1} - 1\right)\theta_1 + \theta_2 = -\frac{E_1}{C_1} \quad . \quad . \quad . \quad . \quad (\text{XIV.77})$$

$$\frac{p^2I_1}{C_2}\theta_1 + \left(\frac{p^2I_2}{C_2} - 1\right)\theta_2 + \theta_3 = -\frac{E_1 + E_2}{C_2} \quad . \quad (\text{XIV.78})$$

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* See a paper by Professor C. E. Inglis, read before the Institute of Naval Architects, April, 1913.

$$\frac{p^2I_1}{C_3}\theta_1+\frac{p^2I_2}{C_3}\theta_2+\left(\frac{p^2I_3}{C_3}-1\right)\theta_3+\theta_4=-\frac{E_1+E_2+E_3}{C_3} \qquad \text{(XIV.79)}$$

The last equation of the series is

$$I_1\,\theta_1+I_2\,\theta_2+I_3\,\theta_3+-+I_n\,\theta_n= -\frac{E_1+E_2+E_3+\ \ldots+E_n}{p^2} \quad . \quad \text{(XIV.80)}$$

From these n equations the n unknown amplitudes of vibration of the masses may be found and from these the stresses in the shafts due to the forced vibrations are calculated.

We have assumed that all the applied torques are in phase; by changing the sign of any of the values of E_1, E_2, etc., we include the cases in which some of the torques are of opposite phase to the others. The most important cases of shaft vibration are those in which the applied torques, or forces, are in phase or opposite in phase and these can be dealt with by the above method. If the applied torques are of various phases they should be divided up into groups, each consisting of torques in phase, or opposite in phase. The amplitudes of the vibrations due to the different groups can then be combined as vectors and the resultant amplitudes determined.

EXAMPLE 1.

A torque of of 81 sin $30t$ lb.-ft. is applied to the middle mass of the example on page 426, and one of 99 sin $(30t+\pi)$ to the right-hand mass. Find the amplitudes of the forced vibrations of the masses and the maximum stress produced in each shaft.

The above equations become

$$\left(\frac{900\times194{\cdot}1}{491{,}200}-1\right)\theta_1+\theta_2=0 \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{900\times194{\cdot}1}{331{,}300}\theta_1+\left(\frac{900\times148{\cdot}4}{331{,}300}-1\right)\theta_2+\theta_3=-\frac{81}{331{,}300} \quad (2)$$

and from (XIV.80)

$$194{\cdot}1\;\theta_1+148{\cdot}4\;\theta_2+271{\cdot}7\theta_3=\frac{-81+99}{900} \quad . \quad . \quad (3)$$

Solving these we find

$\theta_1=0{\cdot}0003442$, $\theta_2=0{\cdot}0002218$, and $\theta_3=-0{\cdot}0002936$ radians.

The angles of twist in the two shafts are $\theta_1-\theta_2=0{\cdot}000122$ and $\theta_2-\theta_3=0{\cdot}000515$ radians. The stress is given by

$$\text{Stress} = \frac{\text{modulus} \times \text{radius} \times \text{angle of twist}}{\text{length}}$$

$$\text{Stress in first shaft} = \frac{12 \times 10^6 \times 3{\cdot}5 \times 0{\cdot}000122}{30}$$

$$= 171 \text{ lb. sq. in.}$$

$$\text{Stress in second shaft} = \frac{12 \times 10^6 \times 3 \times 0{\cdot}000515}{24}$$

$$= 773 \text{ lb. sq. in.}$$

These stresses would be negligible in comparison with those due to the torques transmitted.

EXAMPLE 2.

Find the frequency of free axial vibration of the system in the example on page 426, taking $E = 30 \times 10^6$ lb. sq. in.

Changing the symbols in (XIV.45), (XIV.47), and (XIV.49) we have

$$\left(\frac{n^2M_1}{s_1} - 1\right)x_1 + x_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{n^2M_1}{s_2}x_1 + \left(\frac{n^2M_2}{s_2} - 1\right)x_2 + x_3 = 0 \quad . \quad . \quad . \quad (2)$$

and $M_1x_1 + M_2x_2 + M_3x_3 = 0$ . . . . . . (3)

where x_1, x_2 and x_3 are the linear amplitudes of vibration in ft.

$$M_1 = \frac{1000}{g} = 31{\cdot}06 \text{ engineer's units}$$

$$M_2 = \frac{1200}{g} = 37{\cdot}27 \quad ,, \quad ,,$$

$$M_3 = \frac{1400}{g} = 43{\cdot}\ 5 \quad ,, \quad ,,$$

$$\text{Stiffness} = \frac{\text{area of cross-section} \times E}{\text{length}}$$

$$s_1 = \frac{\pi \times 3{\cdot}5^2 \times 30 \times 10^6}{4 \times 2{\cdot}5} = 11{\cdot}55 \times 10^7 \text{ lb. per ft.}$$

$$s_2 = \frac{\pi \times 3^2 \times 30 \times 10^6}{4 \times 2} = 8{\cdot}4 \times 10^7 \text{ lb. per ft.}$$

Equations (1), (2) and (3) become

$$\left(\frac{31{\cdot}06 \times 10^{-7}}{11{\cdot}55}n^2 - 1\right)x_1 + x_2 = 0 \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{31{\cdot}06 \times 10^{-7}}{8{\cdot}4} n^2 x_1 + \left(\frac{37{\cdot}27 \times 10^{-7}}{8{\cdot}4} n^2 - 1\right) x_2 + x_3 = 0 \quad (5)$$

and $$31{\cdot}06x_1 + 37{\cdot}27x_2 + 43{\cdot}5x_3 = 0 \quad . \quad . \quad . \quad (6)$$

After simplification these reduce to

$$(2{\cdot}689 \times 10^{-7}n^2 - 1)x_1 + x_2 = 0$$

$$3{\cdot}698 \times 10^{-7}n^2 x_1 + (4{\cdot}436 \times 10^{-7}n^2 - 1)x_2 + x_3 = 0$$

and $$0{\cdot}7144x_1 + 0{\cdot}8540x_2 + x_3 = 0$$

Eliminating x_1, x_2 and x_3

$$\begin{vmatrix} 2{\cdot}689 \times 10^{-7}n^2 - 1 & 1 & 0 \\ 3{\cdot}698 \times 10^{-7}n^2 & 4{\cdot}436 \times 10^{-7}n^2 - 1 & 1 \\ 0{\cdot}7144 & 0{\cdot}8540 & 1 \end{vmatrix} = 0$$

and writing y for $10^{-7}n^2$ this becomes

$$(2{\cdot}689y - 1)(4{\cdot}436y - 1{\cdot}854) + 0{\cdot}7144 - 3{\cdot}698y = 0$$

i.e. $$11{\cdot}93y^2 - 13{\cdot}12y + 2{\cdot}568 = 0$$

the roots of which are

$$10^{-7}n_1^2 = y_1 = 0{\cdot}255 \text{ and } 10^{-7}n_2^2 = y_2 = 0{\cdot}846$$

From these $n_1^2 = 2{,}550{,}000$ and $n_2^2 = 8{,}460{,}000$, so that $n_1 = 1597$ and $n_2 = 2909$. These are the angular frequencies in radians per second of the first and second normal modes, respectively, and correspond to actual frequencies of

$$\frac{60n_1}{2\pi} = 15{,}300 \text{ per minute and } \frac{60n_2}{2\pi} = 28{,}400 \text{ per minute.}$$

(*Note.* The value of E may vary from $28{\cdot}5 \times 10^6$ to 30×10^6 lb. per sq. in. according to the composition of the steel used and its actual value should be substituted where accurate results are required.)

224. **Harmonic Orders. Vibration Dampers.** The effective turning moment on the crankshaft of a steam-engine or two-stroke internal combustion engine due to each line of reciprocating parts is a periodic function whose period is that of crankshaft rotation. If this function is expanded in the form of a sine series, this series is the sum of a constant (the mean turning moment) and sine terms whose frequencies are in order 1, 2, 3, 4, 5, etc., times that of crankshaft rotation. In a four-stroke engine the period of the turning moment function is twice that of crankshaft rotation, and the corresponding numbers are $\frac{1}{2}$, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, etc. These numbers are known as the harmonic

orders of the sine terms. The constant mean turning moment does the effective work of driving the crankshaft and the fluctuating turning moments represented by the sine terms are disturbances which cause torsional vibrations in the crankshaft.

A disturbance of the nth harmonic order has a frequency n times the engine speed. In order to determine which critical speeds fall inside the range of running speeds a graph is drawn with engine speeds as abscissae and frequencies as ordinates. The harmonic disturbances will be represented by a series of radiating lines through the origin, the gradient of each being equal to its harmonic order, and the natural frequencies each by a horizontal line. The intersections of these two sets of lines give the critical engine speeds. Fig. 286 shows a diagram called a *critical speed spectrum* which could be projected from the above graph but which is more readily drawn separately. If f_c is a critical frequency and n the harmonic order of any component disturbance, the corresponding critical engine speed is f_c/n. Engine speeds are given against the uprights in the long rectangle. The frequencies of the first and second modes of vibration of the crankshaft have been assumed to be 275 and 1320 per minute respectively. Resonance with the first mode will occur at an engine speed of 275 r.p.m., the harmonic order being 1. Thus we have set the order number 1 under that speed. The other harmonic order numbers are placed so that the product of the number and the corresponding engine speed is constant. Similarly above the rectangle we have marked the order numbers so that the products with corresponding engine speeds are constant and equal to 1320. If the engine is run at 400 r.p.m. there are no harmonics near resonance with the first mode but that of order $3\frac{1}{2}$ is near resonance with the second mode. At 200 r.p.m. the harmonic order $1\frac{1}{2}$ is nearest to resonance with the first mode and those of 6 and 7

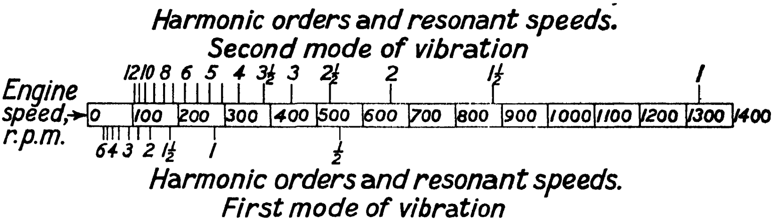

FIG. 286

with the second mode. For a running speed of 100 r.p.m. the orders 2½ and 3 are near to resonance with the first mode and the orders 12 and 12½, the latter not shown, with the second mode. Not all critical speeds are equally important. If the impulses on the various cranks, due to any given harmonic order, are in phase, in general, their effects will be cumulative, and such harmonic orders are known as *major harmonics.* A *major harmonic* is one of an order which is an integral number of times the number of working strokes per revolution of the crankshaft. All other harmonics are *minor harmonics.* If there is no node inside the crank system the crankshaft displacements due to the major harmonics will be in phase and will produce their maximum effects. If there is a node inside the crank system the major harmonics may cancel themselves partly or wholly, whilst some of the minor harmonics (such as the 4½, 7½ and 10½ orders in a 6-cylinder, 4-stroke engine where the disturbances are in phase in sets of three with one set of opposite phase to the other) may become important (see Unwin's "Machine Design"). When important critical speeds occur inside the running range, some, or all, of them may be removed by a change in the proposed stiffness of the shaft, or by changing the moment of inertia of one or more of the crank masses. If it is not possible to design the engine so that there are no important critical frequencies in the running range the fitting of some form of vibration *damper* becomes necessary. "In theory these dampers should be fitted to each crank mass, but, in practice, it is found that the damper mounted anywhere on the engine will sufficiently absorb the vibration to eliminate serious stress."*

There are two general types of damper: (1) The damper, properly named, in which damping is produced by the slipping of a friction element when the amplitude of vibration reaches a fixed limit. This type is not suitable for damping at running speeds because of the considerable loss of energy due to slipping, but is suitable for use where a critical speed has to be passed through in rising to the running speed; (2) a type of damper consisting of an auxiliary vibrating system which is fitted to the main element and alters the natural frequency, or is tuned to the disturbing frequency,* thus absorbing or neutralizing the vibrations. It will be clear that the first type of damper introduces a slight change of frequency when slipping takes place.

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* See a paper by J. Calderwood, B.Sc., on "General Observation on Vibration," read on March 15th, 1941, before the N.E. Coast Institute of Engineers and Shipbuilders.

With the second type of damper, one damper must be fitted to deal with each important critical disturbance. One of this type, however, the *pendulum damper,* described below, controls the disturbance of the particular harmonic order for which it is designed at all engine speeds. Particulars of the various kinds of damper are given in several published works on "Torsional Vibration," one or more of which the reader should consult.

225. **The Pendulum Damper.** A diagram of this is shown in Fig. 287. C, is a mass rotating with the crankshaft at a constant speed of ω radians per second. PQ is a simple pendulum pivoted at P and carrying a mass of m engineers' units at Q. $\overline{OP} = R$ ft., $\overline{PQ} = l$ ft. We shall find the small free vibrations of the pendulum as follows.

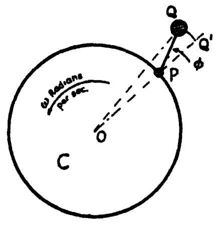

FIG. 287. THE PENDULUM DAMPER IN TUNE

The acceleration of a particle relative to fixed axes is equal to the sum of (1) its acceleration relative to axes rotating uniformly about the origin, (2) an acceleration $\omega^2 r$ towards the origin, where ω is the angular speed and r the distance of the particle from the origin, and (3) the Coriolis acceleration $2\omega v$, where v is the radial component of the velocity of the particle.

In this case, as v is small, the Coriolis acceleration is negligible, and we may solve the problem by treating C as at rest if we introduce at Q a force $m\omega^2\overrightarrow{OQ}$. The component of this perpendicular to PQ is $m\omega^2(R + l)\sin \widehat{OQP}$ approximately. Let ϕ be the angle QPQ', Q' being the equilibrium position of Q, let $\widehat{POQ} = \alpha$, and t the time in seconds. Then $\sin \widehat{OQP} \approx \frac{R}{R+l}\sin\phi \approx R\phi/(R + l)$, and the equation of motion of Q is

$$m\omega^2 R\phi + ml\ddot{\phi} = 0 \quad . \quad . \quad . \quad . \quad (\text{XIV}.81)$$

the solution of which is

$$\phi = \phi_0 \sin \omega\sqrt{\frac{R}{l}}t \quad . \quad . \quad . \quad . \quad (\text{XIV}.82)$$

the second constant of integration vanishing if we take $\phi = 0$ when $t = 0$. The angular frequency of vibration is $\omega\sqrt{\frac{R}{l}}$ which is the same as that of a disturbance of the Nth harmonic order if $N^2 = R/l$. If, then, we tune the pendulum to the frequency of the Nth harmonic order by making $l = R/N^2$ it will respond to a disturbance of that order by making forced vibrations. Owing to these the tension T lb. in the link PQ will exert a variable turning moment on the carrier C of amount $TR \sin \phi \approx TR\phi$ lb.-ft. Resolving the centrifugal force $m\omega^2\overline{O_0Q} \approx m\omega^2(R + l)$ along PQ we have $T = m\omega^2(R + l) \cos \widehat{OQP} \approx m\omega^2(R + l)$. The torque due to this is $TR\phi = TR\phi_0 \sin \omega\sqrt{\frac{R}{l}}t$, i.e., torque $= TR\phi_0 \sin N\omega t$.

Let $F = F_0 \sin N\omega t$ be the disturbing torque. Then the pendulum will swing with an amplitude ϕ_0 such that the pendulum torque is equal to the disturbing torque. We have then,

$$TR\phi_0 = F_0$$

or

$$m\omega^2(R + l)R\phi_0 = F_0$$

The angular amplitude of the pendulum is given by

$$\phi_0 = \frac{F_0}{m\omega^2(R + l)R} \quad . \quad . \quad . \quad \text{(XIV.83)}$$

and the linear amplitude of Q by

$$l\phi_0 = \frac{F_0 l}{m\omega^2(R + l)R} \quad . \quad . \quad . \quad \text{(XIV.84)}$$

For the above approximations to be justified ϕ_0 should not exceed the radian equivalent of 12°, and the value of m must be large enough to ensure this.

Now let us consider the motion of the pendulum when the carrier has an angular motion represented by $\psi = \psi_0 \sin n\omega t$ radians, superposed on its steady angular speed of ω radians per second, where n represents any one of the harmonic order disturbances to the Nth of which the pendulum is tuned (Fig. 288). We shall assume that $\phi = \phi_0 \sin n\omega t$ is the angular displacement at time t seconds, of the pendulum from its mean relative position PQ'. We shall assume also that the mean speed of C is zero and introduce at Q a force $m\omega^2\overrightarrow{OQ}$ whose magnitude is very approximately $m\omega^2(R + l)$, again neglecting the Coriolis acceleration. As before, the tension in PQ is

$$m\omega^2(R + l) \cos \widehat{OQP} \approx m\omega^2(R + l).$$

The displacement of Q from its mean position Q_0 is $(R + l)\psi + l\phi$, Fig. 288.

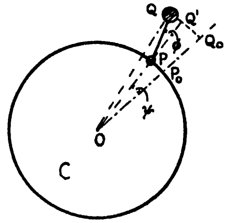

FIG. 288. PENDULUM DAMPER OUT OF TUNE

The component force on Q perpendicular to QP is $m\omega^2(R + l) \sin \widehat{OQP} = m\omega^2 R \sin \phi \approx m\omega^2 R\phi$.

The equation of motion of Q is therefore

$$m\omega^2 R\phi + m(R + l)\ddot{\psi} + ml\ddot{\phi} = 0 \qquad \text{(XIV.85)}$$

Substituting for ϕ and ψ and simplifying we have

$$\phi_0 = \frac{n^2(R + l)}{R - n^2 l}\,\psi_0 \quad . \quad . \qquad \text{(XIV.86)}$$

The tension T in PQ produces a torque on the crankshaft of magnitude $TR \sin \phi \approx TR\phi$. Hence we have, since

$$T = m\omega^2\,(R + l) \cos \widehat{OQP} \approx m\omega^2\,(R + l)$$

$$\begin{aligned} \text{Torque} &= m\omega^2\,(R + l)\,R\phi_0 \sin n\omega t \\ &= \frac{mn^2\omega^2\,(R + l)^2\,R}{R - n^2 l}\,\psi_0 \sin n\omega t \end{aligned}$$

$$\therefore\ \text{Torque} = -\frac{m(R + l)^2}{1 - n^2\dfrac{l}{R}}\,\ddot{\psi} \quad . \quad . \quad . \quad . \qquad \text{(XIV.87)}$$

A mass of moment of inertia I_P attached to the crankshaft would produce a torque of amount $- I_P\ddot{\psi}$ on the crankshaft. The pendulum is therefore equivalent to a crank mass of moment of inertia

$$I_P = \frac{m(R + l)^2}{1 - n^2\dfrac{l}{R}} \quad . \quad . \quad . \quad . \qquad \text{(XIV.88)}$$

and the equivalent moment of inertia of the carrier and pendulum is

$$I_e = I_c + \frac{m(R+l)^2}{1 - n^2 \dfrac{l}{R}} \quad . \quad . \quad . \quad (\text{XIV}.89)$$

where I_c is the moment of inertia of the carrier.

If $n^2 = N^2 = R/l$, the equivalent moment of inertia is infinite and the crankshaft speed will remain constant and unaffected by any disturbance of the Nth harmonic order. Disturbances of any other order will produce forced vibration of the shaft. If $n > N$, I_P is negative, and if $n < N$, I_P is positive. The carrier-pendulum system has thus different moments of inertia for disturbances of different harmonic orders. These variations must be taken into account when determining natural frequencies of a crankshaft system fitted with pendulum dampers.

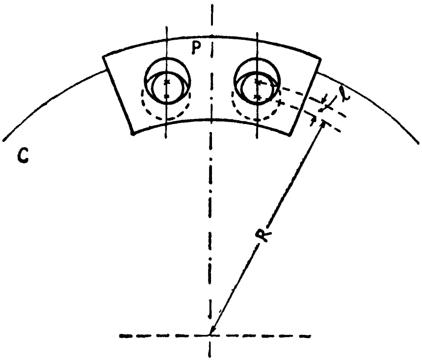

FIG. 289. BILIFAR SUSPENSION IN PENDULUM DAMPER

In actual practice l is not large enough to permit of the use of swinging links such as PQ. One method used (bifilar suspension) is shown in Fig. 289, in which the plate P takes the place of the pendulum bob. The small circles represent rollers held between cylindrical cavities in the plate, shown in full, and in the carrier, shown partly dotted. If D is the diameter of the cavity (which is the same in both P and C) and d is the diameter of the roller, the distance between the centres of the cavities in the plate and in the carrier is $D - d$, so that the length of the equivalent swinging link is $l = D - d$. R is the distance between the axes of C and the cavity in C. There may be several plates equally spaced round the carrier C so as to balance, or there may be only one carried on, and forming part of, the crank balance weight. In either case the plates are constrained to rotate at the same speed as the carrier and the moment of inertia of each plate about its own centre of gravity should be included in the equivalent moment of inertia. If k ft. is the radius of gyration of P about an axis through its centre of gravity

$$I_e = I_c + mk^2 + \frac{m(R+l)^2}{1 - n^2\frac{R}{l}} \quad . \quad . \quad . \quad \text{(XIV.90)}$$

Owing to the change in the effective moment of inertia of the mass to which a pendulum damper is fitted it is possible, in some cases, that some other important critical speed due to a disturbance of order different from that to which the pendulum is tuned may be removed outside the range of running speeds. Thus it is possible for a single damper to deal with disturbances of more than one harmonic order. This is not always possible, and it is necessary, in some cases, to apply to the same crank mass two or more dampers tuned to different harmonic orders.

The above treatment neglects the effects of friction on the pendulum itself and does not apply in cases where the amplitude of vibration of the pendulum exceeds about 12 degrees. For a full treatment of pendulum dampers with small amplitudes of vibration readers should consult a paper by Messrs. Zdanowich and Wilson in the *Journal of the Institute of Mechanical Engineers*, 1940, Vol. 143, No. 3.

In one type of pendulum damper, used in aeroplane engines, the pendulum bob is a cylindrical roller which moves in a cylindrical cavity in the crank balance weight. Owing to the relatively small mass of the roller it must have a large amplitude, as much as 60 degrees. In this case the Coriolis acceleration is not negligible and, as $\sin\phi$ cannot be substituted by ϕ, an elliptic integral appears in the solution to the equation of motion. The result is that, whilst performing its function, the damper introduces harmonic disturbances of its own which, under certain circumstances, may cause trouble. This type of damper is dealt with in a paper by J. P. Den Hartog in the *Stephen Timoshenko Sixtieth Anniversary Volume,* The Macmillan Company, New York.

226. **Equivalent Dynamical Systems.** In determining the motion of a rigid body in plane motion under the action of external forces, it is usually convenient to replace the body by two heavy particles at a fixed distance apart, such that (i) the sum of their masses is equal to the mass of the body, (ii) their centre of gravity coincides with that of the body, and (iii) their moment of inertia about their centre of gravity is equal to that of the body. These conditions were stated in Art. 41 with reference to the motion of a connecting rod.

If the three conditions are satisfied, the equations of motion for the body are identical with those for the particles, and consequently the translatory and rotary motions of the two systems are the same. The two systems are *dynamically equivalent.*

It is usually convenient to assume that one of the particles is at a centre of rotation, or at a pin-joint. Consider again the compound pendulum of Art. 48. With the notation of that article we assume that one particle of mass $\frac{w_o}{g}$ engineers' units is placed at the centre of suspension O (Fig. 26), and the other of mass $\frac{w_c}{g}$ at a point C whose position is to be determined. Conditions (i), (ii), and (iii) give, respectively, the three relations

$$w_o + w_c = W \quad . \quad . \quad . \quad . \quad (\text{XIV}.91)$$

$$w_c \, . \, \overline{OC} = Wh \quad . \quad . \quad . \quad . \quad (\text{XIV}.92)$$

$$w_c \, . \, \overline{OC^2} = W(k^2 + h^2) \quad . \quad . \quad . \quad (\text{XIV}.93)$$

The last of these is true because if conditions (ii) and (iii) are satisfied the two systems have equal moments of inertia about any axis perpendicular to their plane.

From (XIV.92) and (XIV.93), $OC = \dfrac{k^2 + h^2}{h}$, as in Art. 48 . (XIV.94)

From (XIV.92), $w_c = W\dfrac{h^2}{k^2 + h^2}$ . . . (XIV.95)

and from (XIV.91), $w_o = W\dfrac{k^2}{k^2 + h^2}$ . . . (XIV.96)

When finding the inertia force on a connecting rod in a mechanism, such as the simple engine mechanism in Fig. 64, we replace the rod by two particles, one at P, say, and the other at some point to be determined, between P and C. The accelerations of the two points are found from the acceleration image of PC. The product of the mass of each particle and the acceleration is the accelerating force, and the resultant of the two forces is the force necessary to accelerate the rod. (An alternative method of finding this force was given in Art. 89.) A force equal and opposite to the resultant is the inertia force of the bar. The external forces and the inertia forces form a system of forces in equilibrium.

The reader should remember the particulars of the equivalent dynamical system for a thin uniform straight rod OA (Fig. 290) with one equivalent particle at one end O of the rod.

Let l ft. be the length of the rod, W lb. its weight, k ft. its

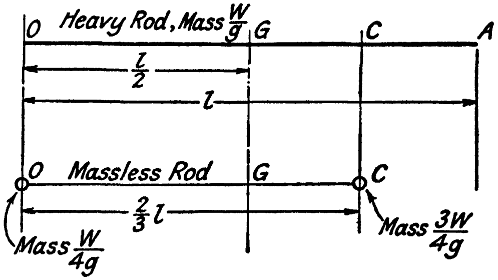

FIG. 290. EQUIVALENT DYNAMICAL SYSTEMS

radius of gyration about its centre of gravity G. Then $k^2 = \frac{1}{12}l^2$ and relations (XIV.65) to (XIV.67) become

$$w_o + w_c = W$$

$$w_c \, . \, \overline{OC} = W \frac{l}{2}$$

$$w_c \, . \, \overline{OC}^2 = W\left(\frac{l^2}{12} + \frac{l^2}{4}\right)$$

$$= W \frac{l^2}{3}$$

From these, $\overline{OC} = \frac{2}{3}l$, $w_c = \frac{3}{4}W$, $w_o = \frac{1}{4}W$.

EXAMPLE.

We shall solve the example worked in Art. 89 using an equivalent dynamical system of which one particle, mass $\frac{25}{4g}$ units, is at A, and the other mass $\frac{75}{4g}$ units, as shown above, is at a point K between A and B, such that $\overline{AK} = \frac{2}{3}\,\overline{AB}$.

The acceleration of A (Fig. 291) is the resultant of $\overrightarrow{o'g'}$, which represents 6 f.s.s. in the acceleration image, and $\overrightarrow{g'a'}$, which represents $a \, . \, \overline{GA} = 3$ f.s.s. This resultant is $\overrightarrow{o'a'}$.

Similarly, the acceleration of B is the resultant of $\overrightarrow{o'g'}$ and $\overrightarrow{g'b'}$ equal and opposite to $\overrightarrow{g'a'}$. k' is the image of K, i.e.

$$\frac{\overline{a'k'}}{\overline{k'b'}} = \frac{\overline{AK}}{\overline{KB}} = \frac{2}{3}$$

By measurement, $o'a'$ represents an acceleration of 6·77 ft. per sec. per sec. The force on the particle at A acts along OA, which is parallel to $o'a'$, and its magnitude is $\frac{25}{4g} \times 6{\cdot}77 = 1{\cdot}31$ lb. The force on the particle at K is the product of the mass $\frac{75}{4g}$ and 3·17 ft. per sec. per sec., and acts along OK parallel to $o'k'$; its magnitude is 1·83 lb. These two forces are set off to scale along OA and OK respectively. The resultant force is 3·10 lb. It meets AB in C and, by measurement, $\overline{CG} = 0{\cdot}87$ ft. These results are in practical agreement with those of Art. 89.

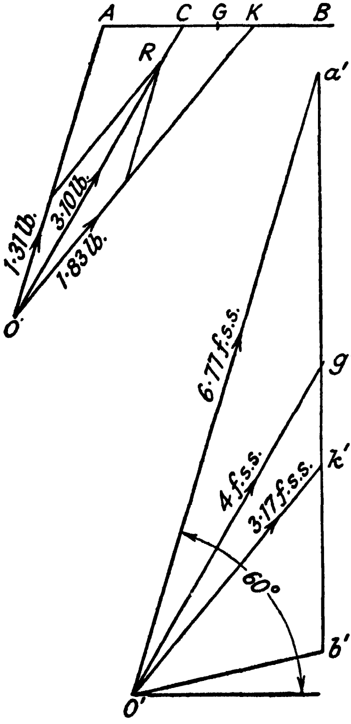

FIG. 291

227. **Force and Couple due to Inertia.** Consider a mechanism in motion under the application of an external force F_P applied to a moving point P on one link of the mechanism, and of other external forces applied at points in the mechanism which are fixed in space. Owing to the motion of the mechanism there will be inertia forces, i.e. reversed mass accelerations, brought into operations. In the absence of friction, and neglecting gravity effects, the inertia forces and the external forces will form a system of forces in equilibrium. In any small displacement of the mechanism the only external force which does work is the force F_P, for the points of application of the other external forces do not move.

Applying the principle of work, we see that the total work done in the small displacement by the force F_P and the inertia forces, is zero. Let the inertia forces be F_1, F_2, F_3, etc., lb. and let v_1, v_2, v_3 etc., be the respective component velocities (f.s.) of their points of application in the direction and sense of the forces. If the senses of a force and the corresponding component velocity are opposed, the work done by the force is negative. If Δt sec. is the interval of time for the small displacement, the total work done by the inertia forces is

$$F_1v_1\Delta t + F_2v_2\Delta t + F_3v_3\Delta t + \ldots = \Delta t\Sigma Fv$$

If v_P is the component velocity of P in the direction and sense of F_P, we have therefore

$$F_Pv_P\Delta t + \Delta t\Sigma Fv = 0$$

i.e.

$$F_Pv_P + \Sigma Fv = 0 \quad . \quad . \quad . \quad . \quad \text{(XIV.97)}$$

the summation in the second term extending over all the inertia forces.

If the effects of gravity are to be included, the summation includes the forces of gravity as well as the inertia forces.

Relation (XIV.97) enables us to find the single force F_P acting at any selected point which will balance the inertia forces; the inertia force at that point is equal and opposite to this. We shall show the method of application in the following example.

EXAMPLE.

In the slotted lever mechanism of Fig. 47, a line diagram of which is given in Fig. 292, the crank OB is 4 in. long, OA is vertical and 8 in. long, and the line of stroke of R is horizontal and 6 in. above O. The rod AP is of 14 lb. weight, 16 in. long, and its centre of gravity is at the middle of its length; the radius of gyration about its c.g. is 6·5 in. The corresponding quantities for the rod PR are 16 lb., 17 in., and 6 in.; its centre of gravity being, also, at its mid point. The slider R weighs 18 lb. OB rotates at 60 r.p.m., clockwise. Find, for the position in which the angle BOA is 150°, the force which the crankpin slide must exert on the lever AB, and hence find the force and couple which the inertia forces produce on the crankshaft at O. Neglect friction.

For the equivalent dynamical systems we assume particles of w_a, w_d, w_r, and w_e lb. weight at A, D, R, and E respectively. Applying the relations (XIV.91) to (XIV.93) we have

Bar AP	Bar PR
$w_a + w_d = 14$	$w_r + w_e = 16$

$$\begin{array}{llll} & w_a\overline{AD} = 14 \times 8 & & w_e\overline{ER} = 16 \times 8{\cdot}5 \\ & w_a\overline{AD}^2 = 14(6{\cdot}5^2 + 8^2) & & w_e\overline{ER}^2 = 16(6^2 + 8{\cdot}5^2) \\ \text{from which} & \overline{AD} = 13{\cdot}3 \text{ in.} & & \overline{ER} = 12{\cdot}72 \text{ in.} \\ & w_d = 8{\cdot}43 \text{ lb.} & & w_e = 10{\cdot}7 \text{ lb.} \\ & w_a = 5{\cdot}57 \text{ lb.} & & w_r = 5{\cdot}3 \text{ lb.} \end{array}$$

As A is fixed, w_A has no acceleration and does not affect the motion. w_r is added to the weight of the slider, which is therefore equivalent to a mass of 23·3 lb. weight.

In Fig. 292 we draw the velocity image on a suitable scale. The velocity of B is $\frac{2\pi}{3} = 2{\cdot}09$ f.s. $= \overline{ob}$. Let C be the point, fixed on the rod AP which coincides with B. oc is perpendicular to AB and bc is parallel to AB. oc is produced and the images of D and P marked on it. pr is perpendicular to PR and or is horizontal. e is the image of E. From this diagram we have $\overline{oc} = 1{\cdot}96$ f.s., $\overline{bc} = 0{\cdot}74$ f.s., $\overline{rp} = 0{\cdot}49$ f.s. Draw the acceleration diagram on a suitable scale. $\overline{o'b'} = \frac{2{\cdot}09^2}{\frac{1}{3}} = 13{\cdot}1$ f.s.s. The Coriolis acceleration of B relative to C is $2\omega v$ (see Art. 19), i.e. $\overline{b'c_1'} = 2\frac{\overline{oc}}{\overline{AB}}$, $\overline{cb} = 2\,\frac{1{\cdot}96}{0{\cdot}971} \times 0{\cdot}74 = 2{\cdot}99$ f.s.s. A line through c' and perpendicular to $b'c'$ is a locus of c'.

Next considering C with relation to A, its acceleration towards A is $\frac{\overline{oc}^2}{\overline{AC}} = \frac{1{\cdot}96^2}{0{\cdot}971} = 3{\cdot}96$ f.s.s. Draw $\overline{o'c_2'} = 3{\cdot}96$ f.s.s. parallel to CA, and through c_2' draw a line perpendicular to CA. This last is the second locus of c', which is, therefore, at the intersection of the loci. On $o'c'$ mark off the images of D and P. From p' set off $\overline{p'r_1'} = \frac{\overline{pr}^2}{\overline{PR}} = 0{\cdot}17$ f.s.s. parallel to PR through r_1' draw a perpendicular to PR to meet the horizontal through o' in r. Mark on $p'r'$ and $o'p'$ the images of E and D respectively. This completes the acceleration image.

The pressure of the crank slider at B on the slotted lever is, as shown, perpendicular to AB; let F_c be its magnitude in pounds. From the velocity diagram the velocity of C in the

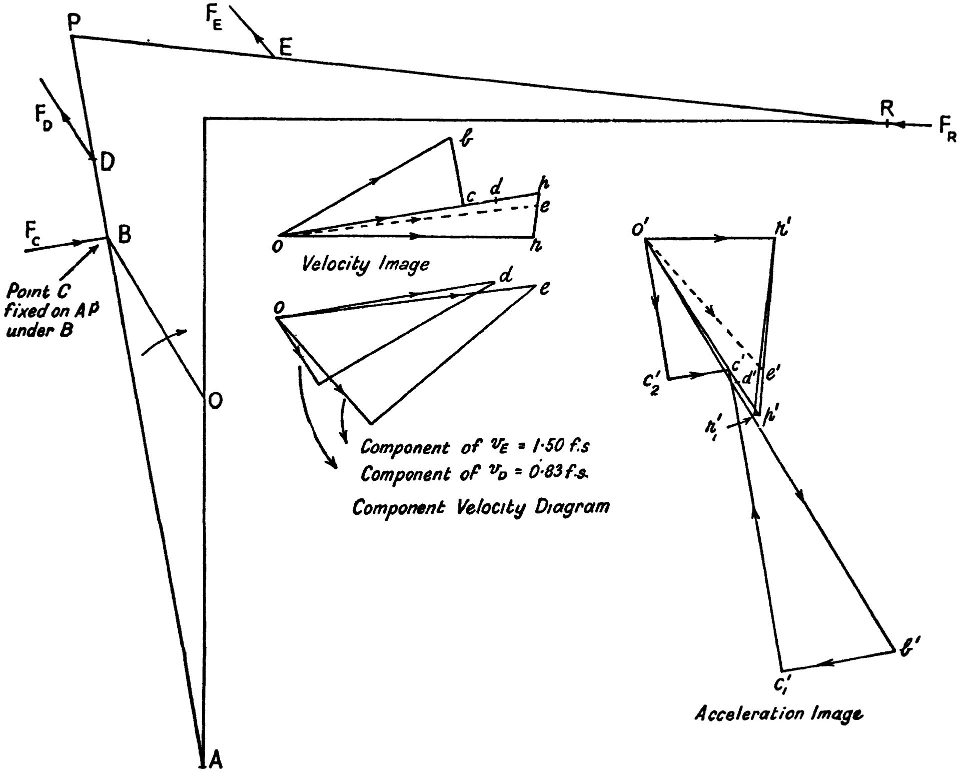
P
F_E
E
R
F_R
F_D
D
F_C
B
Point C fixed on AP under B
O
A
b
c
d
p
e
o
r
Velocity Image
o
d
e
Component of v_E = 1·50 f.s
Component of v_D = 0·83 f.s.
Component Velocity Diagram
o'
r'
c'
d'
e'
c_2'
p'
r_1'
b'
c_1'
Acceleration Image

FIG. 292

direction and sense of F_c is 1·96 f.s. The values of the acceleration and velocities in the table below are measured from the diagrams, the component velocities being found as indicated in the component velocity diagram (Fig. 292). It would be quicker to project the velocities directly from the velocity image to the acceleration image, a separate diagram being used here for clarity.

(1) Point	(2) Mass in engineers' units	(3) Acceleration, f.s.s.	(4) Component velocity in direction and sense of acceleration f.s.	(5) Rate of doing work of the reversed mass-accelerations ft.-lb./sec.
D	$\frac{8\cdot43}{32\cdot2}$	4·79	0·83	– 1·04
E	$\frac{10\cdot7}{32\cdot2}$	4·86	1·50	– 2·42
R	$\frac{23\cdot3}{32\cdot2}$	3·46	2·60	– 6·51

$$\text{Total} = \Sigma Fv = -9\cdot97$$

Numbers in (5) are the products of those in (2), (3), and (4)

Substituting in (XIV.97), we have

$$1\cdot96F_c - 9\cdot97 = 0$$

whence $$F_c = 5\cdot07$$

The force necessary to overcome the inertia forces is, therefore, 5·07 lb. at *C* perpendicular to *AP*. The force on the crank *OB* at *B* is equal and opposite to this. The moment of the couple on the crankshaft *O* is 5·07 times the perpendicular distance from *O*, which is 3·70 in., and the couple on the crankshaft is $5\cdot07 \times \frac{3\cdot70}{12} = 1\cdot56$ lb.-ft.

Inertia couple on crankshaft $O = 1\cdot56$ lb.-ft. anticlockwise.

Inertia force on crankshaft $O = 5\cdot07$ lb. in the opposite sense to F_c in the figure.

228. **Free Oscillations of a Plane Mechanism of Heavy Bars under an Elastic Constraint.** If the lower end of the spring in Fig. 24, instead of being attached to a single mass as shown, is

attached to one of the bars of a mechanism, as, for instance, is the lower end of the spring C in Fig. 227 from which for the moment we assume the spring B to be removed, the system will oscillate when displaced from its position of rest and released. We shall show how to adapt (XIV.3) so as to make it apply in such cases.

Replace each bar of the mechanism by an equivalent dynamical system of two particles, one of which, for preference, should be at a fixed centre of rotation if there is such a centre, or at a pin-joint if the bar has no fixed centre; draw the velocity image of the mechanism. Let O be the point in the mechanism to which the spring is attached, and let A, B, C, D, etc., be the particles forming the equivalent systems, their masses being $\frac{w_a}{g}$, $\frac{w_b}{g}$, $\frac{w_c}{g}$, $\frac{w_d}{g}$, etc., engineers' units. Then if v_a, v_b, v_c, v_d, etc., are the respective velocities of the masses, the kinetic energy of the system of particles is

$$K.E. = \frac{1}{2g}(w_a v_a^2 + w_b v_b^2 + w_c v_c^2 + w_d v_d^2 + \ldots)$$

$$= \frac{v_0^2}{2g}\left[w_a\left(\frac{v_a}{v_0}\right)^2 + w_b\left(\frac{v_b}{v_0}\right)^2 + w_c\left(\frac{v_c}{v_0}\right)^2 + \ldots\right] \quad \text{(XIV.98)}$$

The ratios $\frac{v_a}{v_0}$, $\frac{v_b}{v_0}$, $\frac{v_c}{v_0}$, etc., are readily found from the velocity image.

Let W_0 be the value of the quantity, in the square bracket, then the kinetic energy of the system is $\frac{W_0 v_0^2}{2g}$, and $\frac{W_0}{g}$ is the single inertial mass which placed at O would oscillate with the same periodic time as that of the mechanism. Thus, instead of the equation (XIV.3), we have

$$\frac{d^2x}{dt^2} + \frac{gE}{W_0}x = 0 \quad . \quad . \quad \text{(XIV.99)}$$

where $W_0 = \sum w_a \left(\frac{v_a}{v_0}\right)^2$, the summation extending over all the particles of the equivalent systems. x is the displacement of O, and E is the stiffness of the spring, or other elastic constraint in the direction of the displacement of O.

If the direction of the displacement makes an angle a with the axis of the spring whose stiffness is s, then $E = s \cos a$. The velocity ratios may be found from the velocity image or by the use of instantaneous centres.

EXAMPLE.

Find an expression for the time of free small oscillations of the mechanism in Fig. 227.

Let $\frac{W_H}{g}$ be the mass of the horizontal bar, l ft. its length, a and b the lengths indicated in feet. Replace the rod KM by its equivalent dynamical system, a particle of mass $\frac{W_H}{g} \cdot \frac{k^2}{K^2}$ at K, and one of $\frac{W_H}{g} \cdot \frac{h^2}{K^2}$ at a point L in the bar, where $\overline{KL} = \frac{K^2}{h}$, K and h being the quantities defined in Art. 48.

Let $\frac{W_V}{g}$ be the total mass of the connecting rod and plunger, all of which is moving along the vertical with the velocity of M. Then the equivalent inertial mass at the end of the spring is $\frac{W_0}{g}$, where

$$W_0 = W_H \frac{h^2}{k^2}\left(\frac{\overline{KL}}{a}\right)^2 + W_V\left(\frac{l}{a}\right)^2$$

and $$W_0 = \frac{1}{a^2}(W_H K^2 + W_V l^2)$$

Consider the effect of the two springs. If spring C is extended by a small distance x, the increase in tension is $s_c x$, where s_c is its stiffness. The decrease in tension in B will be $s_b x \frac{b}{a}$, where s_b is its stiffness. The unbalanced couple on the rod will be $s_c xa + s_b x \frac{b}{a} b = \left(s_c + s_b \frac{b^2}{a^2}\right)xa$, a couple which would be exerted by a single spring in the same position as C, but of stiffness $E = s_c + \frac{b^2}{a^2} s_b$. The solution of (XIV.99) is

$$x = A \cos\left(\sqrt{\left(\frac{gE}{W_0}\right)}t + \beta\right)$$

where A and β are constants, and the periodic time is

$$t = 2\pi\sqrt{\frac{W_0}{gE}}$$

so that the time of oscillation of the mechanism

$$t = 2\pi\sqrt{\frac{W_H K^2 + W_v l^2}{g(a^2 s_c + b^2 s_b)}}$$

229. **Equivalent Rotational Systems.** Consider a system of rigid rotors rotating about fixed axes with given velocity ratios as, for example, in the compound wheel train, Fig. 165. Let the shafts be A, B, C, D, etc., the moments of inertia of the shafts and attached masses being I_A, I_B, I_C, I_D, etc., and their velocity ratios with respect to A, 1, r_B, r_C, r_D, etc.

The couple which must be applied to at the shaft A in order to produce a couple C on any of the other shafts is Cr, where r is the velocity ratio between that shaft and A. This follows by the principle of work because the shaft rotates at r times the speed of A.

Suppose the shaft A to have an angular acceleration a radians per second per second, then the angular accelerations of the other shafts will be ar_B, ar_C, ar_D, etc., respectively, and the couples to produce these, applied to their respective shafts, are $I_B ar_B$, $I_C ar_C$, $I_D ar_D$, etc. The equivalent couples on A are $I_B ar_B{}^2$, $I_C ar_C{}^2$, $I_D ar_D{}^2$, etc. Including the couple to accelerate its own rotor, the total couple on A is

$$T_A = \text{total couple on shaft } A$$

$$T_A = (I_A a + I_B r_B{}^2 a + I_C r_C{}^2 a + \ldots)$$

or $$T_A = (I_A + I_B r_B{}^2 + I_C r_C{}^2 + \ldots)a$$

$$\therefore\ T_A = I_e a \qquad \text{(XIV.100)}$$

where $$I_e = (I_A + I_B r_B{}^2 + I_C r_C{}^2 + \ldots). \qquad \text{(XIV.101)}$$

A single rotor of moment of inertia I_e rotating with the shaft A is equivalent, therefore, so far as the dynamics of rotation is concerned, to the system of rotors and shafts I_A, I_B, I_C, etc.

If F_A, F_B, F_C, F_D, etc., are the friction couples on the shafts A, B, C, D, etc., respectively, the equivalent friction couple F_e on the shaft A is

$$F_e = F_A + r_B F_B + r_c F_c + r_e F_e + \ldots . \qquad \text{(XIV.102)}$$

EXAMPLE.

Two shafts *A* and *B* are connected by gearing so that *B* rotates at 4 times the speed of *A*. *A* carries a flywheel and *B* a drum their weights being 2 tons and 1 ton, and their radii of gyration 3 ft. and 1 ft respectively. The friction couple on the shaft *A* is 99·6 lb.-ft. and that on *B* is 49·8 lb.-ft., both assumed constant. Find (1) the total kinetic energy of the system; (2) the time taken to come to rest, running freely, from 200 r.p.m.; (3) the angular acceleration of *A* when there is applied to it a turning moment of 2000 lb.-ft.; (4) the time taken to reduce the speed from 200 r.p.m. to 100 r.p.m., assuming the friction couple to vary directly as the speed, the given value being its value at a speed of 200 r.p.m.

We have $I_A = \frac{2 \times 2240 \times 9}{32·2} = 1252$ engineers' units, $I_B = \frac{2240}{32·2} = 69·56$ engineers' units, and $r_B = 4$.

Hence $I_e = 1252 + 16 \times 65·96 = 2365$ engineers' units.

The friction couple on the shaft *B* is 49 8 lb.-ft. and the equivalent couple on *A* is $4 \times 49·8 = 199·2$ lb.-ft. The total couple applied to *A* to overcome friction is $199·2 + 99·6 = 298·8$ lb.-ft.

(1) The energy at 200 r.p.m. is $\frac{1}{2} I_e \omega_A{}^2$

$$= \frac{1}{2} \times 2365 \times \left(\frac{200}{60} \times 2\pi\right)^2$$

$$= 518{,}700 \text{ ft.-lb.}$$

(2) Retarding couple $= 298·8$ ft.-lb.

Angular retardation $= \frac{\text{Couple}}{\text{Moment of inertia}}$

$$= \frac{298·8}{2365}$$

$$= 0·1264 \text{ radians/sec./sec.}$$

Time to come to rest $= \frac{\text{Angular velocity}}{\text{Angular retardation}}$

$$= \frac{200 \times 2\pi}{60 \times 0·1264}$$

$$= 165·8 \text{ seconds.}$$

(3) Resultant couple $= \text{Applied couple} - \text{Friction couple}$

$$= 2000 - 298{\cdot}8$$

$$= 1701 \text{ lb.-ft.}$$

But $\text{Couple} = I \times \text{Angular acceleration}$

$$\therefore \text{ Angular acceleration} = \frac{1701}{2365}$$

$$= 0{\cdot}7193 \text{ radians/sec./sec.}$$

(4) Friction couple at a speed of ω radians per second

$$= 298{\cdot}8 \times \frac{60\omega}{200 \times 2\pi}$$

$$= 14{\cdot}27\omega \text{ lb.-ft.}$$

$$\text{Angular acceleration} = \frac{d\omega}{dt}, \text{ where } t \text{ is the time in secs.}$$

But $\text{Couple} = \text{Moment of inertia} \times \text{Angular acceleration.}$

$$\therefore\ -14{\cdot}27\omega = 2365\frac{d\omega}{dt}$$

i.e. $$\frac{1}{\omega}\cdot\frac{d\omega}{dt} = -\ 0{\cdot}006034$$

Integrating with respect to t

$$\log_e\omega = -\ 0{\cdot}006034t + \log_e A,$$

where A is an arbitrary constant.

We have then

$$\omega = Ae^{-\ 0{\cdot}006034t}$$

when $t = 0$, $\omega = \dfrac{200 \times 2\pi}{60} = 20{\cdot}95.$

Substituting, we have

$$20{\cdot}95 = Ae_1{}^0 \text{ or } A = 20{\cdot}94$$

$$\therefore \quad \omega = 20{\cdot}95e^{-\ 0{\cdot}006034t}$$

This shows that ω is never zero; however large the value of t, the right-hand side is positive. Actually ω will become smaller and smaller as t increases, and after a time will differ from zero by an insignificant amount. When the speed is 100 r.p.m., $\omega = 10{\cdot}47$ and

$$e^{-0{\cdot}006034t} = \tfrac{1}{2}$$

i.e. $0{\cdot}006034t \log_{10}e = \log_{10}2$

$$\therefore\ t = \frac{0{\cdot}3010}{0{\cdot}006034 \times 0{\cdot}4343}$$

or $$t = \underline{115 \text{ seconds.}}$$

230. **Free Vibrations of a Uniform Horizontal Rod Freely Supported at its Ends.** Consider again the case of Art. 211, assuming the rod to be uniform. In the position $AP'B$, Fig. 276, an element of length Δx in. at P has an acceleration of $\frac{\partial^2 y}{\partial t^2}$ downwards, the differentiation being partial because x is constant, P being fixed in the rod. The shape of the rod at any instant is determined by the inertia forces, i.e. the reversed mass-accelerations acting on it. At P the inertia force is $-\frac{w}{12g}\frac{\partial^2 y}{\partial x^2}$ per inch run. As in Art. 215, $EI\frac{\partial^4 y}{\partial x^4}$ is the load per inch run and neglecting gravity forces

$$EI\frac{\partial^4 y}{\partial x^4} = -\frac{w}{12g}\frac{\partial^2 y}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad (\text{XIV}.103)$$

To solve this we put $y = XT$ where X is a function of x alone and T is a function of t alone.

Substituting and separating the variables

$$\frac{12gEI}{wX}\frac{d^4X}{dx^4} = -\frac{1}{T}\frac{d^2T}{dt^2} \quad . \quad . \quad . \quad . \quad . \quad (\text{XIV}.104)$$

i.e. a function of x only equals a function of t only. This can only be true if each function is a constant. We shall write the constant n^2 taking it as positive, because we know that the beam vibrates, and equating the constant to the right-hand side must give an equation of the type (III.1). Equating each side to n^2 and rearranging

$$\left.\begin{aligned} \frac{d^2T}{dt^2} + n^2T &= 0 \\ \text{and} \qquad \frac{d^4X}{dx^4} &= m^4X \end{aligned}\right\} \quad . \quad . \quad . \quad (\text{XIV}.105)$$

Where $m^4 = \frac{n^2w}{12gEI}$. The solution of the first of these is $T = L \cos nt + M \sin nt$, that of the second is found in Art. 215 for the same boundary conditions; it is $X = D \sin mx$, L, M, and D being arbitrary constants. The solution of (XIV.104) is then

$$\begin{aligned} y &= XT \\ &= D \sin mx(L \cos nt + M \sin nt) \end{aligned}$$

i.e.
$$y = \sin mx(a \cos nt + b \sin nt) \quad . \quad . \quad \text{(XIV.106)}$$

$a = DL$ and $b = DM$ being arbitrary constants.

When $x = l$, $y = 0$ for all values of t, so that from (XIV.106) $\sin ml = 0$. For motion to take place in a normal mode, therefore, $ml = r\pi$, $r = 1, 2, 3, 4$, etc. . . ., that is, since $n = m^2\sqrt{\frac{12gEI}{w}}$, the angular frequencies of the successive normal modes are, writing $c = \frac{\pi^2}{l^2}\sqrt{\frac{12gEI}{w}}$, $n_1 = c$, $n_2 = 4c$, $n_3 = 9c$, $n_4 = 16c$, etc. . . ., $n_r = r^2c$, . . .

When vibrating in the rth normal mode, the shape of the rod is given by

$$y_r = \sin\frac{r\pi x}{l}(a_r \cos cr^2t + b_r \sin cr^2t).$$

Since y_r satisfies (XIV.103) and the boundary conditions for all integral values of r,

$$y = y_1 + y_2 + y_3 + y_4 + \ . \ . \ . \text{ will also satisfy it.}$$

The most general solution under the given conditions is then

$$y = \sum_{r=1}^{r=\infty} \sin\frac{r\pi x}{l}(a_r \cos cr^2t + b_r \sin cr^2t) \quad . \quad . \quad \text{(XIV.107)}$$

which shows the state of the rod when the motion is the sum of all the normal modes each with its own frequency. The amplitudes and phases of the various modes will depend upon the initial conditions. The vibrations will die away, the higher frequencies at the greater rates, and in practice we need only consider the first two or three harmonics as, for example, in the case of an aeroplane strut. The constants a_r and b_r are infinite in number, since r may be any positive integer, and are determined from the initial conditions. From (XIV.107).

$$\frac{\partial y}{\partial t} = \sum_{1}^{\infty} \sin\frac{r\pi x}{l}(-a_r cr^2 \sin cr^2 t + b_r cr^2 \cos cr^2 t) \quad . \quad \text{(XIV.108)}$$

Let us suppose that the rod is held deflected in the shape $y = f(x)$, where $f(x)$ is any possible expression for deflection of the rod, and released at time $t = 0$. At this instant $\frac{\partial y}{\partial t} = 0$ everywhere and from (XIV.108) we see that $b_r = 0$ for all values of r. Also, putting $y = f(x)$, $t = 0$, $b_r = 0$ in (XIV.107)

$$f(x) = \sum_{1}^{\infty} a_r \sin\frac{r\pi x}{l}$$

This is a Fourier series of sines for $f(x)$ over the range $x = 0$ to $x = l$, and it is proved in textbooks of mathematics that

$$a_r = \frac{2}{l}\int_0^l f(x) \sin\frac{r\pi x}{l}\,dx \quad . \quad . \quad . \quad \text{(XIV.109)}$$

from which the amplitudes of those normal modes which are present in the vibrations can be found. We shall show how to do this by assuming that the rod is released from the form of a parabola $y = f(x) = dx(l - x)$ where d is constant. Substituting for $f(x)$

$$a_r = \frac{2d}{l}\int_0^l x(l - x) \sin\frac{r\pi x}{l}\,dx$$

To evaluate the integral we first evaluate $I_1 = \int_0^l lx \sin\frac{r\pi x}{l}\,dx$,

$$I_2 = \int_0^l x \cos\frac{r\pi x}{l}\,dx, \text{ and } I_3 = \int_0^l x^2 \sin\frac{r\pi x}{l}\,dx.$$

Integrating by parts $I_1 = -\frac{l^2}{r\pi}\left[x \cos\frac{r\pi x}{l} - \int \cos\frac{r\pi x}{l}\,dx\right]_0^l$

$$= -\frac{l^3}{r\pi}\cos r\pi$$

$$I_2 = \frac{l}{r\pi}\left[x \sin\frac{r\pi x}{l} - \int sin\frac{r\pi x}{l}\,dx\right]_0^l = \frac{l^2}{r^2\pi^2}(\cos r\pi - 1)$$

and $I_3 = -\frac{l}{r\pi}\left[x^2 \cos\frac{r\pi x}{l} - 2I_2\right]_0^l$

$$= -\frac{l^3}{r\pi}\cos r\pi + \frac{2l^3}{r^3\pi^3}\cos r\pi - \frac{2l^3}{r^3\pi^3}$$

Thus
$$\int_0^l x(l-x)\sin\frac{r\pi x}{l}\,dx = I_1 - I_3$$
$$= \cos r\pi\left(-\frac{l^3}{r\pi}+\frac{l^3}{r\pi}-\frac{2l^3}{r^3\pi^3}\right)+\frac{2l^3}{r^3\pi^3}$$
$$= \frac{2l^3}{r^3\pi^3}(1-\cos r\pi)$$
$$= \frac{4l^3}{r^3\pi^3} \text{ when } r \text{ is odd}$$
$$= 0 \text{ when } r \text{ is even}$$

and
$$a_r = \frac{8dl^2}{r^3\pi^3} \text{ when } r \text{ is odd}$$
$$a_r = 0 \text{ when } r \text{ is even.}$$

The central deflection Δ of the parabola

$$y = dx(l-x) \text{ is } \Delta = \frac{dl^2}{4} \text{ so that when } r \text{ is odd}$$
$$a_r = \frac{32\Delta}{r^3\pi^3}.$$

Substituting these values of a_r, and also $b_r = 0$, in (XIV.107) we have

$$y = \frac{32\Delta}{\pi^3}\left[\sin\frac{\pi x}{l}\cos ct + \frac{1}{3^3}\sin\frac{3\pi x}{l}\cos 3^2ct + \frac{1}{5^3}\sin\frac{5\pi x}{l}\times \cos 5^2ct + \ldots \text{ to } \infty \ldots\right]$$

where c has the value given above. This motion is mainly of the first normal mode. Had we put $f(x) = \Delta\sin\frac{\pi x}{l}$ we should have had motion of the first normal mode alone.

231. **Free Lateral Vibrations of a Taut Wire or String.** Wires are used as tension members in the structure of an aeroplane. Air resistance has a damping effect on mechanical vibrating systems, but in the case of vibrating parts of an aeroplane there is a high resistance which may, however, in some cases have an anti-damping effect by helping instead of hindering the vibrations. With wires of circular section, such as are used on modern monoplanes, this effect does not occur, but streamlined wires were used on biplanes, and in some cases

wires were in a continual state of vibration and produced musical notes. Consider a wire of non-circular section, such as oval or streamline. The wind comes in the direction of the longest diameter which is directed fore and aft. If the wire is vibrating sideways across the direction of the wind there is a sideways component in the relative wind which produces a sideways force on the wire against the direction of motion and thus, with a circular section, helps to damp out the vibrations; but with a streamline section and a slightly oblique relative wind the aerodynamic effect may reverse the direction of the side component of the force and thus have an anti-damping effect. This is equivalent to a forced vibration and is in resonance with the frequency of vibration of the wire.

We shall find the normal modes of vibration of a taut uniform wire weighing w lb. per in., l inches long, and stretched under

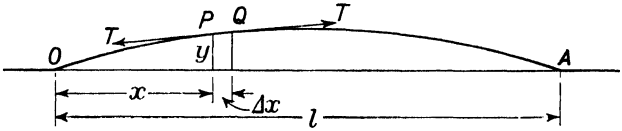

FIG. 293. VIBRATING WIRE OR STRING

a tension of T lb. weight. T is large enough not to vary appreciably owing to small displacements of the wire; OPA, Fig. 293, represents the moving wire. OA is its position of rest. Let x in. and y in. be the co-ordinates of P as shown, and let Δx be the length of an element PQ of the wire. The horizontal motion of the element will be negligible. The element is acted upon by two tensions T, one at P and one at Q along their tangents at these points. The tension at P has a downward component. Since the slope at all parts of the wire is very small this component is $T\frac{\partial y}{\partial x}$. The vertical component of the tension at Q acts upwards and its magnitude is $T\frac{\partial y}{\partial x} + \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)\Delta x$, the differential coefficients being partial because we are dealing with an instantaneous position. Thus the net upward component of force on PQ is $\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)\Delta x = T\frac{\partial^2 y}{\partial x^2}\,\Delta x$. Its acceleration upwards is $\frac{\partial^2 y}{\partial t^2}$, the differentiation being partial because we are dealing with a bit of wire whose x co-ordinate is fixed. Since Force = Mass times Acceleration we have

$$T\frac{\partial^2 y}{\partial x^2}\Delta x = \frac{1}{12}\,\frac{w}{g}\,\Delta x\,\frac{\partial^2 y}{\partial t^2}$$

i.e. $$\frac{\partial^2 y}{\partial x^2} = \frac{w}{12gT}\,\frac{\partial^2 y}{\partial t^2}, \text{ or writing } c = \sqrt{\frac{12gT}{w}}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2 y}{\partial t^2} \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV.110)}$$

We could solve this partial differential equation by the method used in Art. 230, but as we are dealing with the separate normal modes of motion we may assume that $y = y_0 \sin(\omega t + \alpha)$ will represent the motion of a point whose position on the wire is fixed and whose amplitude is y_0. As y_0 varies from point to point it is a function of x and we may write $y = \sin(\omega t + \alpha)f(x)$. α is an arbitrary constant and the value of ω has to be determined. Substituting for y

$$\sin(\omega t + \alpha)\,\frac{d^2}{dx^2}\,f(x) = -\frac{\omega^2}{c^2}\sin(\omega t + \alpha)\,f(x)$$

or $$\frac{d^2 f(x)}{dx^2} + \frac{\omega^2}{c^2}f(x) = 0 \quad . \quad . \quad . \quad \text{(XIV.111)}$$

i.e. an ordinary differential equation whose solution is $f(x) = a\sin\frac{\omega x}{c} + b\cos\frac{\omega x}{c}$. The complete solution of (XIV.110) is therefore

$$y = \sin(\omega t + \alpha)\left(a\sin\frac{\omega x}{c} + b\cos\frac{\omega x}{c}\right) \quad . \quad \text{(XIV.112)}$$

where a, b and α are arbitrary constants. When $x = 0$, $y = 0$ for all values of t, hence $b = 0$ and

$$y = a\sin(\omega t + \alpha)\sin\frac{\omega x}{c} \quad . \quad . \quad . \quad \text{(XIV.113)}$$

Again, for all values of t, $y = 0$ when $x = l$, so that $a\sin\frac{\omega l}{c} = 0$.

As we suppose the wire to be vibrating, $a \neq 0$, so that $\sin\frac{\omega l}{c} = 0$

The values of ω which satisfy this are $\omega_1 = \frac{\pi c}{l}$, $\omega_2 = \frac{2\pi c}{l}$

$\omega_3 = \frac{3\pi c}{l}, \ldots \omega_r = \frac{r\pi c}{l}, \ldots$ The angular frequency of vibration is, for the rth normal mode,

$$\omega_r = \frac{r\pi}{l}\sqrt{\frac{12gt}{w}} \text{ radians per sec.} \qquad (\text{XIV}.114)$$

and the form of the curve of the wire is given at any time t sec. by

$$y_r = a_r \sin\left(\frac{r\pi c}{l}t + \alpha\right)\sin\frac{r\pi x}{l} \qquad (\text{XIV}.115)$$

If the wire is set vibrating by being released from a position of rest $\frac{\partial y}{\partial t} = 0$ for all values of x when $t = 0$, i.e., $\cos\alpha = 0$, thus $\alpha = \frac{\pi}{2}$ and

$$y_r = a_r \cos\frac{r\pi ct}{l}\sin\frac{r\pi x}{l} \qquad (\text{XIV}.116)$$

This gives the state of the wire when vibrating in the rth normal mode. For the general motion in which all possible modes may be present

$$y = \sum_{r=1}^{r=\infty} a_r \cos\frac{r\pi ct}{l}\sin\frac{r\pi x}{l} \qquad (\text{XIV}.117)$$

The values of a_1, a_2, a_3, . . . etc., depend upon the shape of the wire at time $t = 0$.

If it is released from the shape $y = f(x)$ at time $t = 0$, this becomes

$$f(x) = \sum_{r=1}^{r=\infty} a_r \sin\frac{r\pi x}{l}$$

which is the Fourier expansion of $f(x)$ in a series of sines over the range $x = 0$ to $x = l$ and a_r is given by (XIV.109). If the wire has its middle point held sideways at a displacement d in. and is released at time $t = 0$, $f(x) = 2dx/l$ from $x = 0$ to $x = l/2$ and $2d(1 - x/l)$ from $x = l/2$ to $x = l$, and the reader should be able to prove that

$$f(x) = \frac{8d}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l} + \frac{1}{5^2}\sin\frac{5\pi x}{l} - \frac{1}{7^2}\sin\frac{7\pi x}{l} + \ldots\right] \qquad (\text{XIV}.118)$$

The solution of (XIV. 110) is therefore

$$y = \frac{8a}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{1}{3^2}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l} + \frac{1}{5^2}\sin\frac{5\pi x}{l}\cos\frac{5\pi ct}{l}\ \ldots\right] \quad . \quad . \quad . \quad . \quad \text{(XIV.119)}$$

The first term representing vibration in the fundamental mode represents about 90 per cent of the total displacement.

232. **Other Vibrating Systems.** The methods of the two preceding sections are applied to many cases of vibrating bodies. We shall indicate the method of obtaining the equation of motion in the cases of axial vibrations of a uniform bar and torsional vibrations of a uniform straight circular shaft.

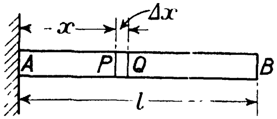

FIG. 294A. TORSIONAL VIBRATIONS OF A HEAVY SHAFT

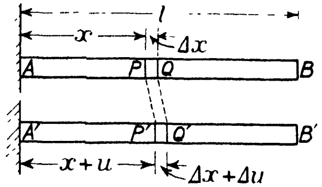

FIG. 294B. LONGITUDINAL VIBRATIONS OF A UNIFORM BAR

Consider first the circular shaft in Fig. 294A. Consider the portion PQ of the shaft of length Δx, P being at a distance x in. from a fixed section at A. Let r in. be the radius and l in. the length of the shaft; let w lb. be the weight of 1 cub. in. of the material. The moment of inertia of the mass of the slice about its axis is $\frac{w}{g}\pi r^2\Delta x \times \frac{r^2}{2} = \frac{\pi}{2g}wr^4\Delta x$ mass-inch² units. Let C be the torque in lb.in. at the section P; $C + \frac{\partial C}{\partial x}\Delta x$ will be that at Q. Also let θ be the angle of twist about the axis of the shaft relative to the fixed section at A. $\theta + \frac{\partial \theta}{\partial x}\Delta x$ is the angle of twist at Q. The slice between P and Q has an angular acceleration $\frac{\partial^2\theta}{\partial t^2}$ radians per sec. Its equation of motion is

Couple producing acceleration = moment of inertia of mass × angular acceleration.

Thus $$\frac{\partial C}{\partial x}\Delta x = \frac{1}{12} \times \frac{\pi}{2g} wr^4 \Delta x \times \frac{\partial^2\theta}{\partial t^2},$$

the multiplier $\frac{1}{12}$ appearing because the units on the left are lb.-in. and those on the right would otherwise be mass-in.^{2} units Simplifying

$$\frac{\partial C}{\partial x} = \frac{\pi w r^4}{24g}\frac{\partial^2\theta}{\partial t^2} \quad . \quad . \quad . \quad . \quad (\text{XIV}.120)$$

If C_0 is the couple which would produce unit twist in the whole shaft when stationary,

$$C = C_0 l \frac{\Delta\theta}{\Delta x} = C_0 l \frac{\partial\theta}{\partial x} \text{ and, substituting,}$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\pi w r^4}{24 g C_0 l}\frac{\partial^2\theta}{\partial t^2} \quad . \quad . \quad . \quad . \quad (\text{XIV}.121)$$

which is the required equation of motion and is identical in form with (XIV.110).

Now consider the bar AB shown in Fig. 294B, which we suppose to be in a state of longitudinal, or axial, vibration. The upper figure shows the bar at rest and the lower figure shows it when displaced during vibration. Corresponding sections are marked by the same letters with dashes in the lower figure. The rod is l in. long, of cross-sectional area A sq. in., of material with Young's modulus of elasticity E lb. per sq. in. and weighing w lb. per cub. in. Let AP be x in. long stretching to $x + u$ in. in the lower figure. Let PQ be a slice of length Δx in. becoming $\Delta x + \Delta u$ in. in the lower figure. This slice has mass $\frac{wA\Delta x}{g}$ engineers' units at time t seconds the length AP has changed to $A'P'$ and the acceleration of P to the right is $\frac{\partial^2 u}{\partial t^2}$ in. per sec. per sec. The length Δx has stretched to $\Delta x + \Delta u$, so that the strain in that length is $\frac{\Delta u}{\Delta x}$, which becomes $\frac{\partial u}{\partial x}$ in the limit. If T_P lb. is the tension at P' we have from stress = strain × Young's modulus

$$T_P/A = E\frac{\partial u}{\partial x} \text{ where } E \text{ is Young's modulus of elasticity in lb. per sq. in.}$$

or $$T_P = AE\frac{\partial u}{\partial x} \text{ lb. and the tension } T_Q \text{ at } Q' \text{ is}$$

$$T_Q = T_P + \frac{\partial T_P}{\partial x}\,\Delta x$$

i.e.
$$T_Q = T_P + AE\,\frac{\partial^2 u}{\partial x^2}\,\Delta x \text{ lb.}$$

The resultant force on the element $P'Q'$ to the right is $T_Q - T$ $= AE\dfrac{\partial^2 u}{\partial x^2}\,\Delta x$ lb. Then

Force = mass × acceleration, gives

$$AE\frac{\partial^2 u}{\partial x^2}\,\Delta x = \frac{wA\,\Delta x}{g} \times \frac{1}{12}\frac{\partial^2 u}{\partial t^2}$$

or
$$\frac{\partial^2 u}{\partial x^2} = \frac{w}{12gE}\,\frac{\partial^2 u}{\partial t^2} \quad . \quad . \quad . \quad . \quad (\text{XIV.122})$$

which is the equation of motion and is of the form (XIV.110).

233. **Rayleigh's Method of Finding the Frequency of Vibration in the First Mode.** The exact determination of the modes of vibration of a system is often very difficult, if not impossible. In such cases a method due to Rayleigh, used with judgment, enables us to determine approximately the frequency of the first normal mode usually called the *fundamental* mode. When we know that each point in the vibrating body is moving with simple harmonic motion, we do not need to form the equation of motion in order to determine the frequency. Thus in the case of torsional vibrations in Art. 50, if we assume the angular frequency of vibration to be ω radians per second and assume the amplitude of swing of the pulley to be θ_0, i.e. $\theta = \theta_0 \sin \omega t$, we have by the equation of energy.

Kinetic energy in unstrained position = Elastic energy at end of swing

$$\tfrac{1}{2}I\omega^2\theta_0^{\,2} = \tfrac{1}{2}C_0\theta_0^{\,2}$$

from which
$$\omega = \sqrt{\frac{C_0}{I}}.$$

Where there are more degrees of freedom than one and we do not know the relative displacements, we make some kind of reasonable assumption about them and use the above form of the energy equation. This is known as Rayleigh's method. The equation may be written

Maximum kinetic energy = maximum potential energy . . . (XIV.123)

We shall apply the method to find the first mode of vibration of the uniform rod of Art. 211. The best choice for the form of the rod at the end of a swing would be a half sine wave but, as this is the exact shape, as shown in Art. 215, we chose another shape to illustrate the method. With the notation of Art. 211 we shall assume that the shape at the end of a swing is given by $y = \frac{4dx}{l^2}(l - x)$. This is a poor choice; it assumes the bending moment to be the same at all points, whereas we know that it is zero at the ends and very little near them.

$$K_{max} = \text{maximum kinetic energy}$$
$$= \int_o^l \frac{w}{24g}\,\omega^2 y^2 dx$$
$$= \int_o^l \frac{w\omega^2}{24g} \times \frac{16d^2}{l^4}\,x^2(l - x)^2\,dx$$
$$= \frac{w\omega^2 d^2 l}{45g}\ \text{in.-lb.}$$
$$P_{max} = \frac{1}{2EI}\int_o^l M^2 dx, \text{ as proved in books on "Strength of Materials"}$$
$$= \frac{1}{2EI} \times E^2 I^2 \int_o^l \left(\frac{d^2y}{dx^2}\right)^2 dx$$
$$= \frac{EI}{2}\int^l \frac{64d^2}{l^4}\,dx = \frac{32EId^2}{l^3}$$

and equating these and solving for ω^2

$$\omega^2 = \frac{1440gEI}{\omega l^4}$$

or
$$\omega = \frac{10{\cdot}95}{l^2}\sqrt{\frac{12gEI}{w}}.$$

The correct result is $\omega = \frac{\pi^2}{l^2}\sqrt{\frac{12gEI}{w}}$

and as $\pi^2 = 9{\cdot}87$ the error is about 11 per cent. Now we shall assume a shape more like that of the rod. There must be points of inflection at the ends and no slope at the middle. If we assume $\dfrac{d^2y}{dx^2} = \dfrac{4d}{l^2}\,x(l-x)$ this will ensure points of inflection at the ends. Integrating and determining the constant so that $\dfrac{dy}{dx} = 0$ when $x = l/2$ we have

$$\frac{dy}{dx} = \frac{4d}{l^2}\left(\frac{lx^2}{2} - \frac{x^3}{3} - \frac{l^3}{12}\right) \text{ and integrating again}$$

$y = \dfrac{4d}{l^2}\left(\dfrac{lx^3}{6} - \dfrac{x^4}{4} - \dfrac{l^3x}{12}\right)$, the constant of integration being zero because $y = 0$ when $x = 0$. The condition $y = 0$ when $x = l$ is also satisfied; this is not accidental but is the result of our assumption. Using these values

$$\begin{aligned} K_{max.} &= \int_o^l \frac{w\omega^2y^2}{24g}\,dx \\ &= \frac{16w\omega^2d^2}{24 \times 144gl^4}\int_o^l (2lx^3 - x^4 - l^3x)^2\,dx \\ &= \frac{31w\omega^2d^2l^5}{216 \times 630g}\text{ in. lb.} \end{aligned}$$

also
$$\begin{aligned} P_{max.} &= \frac{1}{2EI}\int_o^l M^2\,dx = \frac{16EId^2}{2l^4}\int_o^l x^2(l-x)^2\,dx \\ &= \frac{4}{15}EId^2l. \end{aligned}$$

Equating these and solving for ω^2 we find

$$\omega^2 = \frac{4 \times 216 \times 630gEI}{15 \times 31\,wl^4}$$

or
$$\omega = \frac{9{\cdot}88}{l^2}\sqrt{\frac{12gEI}{w}}.$$

This differs by less than 1 per cent from the true value given above.

When choosing a likely shape attention must be paid to points of inflection. A half wave of a sine curve is the obvious selection for the above example but, knowing it to be the true solution, we avoided it in order to illustrate two other shapes, one of which proved to be bad and the other good. It can be shown that Rayleigh's approximation is always a little too high so that the lowest approximation found by assuming different curves is the best. For a rod built in at one end and free at the other the assumed curve should be horizontal at the fixed end and have a point of inflection at the other. Taking the case of a uniform rod, as shown in Fig. 295, the assumed shape $y = d\left(1 - \cos\frac{\pi x}{2l}\right)$ is a good one having zero slope at $x = 0$ and a point of inflection at $x = l$. We shall, however, leave this case to be worked out by the reader and shall assume instead the curve for which $\frac{d^2y}{dx^2} = \frac{4k}{l^2}(l - x)$ giving the necessary point of inflection at $x = l$. Integrating and making $\frac{dy}{dx}$ vanish when $x = 0$, $\frac{dy}{dx} = \frac{4k}{l^2}\left(lx - \frac{x^2}{2}\right)$.

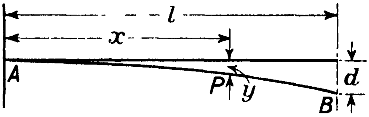

FIG. 295. OSCILLATING CANTILEVER

Integrating again and putting $y = 0$ when $x = 0$, $y = \frac{4k}{l^2}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$. Then

$$K_{max.} = \int_0^l \frac{w\omega^2y^2}{24g}\,dx \text{ in. lb.}$$

$$= \frac{16w\omega^2k^2}{24gl^4}\int_0^l \left(\frac{lx^2}{2} - \frac{x^3}{6}\right)dx$$

$$= \frac{11w\omega^2k^2l^3}{630g}$$

and
$$P_{max.} = \frac{1}{2EI}\int_0^l M^2dx = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$$

$$= \frac{8EIk^2}{l^4}\int_0^l (l - x)^2\,dx$$

$$= \frac{8}{3l}EIk^2.$$

Equating these and solving for ω^2,

$$\omega^2 = \frac{140}{11} \times \frac{12gEI}{wl^4}$$

and
$$\omega = \frac{3{\cdot}57}{l^2}\sqrt{\frac{12gEI}{w}} \text{ radians per sec}$$

The correct result has 3·52 instead of 3·57 (Ex. 14, 57).

For the case of the vibrating wire of Art. 230 we shall assume that when vibrating in its fundamental mode the wire at the end of its swing takes the form $y = d \sin \frac{\pi x}{l}$. This assumption is correct and the method will give us the correct result.

$$\begin{aligned} K_{max.} &= \int_o^l \frac{w\omega^2 y^2}{24g}\, dx \\ &= \frac{w\omega^2 d^2}{24g}\int_o^l \sin^2 \frac{\pi x}{l}\, dx \\ &= \frac{w\omega^2 d^2 l}{48g} \text{ in.-lb.} \end{aligned}$$

The potential energy is Te where e is the total stretch in the wire, that is the difference between the lengths of the curve OPA and the straight line OA, Fig. 293. Thus

$$e = \int_o^l \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx - l, \text{ or since } \frac{dy}{dx} \text{ is small,}$$

$$\begin{aligned} e &= \int_o^l \left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right] dx - l \\ &= \tfrac{1}{2}\int^l \left(\frac{dy}{dx}\right)^2 dx \\ &= \frac{\pi^2 d^2}{2l^2}\int_o^l \cos^2 \frac{\pi x}{l}\, dx \\ &= \frac{\pi^2 d^2}{4l} \end{aligned}$$

$\therefore \qquad P_{max.} = \frac{\pi^2 d^2 T}{4l}$, and, equating $P_{max.}$ to $K_{max.}$ and solving for ω^2,

$$\omega^2 = \frac{\pi^2 T \times 12g}{l^2 w}$$

and

$$\omega = \frac{\pi}{l}\sqrt{\frac{12gT}{w}} \text{ radians per second.}$$

As a final example of the use of Rayleigh's Method we shall take the case of a pulley A of moment of inertia I_A, engineers' units at the end of a shaft SS_c of length l in., and of appreciable moment of inertia I_s engineers' units about the axis of the shaft, Fig. 27. We shall assume that the angle of twist θ at a distance x from the fixed end S is given by $\theta = \theta_A x/l$ as with a weightless shaft. If ω is the angular frequency of the fundamental mode $\theta_A = \theta_o \sin \omega t$ and $\dot{\theta}_A = \omega\theta_o \cos \omega t$ where θ_o is the angular amplitude at A, $\theta = \frac{x}{l}\theta_A = \frac{\omega x}{l}\theta_o \cos \omega t$. The maximum angular speeds at P and A are respectively $\theta_{max} = \frac{\omega\theta_o x}{l}$ and $\omega\theta_o$. The maximum kinetic energy is therefore $\frac{1}{2} I_A \omega^2\theta_o^2 + \int_o^l \frac{I_s}{2l^3}\omega^2\theta_o^2 x^2 dx$, so that

$$K_{max} = \tfrac{1}{2}\omega^2\theta_o^2\left[I_A + \frac{I_s}{l^3}\int_o^l x^2 dx\right]$$
$$= \tfrac{1}{2}(I_A + \tfrac{1}{3}I_s)\,\omega^2\theta_o^2.$$

If C_o is the couple required to give one radian twist to the shaft when at rest, the energy stored up in the shaft when the pulley is displaced through an angle θ_o is $\frac{1}{2}C_o{}^2\theta_o{}^2$. As we have assumed that the end shape when vibrating is the same as when the pulley is held displaced, this the maximum potential energy, hence

$P_{max.} = \frac{1}{2}C_o\theta_o{}^2$ and equating P_{max} to $K_{max.}$ and solving for ω^2

$$\omega^2 = \frac{C_o}{I_A + \frac{1}{3}I_s}$$

or

$$\omega = \sqrt{\frac{C_o}{I_A + I_s/3}}.$$

This result implies that the moment of inertia of the shaft is equivalent to a moment of inertia of $\frac{1}{3}$rd that amount at the pulley. This is approximately true if I_A is much greater than I_s. The factor $\frac{1}{3}$ came in when we integrated x^2 in the expression for K_{max}; with another law instead of $\theta = \theta_A x/l$ we should have obtained another factor, $P_{max.}$ would also have been different and we should have obtained a different result. In the case of a mass of W lb. weight on the end of a spring weighing w lb., the assumption that the speed of an element of spring distance x from the fixed end varies directly as x gives us the result that the mass of the spring may be neglected if we consider the mass at the end to be of $W + w/3$ lb. weight. If the shape of the angular deflection graph against x had not been assumed to be straight, as in the static position, we should, instead of the relation $P_{max.} = \frac{1}{2}C_o\theta_o^2$, use the relation $P_{max.} = \int_o^l \frac{C^2}{2GI}\,dx$, where G is the modulus of rigidity and I the polar moment of inertia of the cross-section of the shaft and $C = C_o l \frac{d\theta}{dx}$. In the above case in the end position $\frac{d\theta}{dx} = \frac{\theta_o}{l}$ and $C = C_o\theta_o$ as is evident. Thus $P_{max} = \frac{C_o^2 l\theta_o^2}{2GI}$, but $C_o = \frac{GI}{l}$, so that $P_{max.} = \frac{C_o\theta_o^2}{2}$ as above.

234. **Vibrations of the Gyro Compass about the Meridian.** Referring to the triangle OPT, Fig. 270, let ω be the angular speed of rotation about the axis NS; this is clockwise when viewed from S and is represented by $\overrightarrow{OT}$. Resolving this along OP and PT we have (1) $\overrightarrow{OP}$ of magnitude $\omega \sin L$, and (2) $\overrightarrow{PT}$ of magnitude ω cos L. (1) is turning the meridian about OP at the rate $\omega \sin L$, and (2) is rotating the horizontal plane at P about PT. Let PR_1, Fig. 296 be the mean position of the rotor axis in which it is pointing to the north along the horizontal line PT and suppose the axis to be displaced slightly and set vibrating. Let PR_2 be its position at time t sec. VPT is the meridian plane and PTM the horizontal plane at P. The horizontal tilt, or azimuth, of PR_2 is $\widehat{R_1PM} = \alpha$, and the vertical tilt or altitude is $\widehat{R_2PM} = \beta$; R_2M is vertical as is OPV. Owing to α changing PR_2 is rotating about PV at the rate $\frac{d\alpha}{dt}$ relative to the earth. We saw above that owing to the earth's rotation PR_2 is turning at the rate $\omega \sin L$ about PV, in the clockwise sense seen from below. The rate of turning

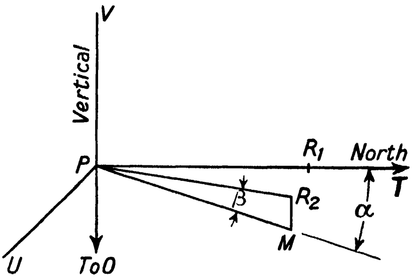

FIG. 296. SMALL VIBRATIONS OF GYRO COMPASS

about PV relative to the fixed stars is $\dfrac{d\alpha}{dt} - \omega \sin L$ in the clockwise sense seen from above. To produce this precession a couple of amount $M\left(\dfrac{d\alpha}{dt} - \omega \sin L\right)$ must act about an axis PU perpendicular to PV and PR_2; M is the angular momentum of spin of the rotor. Taking the sense of spin of the rotor to be that of Fig. 271, the required couple is clockwise seen from U. This is opposite to the sense of the horizontal couple produced by the ballistic (Figs. 271 and 272). Since this is caused by the tilt β we shall assume it to be proportional to β and its magnitude to be $k_1\beta$. We have then

$$\text{Couple} = \text{angular momentum} \times \text{rate of precession}$$

or

$$-k_1\beta = M\left(\frac{d\alpha}{dt} - \omega \sin L\right)$$

i.e.

$$\frac{d\alpha}{dt} + \frac{k_1}{M}\beta = \omega \sin L \quad . \quad . \quad . \quad . \quad \text{(XIV.124)}$$

Owing to change in β, PR_2 is rotating at the rate $\dfrac{d\beta}{dt}$ about UP, counterclockwise from U. Owing to the earth's rotation PM is rotating about PR, with angular speed $\omega \cos L$ clockwise from P. M is moving in the sense of $\overrightarrow{R_2M}$ at speed $\omega \cos L\, \overline{OM} \sin \alpha$, i.e. $\omega \cos L\, \overline{OM}\alpha$ since α is small. OM is rotating away from OR_2 at the angular speed $\omega \cos L\, \overline{OM}\alpha/\overline{OM} = \omega\alpha \cos L$, and the speed of rotation about UP relative to the fixed stars is $\dfrac{d\beta}{dt} - \omega\alpha \cos L$ in the sense of $\dfrac{d\beta}{dt}$ positive. The couple producing this is that of the ballistic about the vertical axis. Calling this $k_2\beta$, since the distance of B from the vertical axis is a small proportion of that from the horizontal axis k_2 will be small relative to k_1. This couple is counterclockwise seen from above which is opposite to the sense required, hence

$$-k_2\beta = M\left(\frac{d\beta}{dt} - \omega\alpha \cos L\right),$$

i.e.

$$\frac{d\beta}{dt} + \frac{k_2\beta}{M} = (\omega \cos L)\alpha \quad . \quad . \quad . \quad \text{(XIV.125)}$$

From (XIV.124),

$$\beta = -\frac{M}{k_1}\left(\frac{d\alpha}{dt} - \omega \sin L\right) \text{ and } \frac{d\beta}{dt} = -\frac{M}{k_1}\frac{d^2\alpha}{dt^2}.$$

Substituting in (XIV.125),

$$-\frac{M}{k_1}\frac{d^2\alpha}{dt^2} - \frac{k_2}{k_1}\left(\frac{d\alpha}{dt} - \omega \sin L\right) - (\omega \cos L)\,\alpha = 0$$

i.e. $$\frac{d^2\alpha}{dt^2} + \frac{k_2}{M}\frac{d\alpha}{dt} + \frac{k_1\,\omega \cos L}{M}\,\alpha = \frac{k_2\omega \sin L}{M}.$$

The term on the right can be removed by writing $\alpha' + \alpha_o$ for α where α_o is constant. This gives

$$\frac{d^2\alpha'}{dt^2} + \frac{k_2}{M}\frac{d\alpha'}{dt} + \frac{k_1\omega \cos L}{M}(\alpha' + \alpha_o) = \frac{k_2\omega \sin L}{M}.$$

Hence putting $\alpha_o = \frac{k_2}{k_1} \tan L$ the above equation becomes

$$\frac{d^2\alpha'}{dt^2} + \frac{k_2}{M}\frac{d\alpha'}{dt} + \frac{k_1\omega \cos L}{M}\,\alpha' = 0 \quad . \quad . \quad . \quad \text{(XIV.126)}$$

The rotor axis will oscillate about a mean position tilted horizontally through an angle $\frac{k_2 \tan L}{k_1}$ from the meridian. In low latitudes this is negligible, but in high latitudes $\tan L$ is large and corrections have to be applied. The error is known as the *latitude error* and is due to the damping effect of the vertical couple produced by the ballistic. The solution of (XIV.126) is by (XIV.57*c*)

$$\alpha' = A e^{-\frac{k_2 t}{2M}} \sin \frac{1}{2M}\sqrt{4Mk_1\omega \cos L - k_2^2}\,.\,t \quad . \quad \text{(XIV.127)}$$

taking $\alpha' = 0$ when $t = 0$. The periodic time of the damped vibrations is

$$T = \frac{4\pi M}{\sqrt{4Mk_1\omega \cos L - k_2^2}} \text{ sec.} \quad . \quad . \quad . \quad \text{(XIV.128)}$$

This period should be many times that of the rolling oscillation of the ship so that on the whole the effect of the roll on the compass vibrations will cancel out. The value of T may be about 1½ hr., whilst that of rolling may only be 25 sec.

EXAMPLES 14.

In these examples take $E = 30 \times 10^6$ lb. per square inch, and $w = \cdot 28$ lb.

1. A beam of steel 24 ft. long is freely supported at the ends, and carries a uniformly distributed load of 400 lb. per foot run. The moment of inertia of its cross-section about the neutral axis is 2,400 $(\text{inch})^4$ units. Find the period of transverse vibration.

2. The beam of example 1 carries a concentrated load of 10 tons at its centre. What is now its period of vibration, (1) neglecting the distributed load, (2) taking account of all the load ?

3. A steel shaft, 4 in. diameter, is of 18 ft. span between two bearings. Assuming the shaft to be freely supported at the bearings, find the lowest whirling speed.

4. The wheel of a single stage turbine weighs 150 lb. and the steel shaft has an average diameter of 2 in., and is supported on spherically seated bearings. The span is 2 ft., and the wheel is at the middle of the span. Neglecting the weight of the shaft, find the whirling speed.

5. The first whirling speed of a uniform steel shaft 4 ft. long and $\frac{1}{2}$ in. diameter, freely supported at the ends, is 1,060 revs. per min. Find the first and second critical speeds of a uniform shaft of the same material when freely supported at the ends, the span being 6 ft. and the diameter $\frac{3}{4}$ in.

6. State the quantities which affect the whirling speed of a shaft. Explain the effects on the whirling speed of varying each of these quantities.

7. Calculate the natural frequency of vibration of an elastic beam freely supported at each end. Such a beam, 30 ft. long, weighs, with its loading, 117·5 lb. per foot run. Its moment of inertia is ·1306 ft. units, and $E = 1{\cdot}2 \times 10^6$ lb. per sq. in. What is the natural frequency ? (U.L.A.)

8. A weight is attached to a spiral spring whose stiffness is 36 lb. per foot. The time of a free oscillation is $\frac{2\pi}{5}$ sec. If, when at rest, it is brought under the action of a periodic force $P = 4 \cos 4t$ lb., find the position of the weight after 3 sec. Neglect the effect of friction and of the mass of the spring itself. (U.L.A.)

9. One end of a uniform spiral spring is fixed and the other end moving, with velocity v ft per sec. If the total mass of the spring is W lb., what is its kinetic energy ?

10. Use the result of the last example, and the energy equation, to prove that, when a weight is oscillating on the end of a spiral spring, the effect of the mass of the spring may be allowed for by adding one third the mass of the spring to the mass of the weight.

11. A wheel of weight W lb. is fixed to a shaft of diameter d in., span L in, supported in two bearings at a distance a from one bearing. Deduce the critical speed and explain how the shaft can run at a higher or lower speed.

A wheel weighing 100 lb. is fixed to the centre of a length of shaft 2 in. diameter, 20 in. span. Find the critical speed.

Would there be any danger if the shaft were run at 4,000 revs. per min. ? (U.L.A.)

12. Find the first two whirling speeds of a mild steel shaft 2 in. diameter, freely supported in bearings 6 ft. apart at the ends. Show that the whirling speeds are the same as the frequencies of transverse oscillation in the first two normal modes.

13. A single cylinder vertical steam engine, 12 in. stroke, weighs, with its foundation, 2,500 lb., the piston and rod being 50 lb., and the connecting-rod 50 lb., of which one third may be regarded as concentrated at the gudgeon-pin and two-thirds at the crank-pin. The connecting-rod is $3\frac{1}{2}$ cranks and the engine is balanced for horizontal forces. The floor upon which the foundation rests is such that the frequency of vertical oscillation is 16 per sec. Determine

the amplitude of vibration when the engine is running at 450 revs. per min (U.L.A.)

14. Explain what is meant by the whirling speed of a shaft, and discuss the connection between this and the frequency of transverse vibrations. Derive the expression for the frequency in the case of a simply supported uniform shaft, uniformly loaded. In practice it is customary to take the critical speed in revolutions per minute as given by $\dfrac{\text{constant}}{\sqrt{\text{maximum static deflection}}}$ What justification is there for this expression? Obtain the probable value of the constant for the uniformly loaded shaft. (U.L.A.)

15. A horizontal beam of elastic material supported or fixed at the ends carries a load at some point of the span. Show that (neglecting the mass of the beam and the moment of inertia of the load about its own centre of mass) the period of free vibration of the load on the beam is the same as that of a simple pendulum whose length is equal to the static deflection under the load.

A steel shaft $2\frac{1}{2}$ in. diameter, with its ends freely supported in bearings 2 ft. apart, carries two steel discs at the middle of the span, weighing together 300 lb. Calculate the whirling speed. (Neglect the mass of the shaft.) (U.L.A.)

16. Find the critical speed of a steel shaft, $\frac{1}{2}$ in. diameter and 15 in. long, which carries a central load of 24 lb. Assume that the end bearings do not fix the direction of the shaft. Find the maximum stress due to dynamical loading if the speed is ·95 of the critical speed and the load is 0·01 in. out of the centre line of the shaft. Take the weight of steel as 0·28 lb per cub. in. (U.L.A.)

17. Deduce an expression for the primary whirling speed of a shaft supported in two short bearings and carrying a heavy weight *W* lb. not at the centre.

Such a shaft is 10 ft. between bearing centres, and carries a weight of 1 ton at a point 3 ft. from one support. It is 4 in. diameter, and the elastic modulus is 30×10^6 lb. sq. in. What is its whirling speed? (U.L.)

18. A steel shaft 2 in. diameter is carried at its end in swivel bearings, 36 in. from centre to centre. A wheel of mass 400 lb. is carried with its centre of mass 16 in. from one end.

Estimate the speed of whirling neglecting the mass of the shaft. Take *E* as 30×10^6 lb. per sq. in. (U.L.A.)

(Solve this example by both methods.)

19. The shaft of the previous example also carries a mass of 500 lb. at a point 10 in. from the other end of the shaft. Find the two whirling speeds and the position of the node for the second whirling speed.

20. A steel shaft 15 in. outside diameter and 12 in. inside diameter is 30 ft. long. It carries three pulleys, one at each end of moment of inertia 60 tons-ft.^{2}. and one of 90 tons-ft.^{2} at a point 12 ft. from one end. Find the first and second critical frequencies and the position of the node for the first critical speed.

21. Find the critical frequencies of torsional oscillations of the given system. Three rotating masses of 1,728 lb., 2,073 lb., and 2,419 lb. respectively and radii of gyration 36 in., 28·8 in., and 36 in., are mounted on a shaft. The first and second masses are 36 in. apart and the shaft between them is of 4·2 in. diameter. The third mass is 28·8 in. beyond the second and the shaft between them is of 3·6 in. diameter. Find also the positions of the nodes at the second critical speed.

22. Find by the method of Art. 218 the first whirling speed of the shaft in Example 17.

23. To reduce vibration an instrument is mounted on a heavy table weighing 100 lb. which is suspended from the roof by three springs, each of stiffness 9 lb. per inch. A dashpot is fitted to assist in damping the vertical vibration;

experiments on the dashpot show that when a force of 15 lb. is applied it moves at a constant rate of 1 ft. per second.

(*a*) Deduce the time required for the amplitude of any vertical natural vibration of the table to be reduced to 1 per cent of its initial value.

(*b*) If the roof from which the table is suspended vibrates vertically with an amplitude of 0·05 in. at a frequency of 1200 per minute, deduce the amplitude of vibration of the table.

The mass of the springs may be neglected. (U.L.A.)

24. A beam 15 ft. long and freely supported at the ends carries masses of 5, 2, and 4 tons respectively at points 3 ft., 7·5 ft., and 12 ft. from one end.

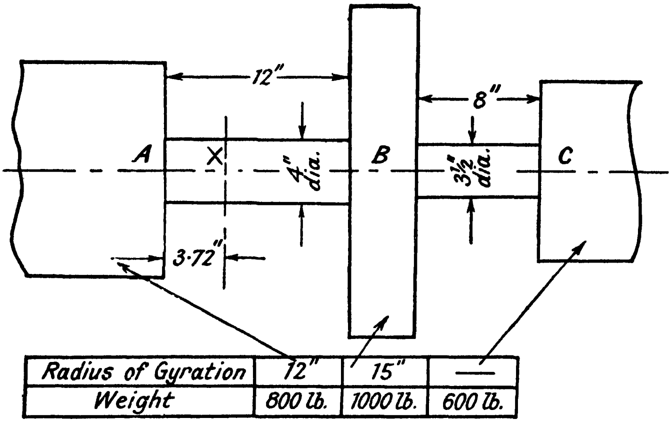

FIG. 297

(The mass of the beam itself may be neglected.) It is found that the deflection of the beam is 0·25 in. when the central load only is applied.

Deduce the natural frequency of vibration of the loaded beam, stating carefully the assumptions you have made.

If it were necessary to take the mass of the beam itself into account, indicate briefly how this might be done.

Note. The deflection y of a freely supported beam, of span l, at a point distant x from one end due to a load W distance a from that end, $a \geq x$, is given by

$$y \quad \frac{Wl^3}{6EI}\cdot\frac{x}{l}\cdot\frac{l-a}{l}\left[1-\left(\frac{x}{l}\right)^2-\left(\frac{l-a}{l}\right)^2\right] \qquad \text{(U.L.A.)}$$

25. A motor-generator set consists of two armatures A and C connected as shown in Fig. 297 with a flywheel between them at B. The modulus of rigidity for the shaft is 12×10^6 lb. per square inch. The system can vibrate torsionally with one node at X, 3·72 inches from A, the wheel B being at an anti-node. Find the position of the other node, the natural frequency of vibration, and the radius of gyration of the armature C. (I.L.A.)

26. For experimental purposes an engine weighing 100 lb. is mounted centrally between the mid-points of two parallel laminated springs supported at the ends; the springs deflect 0·5 in. under the load, transverse vibrations of the springs are controlled by dashpots, and it may be assumed that the resistance to motion is proportional to the velocity. The engine has a balanced crank 3 in. radius, which imparts simple harmonic motion in a vertical plane to a sliding piece which weighs 1 lb.

As the speed of the engine is gradually increased from zero to a final speed of 250 r.p.m., the greatest amplitude of vibration is 1·5 in. Calculate (*a*) the damping force in pounds at a speed of 1 ft. per second, (*b*) the amplitude at the final speed, (*c*) the maximum force on each spring support due to the engine and sliding piece. The weight of the engine includes the necessary proportion of the weight of the springs. (U.L.A.)

27. Show that in a simple system making free vibrations with viscous damping the ratio of the frequency of vibration to the frequency when there is no damping is given by the expression $\sqrt{\left(1 - \frac{c^2}{4km}\right)}$, where c is the damping coefficient (force per unit velocity), k is the elastic stiffness of the system, and m is the mass of the vibrating body. Explain the meaning of the term "critical damping" and show that this condition is reached when $c = 2m\omega$, where ω is the angular frequency of the system without damping. (U.L.A.)

28. A load of 2000 lb. is supported at mid-span by a simply supported horizontal beam of 24 ft. span, of I-section 10 in. deep, having a second moment of 211·5 $(\text{in})^4$ about the neutral axis. Neglecting viscous resistances and the inertia of the beam itself, calculate the natural period of vibration of the system.

An alternating load due to a mass of 12 lb. at a radius of 9 in. and revolving at 360 r.p.m. is superposed on the given load. Calculate the amplitude of the forced oscillation and the additional maximum stress in the beam due to this E for beam $= 30 \times 10^6$ lb. per square inch. (U.L.A.)

29. The lengths of the crank and connecting rod of a small internal combustion engine are respectively 2 in. and 8 in. The connecting rod weighs 1·3 lb. and its centre of mass is 2·6 in. from the centre of the crankpin. The radius of gyration about an axis passing through the mass centre, and perpendicular to the plane in which the rod moves, is 3·9 in. The speed of rotation is 3600 r.p.m.

Determine the magnitude and line of action of the force required to accelerate the connecting rod when the crank is at 45° from the inner dead centre.

Use the following scales—

1 in. to 20 ft. per second for the velocity diagram.

1 in. to 5000 ft. per second per second for the acceleration diagram.

(Klein's construction for the acceleration diagram must *not* be used.) (U.L.A.)

30. A link AB, 5 ft. long, is suspended from a point A and carries a weight of 5 lb. at B. When AB is vertical, its mid-point C is level with, and 4 ft. distant from, a crankshaft E. By means of a crank ED 1 ft. long, and a connecting rod DC 4 ft. long, rotation of the crankshaft causes an oscillation of AB. Determine the pressure on the pin at A in the position in which EDA is a straight line, when the crankshaft speed is 200 r.p.m. (U.L.)

[The method of solution is to draw the triangle of forces for the bar; there are three forces, one at the hinge, one along DC, and the reversed mass-acceleration of B.]

31. State the conditions which must be satisfied when a rigid body having plane motion is replaced by an equivalent dynamical system comprising two masses concentrated at points. A connecting rod weighs 3 lb. and the length between the gudgeon pin and crankpin centres is 8 in. The centre of mass is 3 in. from the crankpin centre, and the radius of gyration about the mass centre is 4 in. Determine the magnitudes of the two equivalent masses assuming that one of them is situated at the gudgeon-pin centre. (U.L.)

32. Show that the time of oscillation of a compound pendulum (e.g. a connecting-rod) when the knife-edge is replaced by a circular rod, radius r ft., is given by

$$T = 2\pi \sqrt{\frac{k_o^2 + (R+z)^2}{g\left(z + \frac{R}{1 - \frac{r}{R}}\right)}}$$

where R is the radius of the part in contact with the circular rod (ft.), k_0 is the radius of gyration about an axis through the C.G. (ft.), z is the distance between the centre of curvature of the part in contact with the rod and the C.G. (ft.).

Hence show that if values of $\frac{1}{T^2}$ are plotted against values of $\frac{1}{1-\frac{r}{R}}$ a straight line will result.

33. A railway carriage door has a radius of gyration of 18 in. about the vertical axis passing through the hinges, and the centre of mass is 13 in. from this axis. When the train is at rest, the door is stationary and at right angles to the side of the carriage.

Determine the angular velocity with which the door shuts if the train moves off with an acceleration of 1·5 ft. per second per second.

Neglect frictional resistance at the hinge and the force due to air pressure. (U.L. II.)

34. Two shafts of 3·5 in. diameter each carry a rotor at one end, and the other ends are connected through a gear box with a speed ratio of 4 : 1. One shaft 20 in. long carries a rotor of 1000 lb. weight with a radius of gyration of 30 in. The other shaft 25 in. long, connected to the quicker-moving side of the gear box, carries a rotor of 125 lb. weight and radius of gyration 15 in. Find the frequency of oscillation and the position of the node.

35. Show how the inertia of a system of masses positively connected together by gear trains, or otherwise, may be referred to any one selected line or shaft.

A hoisting gear with a 5 ft. diameter drum operates two cages by ropes passing from the drum over guide pulleys 3 ft. 6 in. diameter. One cage, loaded, rises whilst the other, empty, descends. The drum is driven by a motor through double reduction gearing. The particulars of the various parts are as follows—

PART	MAXIMUM SPEED	WEIGHT	RADIUS OF GYRATION	FRICTION
	r.p.m.	lb.	in.	
Motor . . .	900	400	3·5	—
Intermediate gear .	275	750	9·0	10 lb.-ft.
Drum and shaft .	50	4500	24·0	75·0 lb.-ft.
Guide pulley, each .	—	400	17·0	10·0 lb.-ft
Rope out . .	—	500	—	—
Ascending cage .	—	1800	—	100 lb.-ft.
Descending cage .	—	800	—	70 lb -ft.

Determine the total torque necessary to produce a cage acceleration of 3 ft. per second per second. (U.L.A.)

36. A uniform shaft, whose inertia may be neglected, carries 4 rotors, whose moments of inertia are respectively 100, 300, 500, and 100 engineers' units. The distances apart of successive masses are in order 18 in., 16 in., and 18 in. Find the positions of the three nodes for the third critical speed. If the diameter of the shaft is 3·5 in. and the modulus of rigidity is 12×10^6 lb. per square inch, find the corresponding frequency.

37. A spring of negligible mass, which stretches 1 in. under a tension of 2 lb., is fixed at one end and carries a mass of M lb. at the other end. If when the mass is at rest a periodic force $x = 2 \cos 6t$ lb. acts upon it and produces resonance, the displacement x ft. in time t sec. is given by $x = at \sin 6t$. Neglect damping forces What are the values of M and a?

38. A mass of 5 engineers' units is suspended from the lower end of a spring, the stiffness of which is such that a force of 1 lb. stretches it by 0·008 in. The upper end of the spring is given a simple harmonic motion $5 \sin t$, where t is the time in seconds. There is a damping resistance proportional to the velocity which has the value 30 lb. at unit velocity. Find the amplitude of the forced vibration after the free vibrations have died out. Find also the phase difference, i.e. the angle by which the forced oscillation lags behind the forcing oscillation.

39. Taking the case of the previous example, but assuming the upper end of the spring to be fixed, write down an expression for the displacement x ft. of the mass at time t sec. when oscillations are occurring. Give the value of the log decrement.

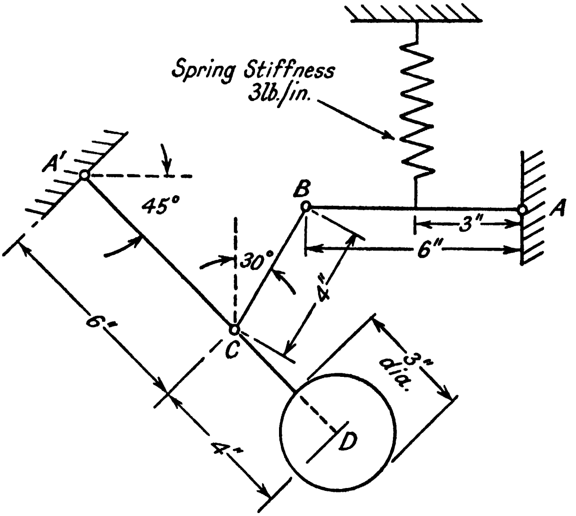

FIG. 298. OSCILLATING MECHANISM

40. Solve example 17, page 152, using an equivalent dynamical system.

41. Solve example 18, page 152, using an equivalent dynamical system. Find the force on the piston necessary to overcome this inertia force.

42. A horizontal bar, 2 ft. long, of uniform material and section weighing 5 lb. is pivoted about one end and carries a disc weight of 7 lb. weight and 6 in. diameter with its axis at the other end parallel to the axis of the pivot. The bar is supported in a horizontal position by two vertical springs, one at 1 ft. and the other at 15 ft. from the pivot. If both springs have the same stiffness, i.e. 6 lb. per inch change of length, find the frequency of oscillation of the system.

43 In the mechanism shown in Fig. 298, A and A' are fixed pivots. The thin uniform bars AB, BC, $A'D$ weigh respectively $1\frac{1}{2}$, 1, and $2\frac{1}{2}$ lb., and in addition the uniform disc at D weighs 2 lb. Find the period of oscillation of the linkage for small displacements from the configuration shown. (U.L.A.)

[*Note*. The relative velocities are most quickly found by the use of instantaneous centres.]

44. Assuming that the bars in the mechanism in Fig. 92 are uniform rods of weights AD 1·2 lb., DE and EF 1 lb. each, find the equivalent dynamical systems for the bars AD, DE, and EF together, and use it to find the tangential force at B to balance the inertia effects due to those bars.

45. Taking what information you need from the answers to the previous example and example 16, page 152, find the force at A along the line of strike to balance the inertia forces.

46. A shaft A of diameter d_A and length l_A carries at one end a disc of polar moment of inertia I_A; the other end is geared to a second shaft, carrying at its end a second disc, the corresponding quantities being d_B, l_B, and I_B. The speed of B is n times that of A, and both the inertia and flexibility of the gears may be neglected.

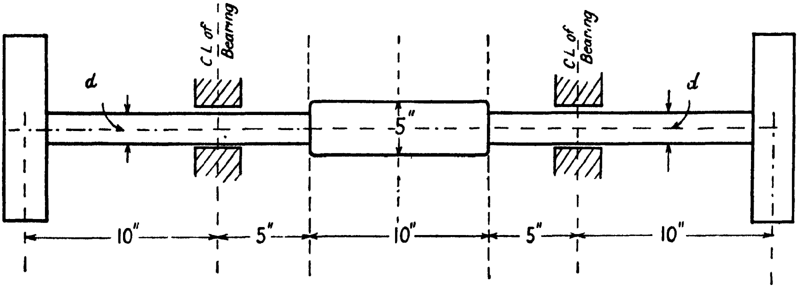

FIG. 299

Show that for torsional oscillation the gears and shaft B may be replaced by an extension of the shaft A such that the total equivalent length of shaft A becomes $l_A + \frac{l_B}{n^2}\left(\frac{d_A}{d_B}\right)^4$ and the disc being replaced by one having a moment of inertia $n^2 I_B$.

In a particular case, in which $n = 3$, the magnitudes of the various quantities are—

	Shaft		Disc	
	Diameter	Length	Diameter	Length
A	in. 3	in. 18	in. 20	in. 6
B	2	18	15	4

The material has a modulus of rigidity of 12×10^6 lb. per square inch and weighs 0·28 lb. per cubic inch. Determine the frequency of torsional oscillations of the system, neglecting the inertia of the shafts. (U.L.A.)

47. A shaft, with similar wheels at the ends, is supported in two bearings, as shown in the figure (299). Each wheel has a weight of 600 lb. and a radius of gyration of $7\frac{1}{2}$ in. The shaft is to run at 1200 r.p.m., and is so driven that there are two torsional impulses per revolution. Determine the diameter, d,

so that the torsional critical speed may be 20% above the running speed; and with the diameter so fixed calculate the whirling speed. Neglect the shaft mass in both cases. $E = 30 \times 10^6$ lb./sq. in; $C = 12 \times 10^6$ lb./sq. in. (U.L.)

48. A vibration indicator comprises a light steel cantilever of section 1 in. broad by $\frac{1}{4}$ in. thick firmly clamped to a frame at one end and carrying a weight of 5 lb. at a distance of 18 in. from the fixed end. A light recording gear gives a magnification of ten times the movement of the weight relative to the frame. When the cantilever is set freely vibrating the damping is such that successive swings in the same direction are in the ratio 1 : 0·8. Estimate the probable amplitude of the recording pointer when the frame of the instrument is placed on a body vibrating according to the law: $x = 0{\cdot}05 \cos 100t$ in. where x is the displacement from mean position and t is in seconds. Assume that the mass effects of cantilever and recording gear are equivalent to 0·5 lb. at the weight position. $E = 30 \times 10^6$ lb./sq. in. (U.L.)

49. Find the frequency of free axial vibrations of the system in Example 47. Find the amplitude of the forced vibration of an end mass when equal and opposite axial forces each of magnitude, $1000 \sin 3{,}000t$ lb. act in opposite senses, one on each mass. Put $d = 3\frac{1}{4}$ in.

50. Find the frequencies of free axial vibrations of the system in Example 21. Find the amplitudes of vibration of each mass when an axial force $F = 6000 \sin 300t$ lb. is applied to the first mass. What would be the amplitude of vibration if the shaft were absolutely rigid? $E = 30 \times 10^6$ lb. per sq. in.

51. To prevent vibration of the beam in Example 28 it is proposed to suspend from the beam, at mid-span, a load of w lb. weight carried by a vertical spring of stiffness s lb. per ft. Find w and s if the amplitude of vibration of w is not to exceed 0·4 in. Is this a practical proposition?

52. Three masses each of moment of inertia I are mounted, one at each end and one at the middle, on a uniform shaft. Show that the critical frequency of the second mode of free torsional vibration is $\sqrt{3}$ times that of the first mode.

If $C/2$ is the couple for unit twist of the whole shaft, find the amplitudes of the forced vibrations due to a torque $F_0 \sin pt$ acting on the middle mass.

53. Find the amplitudes in the last example if the torque is applied to the first mass.

54. Find the amplitudes of vibration of each mass in Example 21 when axial forces of $6000 \sin 300t$, lb. and $6000 \sin\left(300t + \frac{2\pi}{3}\right)$ lb. act together on the first and third masses respectively. [The effect of the first force has been found in solving Example 50; find the effect of the second force and combine the amplitudes by vector addition.]

55. Three masses each of moment of inertia 200 engineers' units are mounted on a shaft, one at each end and one at the middle. The couple for one radian twist of each half of the shaft is 5×10^5 lb. ft. Torques of $450 \sin 100t$, $450 \sin (100t + 2\pi/3)$ and $450 \sin (100t + 4\pi/3)$ lb. ft. respectively are applied to the first, second and third masses. Find the total angle of twist in each half of the shaft due to the forced vibration. [This problem should be solved by the method explained above. We give below an alternative method, using the system of tabulation of Art. 219 adapted to the case of forced vibrations. $p^2 = 10^4$ takes the place of n^2, the first value of θ being unknown is written θ_1, instead of unity, and the amplitude of the forcing couple is added to the value of $Ip^2\theta$ in each case.]

Assuming the maximum values of the forcing couples to be F_1, F_2 and F_3 respectively, we have the tabulation as shown on page 498.

The last entry in the fifth column must be zero, hence

$$6 \times 10^6\,\theta_1 + 5\,F_1 - 3\,F_2 + F_3 = 0$$

I	Ip^2	θ Radians	$Ip^2\,\theta + F$	$\Sigma(Ip^2\,\theta + F)$	C	$\Delta\theta = \dfrac{\Sigma(Ip^2\,\theta + F)}{C}$
200	2×10^6	θ_1	$2 \times 10^6\theta_1 + F_1$	$2 \times 10^6\theta_1 + F_1$	5×10^5	$4\theta_1 + 2 \times 10^{-6}F_1$
200	2×10^6	$-3\theta_1 - 2 \times 10^{-6}F_1$	$-6 \times 10^6\theta_1 - 4F_1 + F_2$	$-4 \times 10^6\theta_1 - 3F_1 + F_2$	5×10^5	$-8\theta_1 + 2 \times 10^{-6}(F_2 - 3F_1)$
200	2×10^6	$5\theta_1 + 2 \times 10^{-6}(2F_1 - F_2)$	$10^7\theta_1 + 8F_1 - 4F_2 + F_3$	$6 \times 10^6\theta_1 + 5F_1 - 3F_2 + F_3$		

By making two of the three values of F equal to zero we can find the effect of the remaining value of F.

$$\begin{aligned} &\text{If } F_1 = F_2 = 0, \quad && \theta_1 = -0{\cdot}1667 \times 10^6 F_3 \\ &\text{If } F_1 = F_3 = 0, && \theta_1 = 0{\cdot}5 \times 10^6 F_2 \\ &\text{If } F_2 = F_3 = 0, && \theta_1 = -0{\cdot}8333 \times 10^{-6} F_1 \end{aligned}$$

The values of θ for the second and third masses are found from the third column by substitution.

	Due to F_1	Due to F_2	Due to F_3
θ_1	− 0·000375	+ 0·000225	− 0·000075
θ_2	+ 0·000225	− 0·000675	+ 0·000225
θ_3	− 0·000075	+ 0·000225	− 0·000375

The amplitude in radians of the first mass is the vector sum of $-0{\cdot}000375_{0^\circ}$ $+ 0{\cdot}000225_{120^\circ} - 0{\cdot}000075_{240^\circ}$ which is found by calculation, or by drawing, to be $0{\cdot}000520_{150^\circ}$.

Amplitude of second mass $= 0{\cdot}000225_{0^\circ} - 0{\cdot}000675_{120^\circ} + 0{\cdot}000225_{240^\circ}$ $= 0{\cdot}000900_{300^\circ}$, and amplitude of third mass $= -0{\cdot}000075_{0^\circ} + 0{\cdot}000225_{120^\circ}$ $- 0{\cdot}000375_{240^\circ} = 0{\cdot}000520_{90^\circ}$, angle of twist of first shaft $= 0{\cdot}000900_{300^\circ}$ $- 0{\cdot}000520_{150^\circ} = 0{\cdot}00138$ radians, angle of twist of second shaft $= 0{\cdot}000900_{300^\circ}$ $- 0{\cdot}000520_{90^\circ} = 0{\cdot}00138$ radians.

56. Apply the tabular method of solution to the problem of Example 54.

57. An unloaded steel strip of uniform rectangular section is clamped at one end. Show that if t is the thickness and l the length in inches the fundamental frequency N vibrations per second is $N = 3{\cdot}25 \times 10^4 t/l^2$. (Strips of steel of suitable dimensions, clamped as above, may be used as vibration detectors.)

58. Find by the exact method the frequency of the first normal mode of vibration of a uniform rod built in at one end and free at the other. The frequency is found from a root of the equation $\operatorname{sech} ml = -\cos ml$. Solve this and determine the angular frequency. Rod l in. long weighing w lb. per in.

59. Using Rayleigh's Method, find the angular frequency of the fundamental mode of vibration of the rod in Ex. 58 with a load W lb. at the free end. Assume the relation $y = d\left(1 - \cos\dfrac{\pi x}{2l}\right)$. Compare your result (1) with the result of the previous example, putting $W = 0$; (2) with the correct result when $w = 0$. Show that the mass of the rod may be assumed to be equivalent to one of about $\frac{1}{4}$ of its amount at the end.

60. Determine by the exact method the frequencies of the modes of motion of a uniform rod of length l in. in a state of transverse vibration, using the notation of Art. 230; the rod is clamped horizontal at both ends. This involves the solution of the equation $\cos ml = \operatorname{sech} ml$. Find the lowest root accurate to 4 significant figures. Give the other roots approximately, these are given in XIV.32.

61. Apply Rayleigh's method to find the fundamental frequency in Ex. 60 assuming (1) that the curve of the rod when vibrating is the deflection curve of the rod when horizontal, and (2) that its shape is $y = d\left(1 - \cos\dfrac{2\pi x}{l}\right)$.

62. By putting $k_2 = 0$ in the equations of motion of the gyro compass (XIV.125) and (XIV.126) show that, when there is no damping, the vertical and horizontal vibrations have the same periodic time and that any point on the rotor axis traces out an ellipse in a vertical plane perpendicular to the meridian.

EXAMPLES 15.

MISCELLANEOUS

1. The crank radius of an engine is 7 in.; the connecting rod is 27 in. long, its weight is 32 lb., its centre of gravity is 8½ in. from the crank end, and its radius of gyration about an axis through its centre of gravity is 6½ in. The crank rotates at 210 r.p.m.

Determine, either graphically or by calculation: (*a*) the inertia force and couple acting on the rod, and (*b*) the magnitude, direction, and sense of the single force due to inertia, when the crank makes an angle of 50° with the inner dead centre position.

State what further particulars you would require and the procedure you would adopt in finding the maximum bending moment on the rod due to inertia.

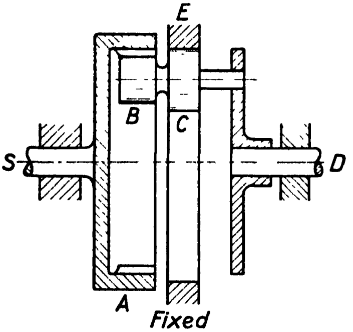

FIG. 300

2. Fig. 300 shows the arrangement of an epicyclic gear in which the shaft *D* drives the shaft *S* with a velocity ratio of 1000 to 1.

The number of teeth in wheel *A* is to be about 200. Find suitable numbers of teeth for all the wheels.

Test whether it is practicable to secure a velocity ratio of 2000 to 1 with this arrangement.

Consider the compound train *ABCE*, *A* being the first wheel. If the numbers of teeth in the wheels are *a*, *b*, *c* and *f* respectively $e = \frac{a \times c}{b \times f}$ (+ve).

Taking *D* as the arm and n_A, n_B, n_C, n_D and n_E as corresponding revs. per min., then $\frac{n_E - n_D}{n_A - n_D} = e$.

Since *E* is fixed, $n_E = 0$, so that

$$\frac{n_D}{n_A} = \frac{n_D}{n_S} = \frac{e}{e-1}$$

$$\therefore \quad e = \frac{n_D}{n_D - n_S} = \frac{a \times c}{b \times f}$$

If $\frac{n_D}{n_S} = 1000$, then $e = \frac{1000}{999} = \frac{a \times c}{b \times f}$

$$\frac{1000}{999} = \frac{50 \times 20}{37 \times 27} = \frac{200 \times 20}{74 \times 54} = \frac{200 \times 20}{222 \times 18}.$$

Thus a possible combination is $a = 200$, $b = 18$; $c = 20$; $f = 222$.

If $\frac{n_D}{n_S} = 2000$ $e = \frac{2000}{1999}$.

Since 1999 is a prime number, it will not be practicable to obtain even an approximate value with integral numbers, as a few trials will show. If, however, e is taken as $\frac{2000}{2001}$, then $e - 1 = -\frac{1}{2001}$ and $\frac{e}{e-1} = -2000 = \frac{n_D}{n_S}$. This gives the required velocity ratio, but S will now rotate in the reverse direction to D.

$$\frac{2000}{2001} = \frac{20 \times 100}{23 \times 87} = \frac{20 \times 200}{23 \times 174}.$$

Thus a possible combination is $a = 200$, $b = 23$, $c = 20$, $f = 174$. If it is essential that S and D rotate in the same direction, take $e = \frac{2001}{2000}$. This gives a velocity ratio of 2001, which will be sufficiently near for most practical purposes. The combination would then be altered to $a = 174$, $b = 20$, $c = 23$, $f = 200$.

In order to equalize the centre distances for B, C the pitch of teeth on wheels C, E must be smaller than on A, B in the ratio $\frac{a - b}{f - c}$.

3. A flywheel weighing 200 lb., radius of gyration 10 in., is attached to one end of a steel shaft 2 in. diameter and 30 in. long. When flywheel and shaft are rotating at N r.p.m. the other end is suddenly fixed.

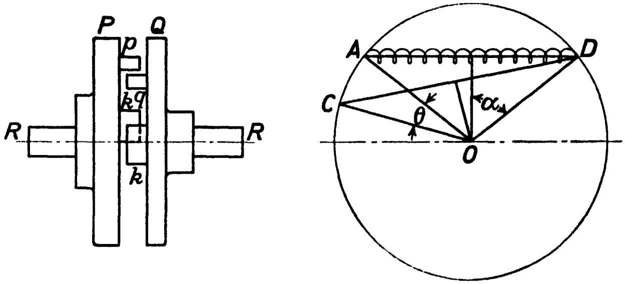

FIG. 301

(*a*) Find the maximum permissible value of N if the resultant maximum shear stress f in the shaft is not to exceed 15,000 lb. per sq. in. Couple for 1 radian twist $= 10^5 \,.\, \frac{d^4}{l}$ lb. ft.; d and l in inches.

Angle of twist $= \frac{f \,.\, l}{6 \times 10^6 \,.\, d}$ radian.

(*b*) What minimum fixing torque is required?

(*c*) If a clutch capable of exerting a resisting torque of this amount is fitted near the end of the shaft, through what angle will slip take place if the end is fixed when the speed of the flywheel is 240 r.p.m.?

(*d*) What will be the frequency of oscillation of the flywheel after the end of the shaft is fixed?

(*e*) Calculate the external diameter of a suitable multi-plate clutch (3 plates), given width of clutch surfaces = 3½ in., $u = 0{\cdot}2$, $p = 30$ lb. per sq. in. What is the necessary end thrust?

4. A disc Q, fixed to the shaft R (Fig. 301), is connected with the flywheel P, which can rotate relatively to Q, by means of a spring AD, the end D being connected to a pin q on Q and the end A being connected to a pin p on P. Stops kk maintain the angle AOD subtended by AD at the centre of the shaft at 2α when the tension in the spring is F. If the stiffness of the spring (lb. per ft. extension) is s, (*a*) show that the torque on P when A moves to C ($AOC = \theta$) is

$$T = r\left\{s \,.\, \sin(2\alpha + \theta) - (2s \,.\, \sin\alpha - F)\cos\left(\alpha + \frac{\theta}{2}\right)\right\}.$$

(*b*) Given that $s = 400$ lb. per ft.; $r = 0{\cdot}5$ ft.; $\alpha = 45°$; $F = 50$ lb., calculate values of T for values of θ from 0° to 70°, plot a curve showing the variation of T with θ, and find the limiting value of θ for which $dT/d\theta$ is positive.

(*c*) Repeat (*b*), taking $s = 200$ and $\alpha = 30°$. Compare the result with that for a flat spiral spring having the same initial torque.

5. By means of a torque applied to a drum 9 ft. diameter, a full cage weighing 3 tons is lifted and at the same time an empty cage weighing 1 ton is lowered at the same speed. The total depth of wind is 2800 ft. and the centre of the drum is 3000 ft. from the bottom of the shaft. The weight of the drum, including the layer of rope, is 5½ tons and its radius of gyration is 3·6 ft.; weight of rope is 1·6 lb. per ft.; frictional resistance to motion is 750 lb. With the full cage at the bottom of the shaft a speed of 27 ft./sec. is to be attained from rest in 4½ sec. Afterward the cages are to move with constant speed until power is cut off and a constant braking torque is then applied which brings the load to rest in a distance of 30 ft.

Find expressions connecting the torque on the drum with the distance travelled (*a*) during the period of acceleration (assumed constant), (*b*) during the period of constant speed lifting. Find also (*c*) the magnitude of the braking torque and (*d*) the total time taken for the wind.

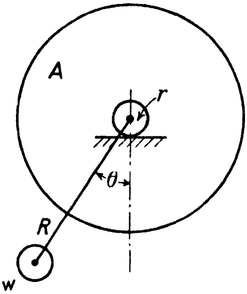

FIG. 302

6. The moment of inertia of an armature A, weight W, of which the centre of gravity lies in the shaft axis, is to be determined by the method shown in Fig. 302. A known weight w is attached at radius R and the time of oscillation T is determined while the journal, of radius r, rolls through small distances on hardened horizontal surfaces. The radius r is small compared with R.

Derive an expression for the moment of inertia of the armature in terms of the quantities given above, and show that small variations of r have the least effect on T when

$$R = \left(\frac{W}{w} + 1\right)r.$$

APPENDIX A

NOTE ON THE PARABOLA REFERRED TO IN ART. 182

FIG. 303 shows a parabola AOA', of which OX is the axis, and OY the tangent at the vertex. A is any point on the curve, AN the ordinate, AB the normal, and NB the subnormal. The angle ABN is α_0. Taking OX and OY as axes, we have at A

$$\frac{dy}{dx} = \cot \alpha_0$$

$$= \frac{\overline{BN}}{\overline{AN}}$$

But if $y^2 = 2hx$ is the equation to the parabola,

$$2y\frac{dy}{dx} = 2h$$

$$\therefore \frac{dy}{dx} = \frac{h}{y}$$

$$\therefore \frac{h}{y} = \frac{\overline{BN}}{\overline{AN}}$$

i.e. $\overline{BN} = h$

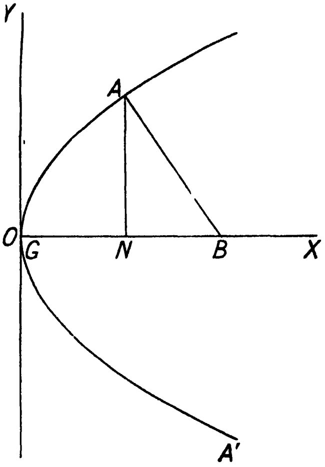

FIG. 303

or the subnormal is constant and equal to h.

Again, if R is the radius of curvature,

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

But $\dfrac{dy}{dx} = \dfrac{h}{y}$, and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{h}{y}\right) = -\dfrac{h}{y^2}\dfrac{dy}{dx} = -\dfrac{h^2}{y^3}$

Hence
$$R = \frac{\left(1 + \frac{h^2}{y^2}\right)^{\frac{3}{2}}}{-\frac{h^2}{y^3}}$$
$$= -\frac{(y^2 + h^2)^{\frac{3}{2}}}{h^2}$$
$$= -\frac{\overline{AB}^3}{\overline{BN}^2}$$
$$= -\overline{BN} \times \left(\frac{\overline{AB}}{\overline{BN}}\right)^3$$
$$= -\frac{h}{\cos^3\alpha_o}$$

which is the result referred to

APPENDIX B

PISTON DISPLACEMENT, VELOCITY, AND ACCELERATION IN THE SIMPLE ENGINE MECHANISM

THE distance x from the crankshaft to the crosshead in the simple engine mechanism is equal to $\overline{AC}$ (Fig. 16), therefore from Art. 40,

$$x = r\left(\cos\theta + n\sqrt{1 - \frac{1}{n^2}\sin^2\theta}\right) \text{ where } n > 1$$

But by the binomial theorem, $\sqrt{1 - \frac{1}{n^2}\sin^2\theta}$ may be expanded in the form

$$\sqrt{1 - \frac{1}{n^2}\sin^2\theta} = 1 + \frac{A}{n^2}\sin^2\theta + \frac{B}{n^4}\sin^4\theta + \frac{C}{n^6}\sin^6\theta + \dots$$

where A, B, C, etc., are the coefficients of x, x^2, x^3, etc., in the expansion of $(1 - x)^{\frac{1}{2}}$. We have then

$$x = r\left(n + \cos\theta + \frac{A}{n}\sin^2\theta + \frac{B}{n^3}\sin^4\theta + \frac{C}{n^5}\sin^5\theta + \dots\right) \quad 1$$

Again,*

$$\left.\begin{aligned} \sin^2\theta &= \frac{1 - \cos 2\theta}{2} = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta \\ \sin^4\theta &= -\tfrac{1}{8}\cos 4\theta + \tfrac{1}{2}\cos 2\theta - \tfrac{3}{8} \\ \sin^6\theta &= -\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta - \frac{15}{32}\cos 2\theta + \frac{5}{16} \\ \&c. \qquad & \qquad \&c. \end{aligned}\right\} \quad . \quad 2$$

If we substitute in (1) from (2) we shall obtain an expression of the type

$$\left.\begin{aligned} x = r(n &+ c + \cos\theta + a_2\cos 2\theta + a_4\cos 4\theta \\ &+ a_6\cos 6\theta + \dots) \end{aligned}\right\} \quad . \quad 3$$

in which c, a_2, a_4, a_6, etc , are constants. The determination of the values of c, a_2, a_4, a_6, etc., is a tedious business, unless we make the assumption that only a few of the terms at the beginning of the series are important. It may be seen from (1) that if n is 4, or over, the terms rapidly diminish in importance, because of the quantities n, n^3, n^5, n^7, etc., in the denominators. The series (3) is one of a general type known as *Fourier's series.* Fourier developed a method of expressing any periodic function of θ in terms of a series of sines and cosines in the form

---

* For proofs of these, readers should consult some standard textbook on trigonometry.

$$\begin{aligned} f(\theta) &= \text{periodic function} \\ &= a_0 + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \ \ldots \\ &\qquad + b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + \ \ldots \end{aligned}$$

In this case, b_1, b_2, b_3, etc., are all zero, and a_5, a_7, etc., are also zero, and thus we have the expression (3).

We shall determine the values of a_2 and a_4 in (3) on the assumption that a_8, a_{10}, etc., are all so small as to be negligible.

The values of A, B, and C in (1) are respectively the coefficients of x, x^2, and x^3 in the expansion of $(1-x)^{\frac{1}{2}}$. Thus

$$A = -\tfrac{1}{2},\ B = \frac{\frac{1}{2}(-\frac{1}{2})}{\underline{|2}} = -\tfrac{1}{8},\ \text{and } C = -\frac{\frac{1}{2}(-\frac{1}{2})\,(-\frac{3}{2})}{\underline{|3}} = -\tfrac{1}{16}$$

From (1) we have the approximate relation

$$x = r\left(n + \cos\theta - \frac{1}{2n}\sin^2\theta - \frac{1}{8n^3}\sin^4\theta - \frac{1}{16n^5}\sin^6\theta\right)$$

and substituting from (2),

$$\begin{aligned} x = r\Big[n + \cos\theta - \frac{1}{2n}(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta) + \frac{1}{8n^3}(\tfrac{1}{8}\cos 4\theta \\ - \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8}) - \frac{1}{16n^5}\Big(\frac{5}{16} - \frac{15}{32}\cos 2\theta \\ + \frac{3}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta\Big)\Big] \end{aligned}$$

Comparing this with the relation (3) we have,

$$c = -\tfrac{1}{4}n + \frac{3}{64n^3} - \frac{5}{256n^5}$$

$$a_2 = \frac{1}{4n} - \frac{1}{16n^3} - \frac{15}{512n^5}$$

$$a_4 = \frac{1}{64n^3} - \frac{3}{256n^5}$$

If the connecting-rod is 4 cranks long, $n = 4$, and

$$c = -\frac{1}{16} + \frac{3}{64^2} - \frac{5}{256 \times 1024} = -\cdot 0618$$

$$a_2 = \frac{1}{16} + \frac{1}{1024} - \frac{15}{512 \times 1024} = \cdot 0624$$

and $$a_4 = \frac{1}{64^2} - \frac{3}{256 \times 1024} = \cdot 00023$$

Substituting the above in (3),

$$x = r(3\cdot 938 + \cos\theta + \cdot 0624 \cos 2\theta + \cdot 00023 \cos 4\theta)$$

From this, we have on differentiation,

$$\text{Velocity} = \frac{dx}{dt} = -\omega r(\sin\theta + \cdot 1248 \sin 2\theta + \cdot 00092 \sin 4\theta)$$

and again,

$$\text{Acceleration} = \frac{d^2x}{dt^2} = -\omega^2 r(\cos\theta + \cdot 2456 \cos 2\theta$$
$$+ \cdot 00368 \cos 4\theta)$$

In the case of the straight-sided cam of Art. 172 when the roller is on the nose of the cam, we have the case of crank and connecting-rod where n is less than unity. In this case the above series for x becomes divergent and cannot be used.

Referring to Fig. 16, if $n = \frac{1}{p}$, and $r = p \,.\, l$, we have

$$x = l(p \cos\theta + \sqrt{1 - p^2 \sin^2\theta})$$

$$\sqrt{1 - p^2 . \sin^2\theta} = 1 - \frac{p^2}{2} . \sin^2\theta - \frac{p^4}{8} . \sin^4\theta - \frac{p^6}{16} . \sin^6\theta - \ldots$$

If $p \ngtr 1\cdot 5$ and $\theta \ngtr 30°$, $\frac{p^6}{16} . \sin^6\theta \ngtr \cdot 015$. By ignoring this term we introduce a maximum error of $2\frac{1}{2}$ per cent in the value of x, which is of little importance in practice. The corresponding error in a is under 2 per cent.

$$\therefore\ x = l\left(p . \cos\theta + 1 - \frac{p^2}{2} . \sin^2\theta - \frac{p^4}{8} . \sin^4\theta\right) \text{approx.}$$

which may be written

$$\begin{aligned} x = l\Big\{ p \,.\, \cos\theta + \left(1 - \frac{p^2}{4} - \frac{3p^4}{32}\right) \\ + \cos 2\theta\left(\frac{p^2}{4} + \frac{p^4}{16}\right) - \frac{p^4}{64}\cos 4\theta \Big\} \end{aligned}$$

$$\therefore \quad v = -l \,.\, \omega \left\{ p \,.\, \sin\theta + \left(\frac{p^2}{2} + \frac{p^4}{8}\right)\sin 2\theta - \frac{p^4}{16} \,.\, \sin 4\theta \right\}$$

$$a = -l \,.\, \omega^2 \left\{ p \,.\, \cos\theta + \left(p^2 + \frac{p^4}{4}\right)\cos 2\theta - \frac{p^4}{4}\cos 4\theta \right\}$$

$$\therefore \quad a = -r \,.\, \omega^2 \left\{ \cos\theta + \left(p + \frac{p^3}{4}\right)\cos 2\theta - \frac{p^3}{4} \,.\, \cos 4\theta \right\}$$

The exact expression for a, which is less convenient for calculation, is

$$a = -r \,.\, \omega^2 \left\{ \cos\theta + \frac{p \,.\, \cos 2\theta}{\sqrt{A}} + \frac{p^3}{8} \,.\, \frac{1 - \cos 4\theta}{A^{\frac{3}{2}}} \right\}$$

where $A = 1 - p^2 \sin^2\theta$.

APPENDIX C

THEORY OF UNITS AND DIMENSIONS

THE units of all the quantities used in the subject of mechanics can be expressed in terms of three suitably chosen units. These three units are arbitrary, and are known as *fundamental units.* All the other units which are derived from the fundamental units are known as *derived units.* In this book we have used the engineers' system of units, the fundamental units in which are the unit of length of one foot, the unit of time of one second, and the engineers' unit of mass. With this system the derived unit of velocity is $\frac{1 \text{ ft.}}{1 \text{ sec.}} = 1$ ft. per sec., and that of acceleration is $\frac{1 \text{ ft. per sec.}}{1 \text{ sec.}} = 1$ ft. per sec. per sec. The three fundamental units are fixed by the definitions, the foot is a standard length, the second a standard time, and the engineers' unit of mass is the mass of a body whose weight in pounds at any place is numerically equal to its acceleration due to gravity in feet per second per second at the same place.

It is sometimes erroneously assumed that the engineers' unit of mass is the mass of g lb., which is variable. It will be apparent that though the engineers' unit of mass is defined in terms of the ratio of two variable quantities, the ratio itself is constant. Any system of units having the same values at all times and places is known as an *absolute system.* The engineers' system of units is an absolute system. Let M, L, and T be the fundamental units of mass, length, and time respectively, in any system of units, then the derived units of the system are given in the table on page 510.

The derived unit in each of these cases is of the type $M^xL^yT^z$, where x, y, and z are positive or negative whole numbers. The values of x, y, and z, for any quantity, are the indices of M, L, and T in the third column of the above table. The indices x, y, and z are known as the dimensions of the quantity with reference to mass, length, and time respectively. Usually we look upon the expressions in the third column of the table as representing the dimensions of the corresponding quantities

Quantity	Formula Defining the Quantity	Derived Unit
Area	Area = length × length	$L \times L = L^2$
Volume	Volume = length × length × length	$L \times L \times L = L^3$
Velocity	Velocity $= \frac{\text{length}}{\text{time}}$	$\frac{L}{T} = LT^{-1}$
Acceleration	Acceleration $= \frac{\text{velocity}}{\text{time}}$	$\frac{LT^{-1}}{T} = LT^{-2}$
Angle	Angle $= \frac{\text{arc}}{\text{radius}}$	$\frac{L}{L} = 1$
Angular velocity	Angular velocity $= \frac{\text{angle}}{\text{time}}$	$\frac{1}{T} = T^{-1}$
Angular acceleration	Angular acceleration $= \frac{\text{angular velocity}}{\text{time}}$	$\frac{T^{-1}}{T} = T^{-2}$
Force	Force = mass × acceleration	$M \times LT^{-2} = MLT^{-2}$
Couple or torque	Couple = force × distance	$MLT^{-2} \times L = ML^2T^{-2}$
Work or energy	Work = force × distance	ML^2T^{-2}
Power	Power $= \frac{\text{work}}{\text{time}}$	$\frac{ML^2T^{-2}}{T} = ML^2T^{-3}$
Momentum	Momentum = mass × velocity	$M \times LT^{-1} = MLT^{-1}$
Moment of inertia	Moment of inertia = mass × (distance)^2	$M \times L^2 = ML^2$
Angular momentum	Angular momentum = couple × time	$ML^2T^{-2} \times T = ML^2T^{-1}$

in the first column. Thus the dimensions of force are MLT^{-2}, or one dimension each of mass and length, and minus two of time. In any equation which is applicable in any system of units, the dimensions of all terms in the equation must be the same. As an example, consider the well-known relation

$$v^2 = u^2 + 2fs$$

in which v and u are velocities, f an acceleration, and s a length. The dimensions of the three terms are

$$\left(\frac{L}{T}\right)^2, \left(\frac{L}{T}\right)^2 \text{ and } \frac{L}{T^2} \times L$$

all of which reduce to L^2T^{-2}. Again, in the relation

Couple = moment of inertia × angular acceleration,

the dimensions of the terms are

$$ML^2T^{-2} \text{ and } ML^2 \times T^{-2} \text{ respectively,}$$

which agree.

A knowledge of the theory of dimensions often helps us to remember the form of an expression. Thus, suppose we are in doubt as to which of the two formulae $t = 2\pi\sqrt{\frac{l}{g}}$, and

$t = 2\pi\sqrt{\frac{g}{l}}$, is correct for the time of oscillation of a simple pendulum. The dimensions of the two sides of the first relation are

$$T \text{ and } \sqrt{\frac{L}{LT^{-2}}}$$

or $$T \text{ and } T, \text{ which agree.}$$

The dimensions in the second case are

$$T \text{ and } \sqrt{\frac{LT^{-2}}{L}}$$

or $$T \text{ and } T^{-1}, \text{ which do not agree.}$$

Thus, the first formula is the correct one.

The above test can only be applied in the case of formulae or equations which are true in all systems of units, and the test does not apply in cases where values have been given to some of the quantities involved, or where the formula is empirical and only true in one set of units. Consider the formula $d = 1{\cdot}2\sqrt{t}$, giving the diameter of a rivet in terms of the thickness of the plates to be riveted together. The dimensions of the two sides are

$$L \text{ and } L^{\frac{1}{2}}, \text{ which do not agree.}$$

This formula is only applicable when d and t are in inches.

Formula (XII.34) may be shown to have both sides of the same dimensions, thus—

$$\omega = \sqrt{\frac{gW}{wr}}$$

The dimensions of the two sides are

$$T^{-1} \text{ and } \sqrt{\frac{\frac{L}{T^2} \cdot \frac{ML}{T^2}}{\frac{ML}{T^2} \,.\, L}}$$

i.e. $$T^{-1} \text{ and } T^{-1}$$

In certain cases it is possible to make use of the theory of dimensions to predict the form of a mathematical relation between physical quantities. Suppose we have an elastic string of uniform density and perfectly flexible which is set vibrating in one of its normal modes. Let N = frequency, i.e. number of oscillations per minute, P = tension in lb. weight, L = length of string in feet. M = the mass of the whole string in engineers' units. Let P be so great that gravitational forces can be ignored. N can be varied by changing one or more of the quantities P, L or M, and does not depend on any other variable. N is, therefore, a function of these three variables. We assume the relation to be $N \propto P^a L^b M^c$, where a, b, and c are constants, i.e. $N = kP^a L^b M^c$, where k is an absolute constant.

Now since $N = \dfrac{\text{number of vibrations}}{\text{time}}$ its dimensions are $\dfrac{1}{T}$, and P being a force its dimensions are $\dfrac{ML}{T^2}$.

Thus the dimensions of the two sides are

$$\frac{1}{T} \text{ and } \left(\frac{ML}{T^2}\right)^a L^b M^c$$

i.e. T^{-1} and $M^{a+c}L^{a+b}T^{-2a}$.

And equating indices of M, L, and T on the two sides, we have $a + c = 0, a + b = 0. 2a = 1$, from which $a = \frac{1}{2}, b = c = -\frac{1}{2}$, and

$$N = kP^{\frac{1}{2}}L^{-\frac{1}{2}}M^{-\frac{1}{2}}$$

$$N = k\sqrt{\frac{P}{LM}}$$

Or, putting

$$M = \frac{W}{g} = \frac{\text{weight in lb. of whole string}}{\text{acceleration due to gravity}}$$

$$N = k\sqrt{\frac{Pg}{WL}} \text{ in lb.}$$

Again, putting $W = wL$, where w = weight in lb. per foot run of string,

$$N = k\frac{1}{L}\sqrt{\frac{Pg}{w}} \text{ oscillations per minute}$$

It is, of course, impossible to determine by this means the value of the constant k.

APPENDIX D

MORE CORRECT THEORY OF THE WORM AND WORM WHEEL WHEN THE WORM THREAD IS TRAPEZOIDAL IN SECTION

THE calculations in Paragraphs 163 and 164 ignore the effect of the thread angle and assume that the worm is square-threaded, which is not the case in practice, although in the case of worms with large helix angles and low friction coefficients the errors involved are small.

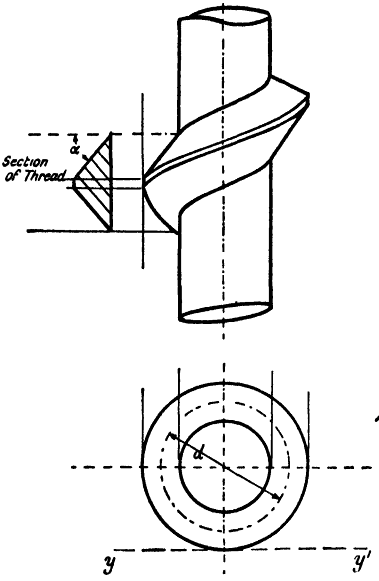

FIG. 304

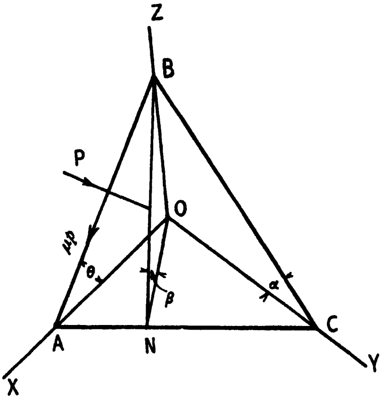

FIG. 305

Fig. 304 shows in plan and elevation one turn of a worm, and Fig. 305 is a pictorial view in which OX is parallel to yy' and OY is perpendicular to yy'.

If θ (Fig. 305) is the helix angle (or lead angle) of the worm $\left(\tan^{-1}\dfrac{\text{lead}}{\pi d}\right)$ and α is the thread angle (inclination of axial section of thread to cross section of worm) then the inclination β of a tangent plane to the worm surface may be found as follows—

Referring to Fig. 305, angle $BAO = \theta$, angle $BCO = \alpha$, angle $BNO = \beta$, where ON is at right angles to AC.

BO is parallel to the axis of the worm, AO is parallel to the axis of the worm wheel and the direction of the radial thrust between worm and wheel is parallel to OC.

Taking BO as of unit length, then $AO = \cot\theta$, $CO = \cot\alpha$, $NO = \cot\beta$.

Since $AOC = 90^\circ$, we get $AC = \sqrt{\cot^2\theta + \cot^2\alpha}$.

$$\text{Since } \frac{ON}{AO} = \frac{OC}{AC}, \therefore ON = \frac{\cot\theta \,.\, \cot\alpha}{\sqrt{\cot^2\theta + \cot^2\alpha}} = \cot\beta.$$

$$\therefore \quad \tan\beta = \sqrt{\tan^2\theta + \tan^2\alpha} \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{and} \quad \sec\beta = \sqrt{1 + \tan^2\beta} = \sqrt{\sec^2\theta + \tan^2\alpha} \quad . \quad (2)$$

Let P = normal thrust on worm surface (at right angles to BN)

μ = coefficient of friction between surfaces

R_W = radial thrust on wheel (parallel to OC)

E_W = end thrust on wheel (parallel to OA)

E_w = end thrust on worm (parallel to OB)

$F_w = \sqrt{R_W{}^2 + E_W{}^2}$ = resultant thrust on worm at right angles to worm axis (total radial load on worm bearings).

The normal pressure P acts at an angle β with BO. The resistance due to friction is $\mu \,.\, P$ and acts in a direction parallel to the direction of relative sliding motion, i.e. along AB, hence μP is equivalent to $\mu P \cos\theta$ parallel to AO and $\mu P \sin\theta$ parallel to BO.

$$\therefore \; P\cos\beta - \mu \,.\, P\sin\theta = E_w$$

$$\therefore \; P = \frac{E_w}{\cos\beta - \mu \,.\, \sin\theta} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let E_W = tangential horizontal force at same radius as P required to be applied to the worm to overcome the tangential resistance to motion of the worm (parallel to OA) = end thrust on worm wheel

then
$$\begin{aligned} E_{\mathrm{W}} &= P \sin\beta \cos NOA + \mu \,.\, P \,.\, \cos\theta \\ &= P\{\sin\beta \,.\, \cot\beta \,.\, \tan\theta + \mu \,.\, \cos\theta\} \\ &= P\{\cos\beta \,.\, \tan\theta + \mu \,.\, \cos\theta\} \end{aligned}$$

$$\therefore \text{ From (3), } E_{\mathrm{W}} = E_w \,.\, \frac{\cos\beta \,.\, \tan\theta + \mu \,.\, \cos\theta}{\cos\beta - \mu \,.\, \sin\theta}$$

$$= E_w \,.\, \frac{\tan\theta + \mu \,.\, \sec.\,\beta \,.\, \cos\theta}{1 - \mu \,.\, \sec.\,\beta \,.\, \sin\theta} \quad . \quad . \quad (4)$$

$$\begin{aligned} R_{\mathrm{W}} &= \text{thrust along } OC = P \,.\, \sin\beta \,.\, \cos NOC \\ &= P \,.\, \sin\beta \,.\, \frac{ON}{OC} \\ &= P \,.\, \cos\beta \,.\, \tan\alpha \\ &= E_w \,.\, \frac{\tan\alpha}{1 - \mu \,.\, \sec\beta \,.\, \sin\theta} \quad . \quad . \quad . \quad (5) \end{aligned}$$

The mechanical efficiency η is given by

$$\begin{aligned} \eta &= \frac{\text{mechanical advantage}}{\text{velocity ratio}} \\ &= \frac{E_w}{E_{\mathrm{W}}} \,.\, \frac{p}{\pi d} = \frac{E_w}{E_{\mathrm{W}}} \,.\, \tan\theta \\ &= \frac{1 - \mu \,.\, \sec\beta \,.\, \sin\theta}{\tan\theta + \mu \,.\, \sec\beta \,.\, \cos\theta} \,.\, \tan\theta \\ &= \frac{1 - \mu \,.\, \sec\beta \,.\, \sin\theta}{1 + \mu .\, \sec\beta \,.\, \cos\theta \,.\, \cot\theta} \quad . \quad . \quad (6) \end{aligned}$$

If $\alpha = 0$ (square thread), then $\beta = \theta$ and (6) becomes

$$\eta = \frac{\tan\theta\,(1 - \mu \,.\, \tan\theta)}{\tan\theta + \mu} = \frac{\tan\theta}{\tan(\theta + \phi)}, \text{ as in Art. 124} \qquad (7)$$

(6) may be derived from (7) by substituting for μ the expression

$$\mu\sqrt{1 + \cos^2\theta \,.\, \tan^2\alpha}$$

The following example will illustrate the application of the above expressions—

Give lead of worm $= 7''$; h.p. transmitted $= 35$; revs. per min. of worm $= 1800$; $\theta = 40°$; $\alpha = 22\frac{1}{2}°$; $\mu = 0{\cdot}05$. $\tan\theta = 0{\cdot}839$; $\cos\theta = 0{\cdot}766$; $\sec\theta = 1{\cdot}305$; $\sec^2\theta = 1{\cdot}704$; $\sin\theta = 0{\cdot}643$; $\tan\alpha = 0{\cdot}414$; $\tan^2\alpha = 0{\cdot}171$.

From (2), $\sec\beta = \sqrt{1{\cdot}704 + 0{\cdot}171} = 1{\cdot}369$

$$\therefore \text{ From (6)} \quad \eta = \frac{1 - \cdot 05 \times 1\cdot 369 \times 0\cdot 643}{1 + \cdot 05 \times 1\cdot 369 \times \cdot 766 \times 1\cdot 192} = 0\cdot 89$$

$$\therefore \frac{E_w \times \cdot 89 \times 7 \times 1800}{12 \times 33000} = 35$$

$$E_w = 1240 \text{ lb. (end thrust on worm)}$$

$$\text{From (5)} \quad R_W = 1240 \, . \, \frac{0\cdot 414}{1 - 0\cdot 05 \times 1\cdot 369 \times 0\cdot 643}$$

$$\therefore R_W = 537 \text{ lb. (radial thrust on wheel)}$$

$$\text{From (4)} \quad E_W = \frac{\cdot 839 + \cdot 05 \times 1\cdot 369 \times 766}{1 - \cdot 05 \times 1\cdot 369 \times \cdot 643} \times 1240$$

$$\therefore \underline{E_W} = 1160 \text{ lb. (end thrust on wheel)}$$

$$\underline{F_W = \sqrt{R_W{}^2 + E_W{}^2} = 1278 \text{ lb.}}$$

In practice ball thrust bearings would be provided to take the end thrusts on worm and wheel and friction of the worm bearings due to F_W must also be allowed for.

The expression for mechanical efficiency applies also to screw threads but is of less value, since it ignores the friction at the base of the nut, and screws with triangular or trapezoidal threads are rarely used for driving purposes against heavy loads (see Ex. 37, p. 212).

The assumption made above that the friction force at any point of the worm surface acts along the helix through that point is only approximate in the case of the worm and worm wheel, since there is some slight motion of the point of contact in a radial direction introducing sliding in this direction. If the wheel were infinitely large this sliding would not take place, but it becomes important in, for instance, a worm drive for the rear axle of a motor vehicle. The spiral angle (θ) of the helix also varies slightly owing to varying radii of the points of contact. In the above investigation θ is given the value it has at the mean radius of contact (see references on p. 274).

APPENDIX E

TORQUE TRANSMITTED AND TOOTH LOADS IN EPICYCLIC GEARS

CONSIDER the reverted epicyclic train shown in Fig. 306. As before, in Arts. 149 and 151,

$$e = \frac{r_A \cdot r_C}{r_B \cdot r_D} \text{ and } 1 - \frac{1}{e} = \frac{r_B \cdot r_D - r_A \cdot r_C}{r_A \cdot r_C}$$

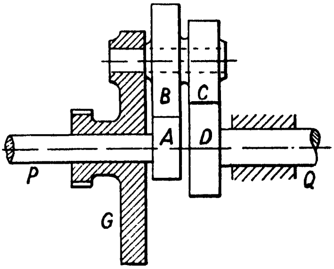

FIG. 306

Let T_P = torque on driving shaft P
T_Q = torque on driven shaft Q
T_G = torque on arm G

Tangential load on $A = \frac{T_P}{r_A}$ = tangential load on B

Tangential load on $C = \frac{r_B}{r_C} \cdot \frac{T_P}{r_A}$ (opposite to load on B, since net torque on BC is zero)

Tangential load on $D = \frac{r_B}{r_C} \cdot \frac{T_P}{r_A}$

$$\therefore \text{ Torque on } D = \frac{r_B \cdot r_D}{r_A \cdot r_C} \cdot T_P = \frac{T_P}{e}$$

$$\therefore \qquad \underline{T_Q = T_D = \frac{T_P}{e}}$$

$$\text{Tangential load on pin} = \frac{r_B}{r_C} \cdot \frac{T_P}{r_A} - \frac{T_P}{r_A} = \frac{T_P}{r_A}\left(\frac{r_B}{r_C} - 1\right)$$

$$\therefore \text{ Torque on } G = T_P \cdot \frac{r_B + r_A}{r_A} \cdot \frac{r_B - r_C}{r_C}$$

$$= T_P \cdot \frac{r_B(r_B + r_A - r_C) - r_A \cdot r_C}{r_A \cdot r_C}$$

$$= T_P \cdot \frac{r_B \cdot r_D - r_A \cdot r_C}{r_A \cdot r_C}$$

$$\therefore \underline{T_G = T_P\left(1 - \frac{1}{e}\right)}$$

It will be seen that the values of T_Q and T_G are quite independent of the relative speeds of rotation of P, Q and G, but it is important that the correct sign be attached to e according to the relative directions of rotation of the first and last wheels of the train with G fixed. It will be noted that the algebraic sum of the torques transmitted by Q and G will always equal the torque on the driving shaft P.

The actual loads on the teeth of wheels B and C, and the corresponding loads on the bearings, are obtained by multiplying the tangential loads by sec α, where α is the pressure angle of the involute profile.

NOTE ON CENTRIPETAL FORCE

In problems involving the forces acting on a body travelling in a circular path, it is usual to reduce these to statical problems by assuming that a "centrifugal" force acts on the body, and is balanced by the remaining forces acting. This procedure, although not strictly necessary, is convenient, but needs some justification.

It must be borne in mind that the centripetal force necessary to keep a body moving in a circular path must be applied by some other body either directly, as by the link *AB* in Fig. 222 or the rails in the case of a locomotive, or indirectly, as by gravitational attraction, and in the former case this force is applied *at the point or points of contact* of the restraining body with the body affected.

D'Alembert's Principle (Routh's *Rigid Dynamics*, Chap. II) states that "If forces equal to the effective forces but acting in exactly opposite directions were applied at each point of a system, these would be in equilibrium with the impressed forces." The "effective forces" referred to are the inertia forces due to acceleration. We are thus justified in reversing the centripetal force and treating the body as if it were at rest and acted upon by "centrifugal" force.

The overturning, when on a curved track, of a locomotive or other vehicle *outward*, under the action of a force acting *inward*, which, apart from the resultant mass acceleration, may be the only horizontal force acting, needs, however, some explanation.

A reference to the diagram (Fig. 307), which shows a vehicle acted on by a single force F acting along XX, will make the matter clear. If the equal and opposite forces FF are applied at the centre of gravity G, the equilibrium will not be affected, so that the inward force F acting along XX is equivalent to an inward force F acting at G, and an overturning couple Fh tending to overturn the vehicle *outward*. So far as overturning is concerned, the result is the same as if an *outward* force F were acting at G, and the usual assumption is justified.

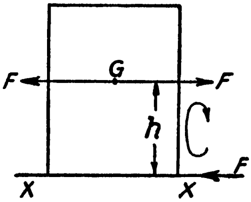

Fig. 307. Overturning Effect on a Curved Track

ANSWERS TO EXAMPLES

EXAMPLES 1. PAGE 22

1. (1) ·455 and 13·6; (2) 20·3 and ·169.
2. 5·76 tons.
3. 1·76 f.s.s.; 3·52 f.s.s.; 53·0 sec.
4. 73·0 at 14° north of east.
5. 48° 15′.
6. 6·08 $\div \sqrt{l}$, l = length in ft.
7. 29·7 miles; 3·22 hrs.
8. 4·66 f.s.s. at 64° to radius; 11·4 f.s.s. at 130° to radius.
9. 95·7 rads. per min.; (1) 8·78; (2) 21·0.
10. Radial acceleration 5·74 and tangential acceleration—2·56 f.s.s. Radial force, 8·96 tons.
12. 14·4 sec.; 357 ft.
13. (*a*) 597,300 lb.-ft.; (*c*) 1225 h.p.; (*d*) 0·0105.
14. (*a*) ·0126 sec.; (*b*) $\frac{1}{16}$ in.
15. 90 sec.; 79·2 miles per hr.
16. 5,200 lb.
18. 348 f.s.; 76 f.s.s.
19. 144 f.s.s.
22. 98 sec., 778 ft.
24. Distance is $\frac{W^2}{W + w} \cdot \frac{n.h}{(R - \overline{W + w})}$ ft.; loss of energy per blow is $\frac{w}{W + w} . W.h$ ft.-lb.; 14·1 ft.; 15,000 ft.-lb.
25. 283 ft. sec.; 35·4 sec.
27. 18·5 ft.
28. 0·213 ft.; 2 ft.-tons.

EXAMPLES 2. PAGE 59

1. 1·839 $(49 - x^2)$ lb.; 82·8 lb.; 60·7 lb.; 23·9 lb.
2. 29·0 lb.; 59° 2′.
3. 10·53 lb. sq. ft.
4. 4,162 lb.; 4,920 h.p.; 1,398 ft.-lb.; 9·71 ft.-lb.; 1,387 ft.-lb.
5. 16,200 lb.; 21,500 lb.
6. I.H.P. = 444; $T_{(\text{mean})}$ = 15,560 lb.-ft.; 23,450 lb.-ft.
7. 22,650 lb.-ft.
8. 752 Eng. units.
9. Angular velocities (rad. sec.) = 1·57, ·804, 0, – 1·57. Angular accelerations (rad. sec.^{2}) = 0, – 8·33, – 9·61, 0.
10. 527 lb.
11. 1,896 lb.
12. 2,505 lb.-ft.; 22·8 h.p.
13. 1·34 tons.
14. 3,640 lb.; 2,180 lb.
15. 488 lb.
16. 17·6 h.p.; 291 lb.-ft.
17. (*a*) 87·3; (*b*) 942.
18. Av. torque = ·637 $P.r$; 39·6° and 140·4°; ·42 $P.r$; angular acceleration of flywheel $= \frac{P.r}{I}(\sin\theta - \cdot637)$; $I = 1{,}920 . \frac{P.r}{N^2}$; angular acceleration of flywheel $= \frac{N^2}{1920}(\sin\theta - \cdot637)$.

19. H.P. = 231 ; $W = 2{,}248$ lb. ; $-4{\cdot}59$ rads. sec.^{2}, 1·49 rads. sec.^{2}.
20. 159·6 h.p. ; $W = 3{,}510$ lb. ; -4 rads. sec.^{2}, 0.
21. Effy. $= \dfrac{W}{1{\cdot}13W + 5{\cdot}3}$; 32·9 lb., ·87.
23. Potential energy (ft.-lb.) 0, 0·171, 0·684 ; kinetic energy (ft.-lb.) 0·684, 0·513, 0.
24. $\dfrac{16k^2.W}{9h^2 + 16k^2}$, $\dfrac{9h^2.W}{9h^2 + 16k^2}$; $\dfrac{h^2.W}{h^2 + 16k^2}$, $\dfrac{16k^2.W}{h^2 + 16k^2}$.
25. 18·37 lb. at 97·8°; Couple $= 7{\cdot}794a$; distance from $A = {\cdot}425a$ to right.
28. 19 h.p.
29. $k = 6{\cdot}65$ in. ; $\dfrac{R}{W} = {\cdot}0334$.
30 99·4 lb.-ft.
31. $\dfrac{W}{\sqrt{\left(\dfrac{D}{x}\right)^2 - 1}}$, $\dfrac{W}{\sqrt{1 - \left(\dfrac{x}{D}\right)^2}}$.
33. 11 h.p.
36. (*a*) 2,534 lb. ft.; (*b*) 2,380 lb.; (*c*) 25·4.

EXAMPLES 3. PAGE 85

3. ·453 sec. ; (1) 1·44 f.s. ; (2) $-25 \cos (13{\cdot}86\, t + a)$ lb.
4. ·927 sec. Two particles of masses $\frac{2}{7}$ths and $\frac{5}{7}$ths respectively of that of the sphere, the former particle at the centre of suspension and the latter on the diameter through the former at a distance of 2·4 in. from, and on the opposite side of, the centre.
5. ·301 sec.
6. ·245 sec.
9. $\dfrac{dl}{dh} = 1 - \dfrac{k^2}{h^2}$.
10. 2·10 in.
11. $4{\cdot}93 \cos (3t + 1{\cdot}173)$; $7{\cdot}60 \cos (3t + 2{\cdot}505)$.
12. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$; slope $= -\dfrac{3}{2} \cos 3t_0$.
13 $9y^2 - 6xy + 4x^2 = 27$.
14. (*a*) 2·92 in. 31° in advance of the first crank. (*b*) 2·58 in. 61° in advance of the first crank.
16. 2·47 ft. ; 95·3 engineers' units.
17. Taking the modulus of rigidity as 12,000,000 lb.-sq. in. the frequency is 11·3 per sec.
18. 2·72 sec.
19. $\cdot 403 \sin 7t$.
20. Displacement $= \dfrac{n^2 A}{n^2 - a^2} \sin at$, where $n^2 l = g$.
22 14 in.

EXAMPLES 4. PAGE 111

8. $\overline{OB} : \overline{OA} :: 7 : 3$ (Fig. 46) ; 1·79 to 1.
9. (*a*) At the wheel centre ; (*b*) on a vertical radius ·636 in. from ground.
10. If there is no slip centrodes are (1) the curve and (2) the circle. If there is *r* per cent slip the centrodes are (1) a curve parallel to the given curve *r* per cent of the radius from it, and (2) a circle concentric with the given circle *r* per cent less or greater in radius.
12. 1·05 f.s.
13. (1) Line of tight side ; (2) the common tangent ; (3) the common normal.

15. 69° ; 5·5 and ·867 rads. per sec. respectively when links not crossed.
16. Rubbing speeds at A, B, C, D are respectively ·88, ·82, ·40, ·46 ft.-sec.
17. 12·5 rads. per sec.
21 $\overline{OB} : \overline{OA} :: 1 : 2$ (Fig. 47).
22. $2\,(\omega_1 + \omega_2)r_2$ and $\sqrt{3}\,(w_1 + \omega_2)r_2$.
23. 11.75 ft.-lb.
25. $d = 2r$; 3 : 1.

Examples 5. Page 150

1. V_E = 4·96 f.s. ; A_E = 48·4 f.s.s. V_F = 6·60 f.s. ; A_F = 32·6 f.s.s. V_P = 6·71 f.s. ; A_P = 28·0 f.s.s.
2. 8·4 f.s., 134 f s.s.
3. ·75 f.s., 10·7 f.s.s., ·60 f.s.
5. Throw 1·5 in. angle of advance 90°.
6. Throw 1·45 in. angle of advance 36·5°.
7. 7·15 f.s. along a line making 4° 36′ with the line of centres.
8. 6·75 f.s. ; 24·3 f.s.s.
9. 1·55 ; 3·56 f.s.
10. 0·80 f.s.
11. 3·095; 1,114 lb.
12. 7·64 rads. per sec. or ·75 on return stroke.
13. 11·2 f.s. ; 12·3 f.s.s.
15. 195 lb. along a line which makes 31° with the axis of the bar and passes 35 in. from its centre of gravity.
16. 32·4; 62 f.s.; 95 f.s.; 2,320 f.s.s.; 5,670 f.s.s.
17. 174 lb. along line making $60\frac{1}{2}$° with coupler at a distance ·17 in. from G.
18. 1,090 lb., angle to rod = 23°, distance of line of action from CG = 7·25 in.
23. $\dfrac{(a-b)^2}{a+b}$.
24. $\dfrac{b^2}{2a}$.
25. 82·6 f.s.s. ; 43·1 f.s.s. ; 79·8 f.s.s.
27. ·96 f.s. ; 14 f.s.s.
28. ·998 ; ·54 ; ·91 ; ·78.
29. $\frac{11}{16}$ in. ; 46·5 ; ·89 ; $1\frac{1}{16}$ in. ; $\frac{3}{16}$ in
30. 36° ; $\frac{3}{4}$ in. ; $\frac{1}{4}$ in. ; ·7 ; ·95.
31. $\frac{11}{32}$; 71·5° ; ·5 ; $1\frac{21}{32}$ in. ; $-\frac{5}{8}$ in.
32. 3·53 f.s. ; 3·6 f.s. ; 3·4 f.s. ; 3·1 f.s.
34. 5 in. ; 40° ; 1·33 in. ; ·25 in. ; ·9.
35. 15 in. ; 62° ; 6·375 in. ; − 2·1 in. ; ·6.
36. ·61, ·48 ; ·61, ·47 ; ·754, ·65 ; ·244, ·164.
37. − ·17 in. to − ·97 in. ; ·53 in. ; ·75 in.
38. 1 in. ; $-\frac{7}{16}$ in.
39. ·8 ; ·54 ; ·77.
42. Length 10·7 in., angle of advance 7°. (The proportions of the given mechanism are not practical ones.)
46. 0, 52·6 f.s.s.; 2·09 f.s., 57·0 f.s.s.
47. 6·04 f.s, 102 f.s.s : 6·04 f.s., 62 f.s.s.
48. $5\frac{1}{4}$ in.; $1\frac{5}{8}$ in.; 41°.
49. 111 f.s.s.
50. (*a*) 236, 277, 28·8; (*b*) 236, 209, 18·3; Velocities (*a*) 534, (*b*) 466; Accelerations, (*a*) 54,700, (*b*) 62,400, approx.

Examples 6. Page 176

3. $\sqrt{2}$, $\frac{1}{2}\sqrt{2}$.
5. Depth of G below CD = ·866 ft.

7. (1) $\overline{FD} = \overline{AF}$ with AF perpendicular to required straight line. (2) D will move on a straight line perpendicular to the diameter through A of the circle.

11. $V_D : V_E : : \overline{AD} : \overline{AE}$.
12. Coupler = ·843 × Inclined Bar.
13. ·4.
14. (1) Watt's Parallel Motion, (2) the pantograph.
15. (1) They are equal; (2) $\dfrac{1 - \sin^2 \varphi \cos^2 \theta}{1 - \sin^2 \varphi \sin^2 \theta}$, φ angle between shafts.
18. $\omega = \omega_1 \,.\, \dfrac{\sqrt{3}}{6}\,(4 - \cos^2 \theta)$ where ω_1 = angular speed of first shaft.
19. Force $= T \dfrac{\overline{QR}}{\overline{PQ}\,.\,\overline{PR} \cos \varphi}$.
20. ·13 in.
21. ·28 in.; ·45 in.
22. See answer to 11.
24. Any other point on OB. Four bar mechanism.
25. 15 f.s., 45 f.s.s.

Examples 7. Page 209

4. $C = \frac{2}{3}\mu N r$.
5. $\frac{2}{3}\mu P$ cosec $\alpha \dfrac{R^2 + Rr + r^2}{R + r}$.
6. 121·5 lb.
8. ·259.
9. 19·1 lb.
10. (*a*) 105 lb., (*b*) 7·3 lb.
11. ·464.
12. (*a*) A push of 1·15 tons. (*b*) A pull of ·54 tons.
14. 1·30.
15. (*a*) 3·50. (*b*) 3·41.
16. x should not be greater than 11¼ in. If x is greater than this the drawer will jam.
22. ·0202 tons.
23. ·427; the gear is self-locking.
24. 2⅝ tons, 6·16 ft. sec. sec.
25. (1) Torque = ·0397W lb.-in. (2) $\dfrac{1 + 25{\cdot}1\mu}{25{\cdot}1 - \mu}$ W lb.-in. where W is the load in lb. and $\mu = \tan \varphi$.
32. 3·06; ·377; 4·32; ·152; 1·28.
33. $1{\cdot}32 \times 10^{-6}$.
36. 0·836; 0·13; 780 lb.; 80°.
37. Yes; 0·096.
38. 2·53 ft.; 0·173 ft. per sec.
40. 2·06 in., say, 2 in.
42. 128 lb.-ft.; 29·25.
44. 4·4 in.; 513 lb.
45. 307 r.p.m.; 97·5 lb.-ft.

Examples 8. Page 238

1. (1) 120 : 21; (2) 5·05 : 1.
3. 2·06.
4. 22·96 ft., 21·70 ft.
6. 98·3 f.s.
8. 2·03 : 1; 263 lb. or 130 lb.
9. (*a*) 47·6 lb.-ft.; (*b*) 34·0 lb.-ft.
10. Diameters 15, 54, 23, 72 in.; widths 4¼ in. and 10 in.; lengths 22·4 ft., 20·8 ft.
11. 7·71 in. or 7¾ in.
12. 47·7 ft.

13. (*a*) $z = \frac{105}{x + 30}$, $y = \frac{21}{4}\frac{x + 10}{x + 30}$ (Fig. 147). (*b*) As in (*a*) with z and y interchanged.
14. 96·6 in. ; ·1 in. slack.
15. 5·28 in.
21. (*a*) 3·95 in. say 4 in.; (*b*) 254 lb.
22. 1·66 in. 180 lb.
24. (*a*) 67 h.p.; (*b*) 20%; (*c*) 68% increase.
25. (*a*) 0·0545; (*b*) 0·125.
27. (*a*) 10·2 sec.; (*b*) 1·354 f.p.s.; (*c*) 32·3 ft.

EXAMPLES 9. PAGE 278

2. (*a*) 6 ft. 0½ in. ; 6·41 in. (*b*) 3,420 lb.-ft. (*c*) 12,800 lb. (*d*) 25·6 in. (*e*) ·757 in. (*f*) 5,300 lb. (*g*) 3,180 B.Th.U. (*h*) 643 lb.
5. 7 in. ; 5 in. and 10 in. ; $\alpha \nless 17 \cdot 5°$.
7. 28 ; 126 ; 15·4 in.
8. 16 ; 72 ; 8·8 in.
9. 4·72 in. ; 22° 1′ to horizontal.
12. (*a*) 1,100 lb. ; (*b*) 1,000 lb. ; (*c*) 119 B.Th.U. ; 535 lb. ; 1,135 lb.
13. 202·5 ; 365·6 ; 637·5 ; 1,000.
14. (*a*) $A = 18$, $B = 40$, $C = 26$, $D = 32$, $E = 34$, $F = 24$, $G = 42$, $H = 16$. (*b*) 507 lb.-ft. ; 274 lb.-ft. ; 157 lb.-ft. ; 94·6 lb.-ft.
16. Main gear 54·9 tons at 26·9° to vertical. Intermediate gear 21·38 tons at 30·8° to vertical.
17. ·424.
18. $N_2 \nless \dfrac{32}{\dfrac{r_2}{2r_1} + 1}$.
19. $\dfrac{2\pi vd(n_1 + n_2)}{pn_1n_2}$
20. 2·094 ft. per sec.
21. $A = 15$, $B = 40$, $C = 10$, $D = 45$.
24. $A = 18$, $B = 189$, $C = 23$, $D = 184$.
26. 180 r.p.m. in same direction as the arm.
28. 7·61 h.p. ; B rotates anti-clockwise.
29. − 62 ; 3,960 sec. α lb. ; − 50 ; 3,300 sec. α lb.
30. $29\frac{41}{73}$ r.p.m.
31. 60 r.p.m. opposite to arm.
32. 4·943.
33. 2·75 ; − 4.
35. 12, 54; 21·5 in.

EXAMPLES 10. PAGE 307

2. $d = \cdot 945$ in. ; $r_2 = \cdot 955$ in. ; $\theta_1 = 25 \cdot 3°$; lift from M to $P = \cdot 207$ in.; acceleration at $P = 1{,}227$ ft. sec.^{2} to − 237 ft. sec.^{2} ; acceleration at $Q = -590$ ft.-sec.^{2} ; 79·6 lb. per inch ; ·22 in.
4. (*a*) $r_2 = \cdot 154$ in. ; $d = 1 \cdot 846$ in. (*d*) 410 ft.-sec.^{2} ; 365 ft.-sec.^{2} to − 196 ft.-sec.^{2} ; − 270 ft.-sec.^{2}.

EXAMPLES 11 PAGE 335

5. 227 and 264 ; 237 and 249·5.
6. 162 r.p.m. ; 178 r.p.m. ; 16 r.p.m.
7. 76·2 r.p.m. ; 92·5 r.p.m. ; 80·7 and 71·2 r.p.m. ; 98 and 86·5 r.p.m.
10. $S = 42$ lb. per inch ; 308 r.p.m.
11. 131·8 r.p.m. ; 154·2 r.p.m.
12. 60·8 and 39·6 lb. per inch respectively.
13. 11·7 in. ; 297 r.p.m.
14. 103·7 lb. per inch ; 0·48 sec. (neglecting inertia of arms).

EXAMPLES 12. PAGE 369

1. 1·65 lb. at 119° to A.
2. 2·5 lb.; $\theta = 284°$.
3. Wt. in XX = 3·77 lb., $\theta = 238°$; wt. in YY = 2·85 lb., $\theta = 15°$.
4. (1) A to B = 188·5°; A to D = 317°. (2) 7·7 in. (3) 19·2 lb.
5. r = 37 in., 43·2° from 90 lb. weight; 4,019 lb.-ft.
6. Wt. at end = 10·7 lb., 159° 26′ to 10 lb.; wt. at centre = 7·5 lb., opposite to 5 lb.
7. 90 lb.; 844 lb.
8. Primary forces and couples balanced. Maximum secondary force = 178 lb.; maximum secondary couple = 2,000 lb.-ft.
9. (a) At A, wt. = 464 lb., $\theta = 127°$; at D, wt. = 769 lb., $\theta = 27·5°$; (b) 212,800 lb.-ft.
10. Primary and secondary couples balanced; primary forces balanced; secondary force $= \dfrac{4W.r.\omega^2}{n.g}.\cos 2\theta = 255 \cos 2\theta$; 255 lb., 128 lb.
11. No. 4 to No. 2 = 108·3°; No. 1 to No. 3 = 108·3°; ·69 ton.
12. 286 lb. each at 113·2° to the nearest crank but one.
13. 10·35 tons.
14. (a) 530 lb. at 118° to nearest crank. (b) 68·3 m.p.h.; 1,170 lb.-ft.
16. 3,850 lb.
17. 180 lb. opposite the crank. (Complete primary balance only, Art. 194.)
21. 8·44 lb.
22. ·119 tons maximum; couple = 0; 463 lb. opposite each crank.

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3. ·0349v m.p.h.
4. $\alpha = 60°$ when inner back wheel moves on circle of 15 yd. radius.
6. ·0335 rads. sec.
7. 13·6 r.p.m.
10. 2·56 tons-ft. about a horizontal axis perpendicular to direction of motion.
12. 65·8 lb.-ft. (a) Effect is to tend to lift back wheels, (b) effect is to tend to lift outer wheels.
13. 15 ft.-tons for angle of pitch of 7° above and below horizontal.
14. 10·3 lbs., outer, and 8·73 lb.
18. 1·732 : 1, 1·935 : 1.
19. 1·542 lb. each, upward on bearing on inside of curve due to precession only.
20. (i) 1,250 lb. (ii) 427 lb.-ft.
21. A rate gyro for measuring rate of turn to right or left, clockwise viewed from left of pointer scale.
22. 0·913 lb.-ft., 0·646 lb.-ft., horizontal.

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1. ·0582 sec.
2. ·125 sec., ·138 sec.
3. 409 r.p.m.
4. 4,400 r.p.m.
5. 707 and 2,827 r.p.m.
7. 259 r.p.m.
8. − ·26 ft.
9. $\dfrac{Wv^2}{6g}$.
11. $N = \dfrac{72d^2}{a(L-a)}\sqrt{\dfrac{EI}{W}}$ neglecting mass of shaft, E is Young's Modulus in lb.-sq. in.; 7,040 r.p.m.; No, the speed is safe outside limits of from 25 to 35 per cent above or below the first critical speed.
12. 1,912 and 7,628 r.p.m.
13. 0·0315 ft.
15. 4,800 r.p.m.
16. 1,890 r.p.m., 37,000 lb.-sq. in.

17. 484 r.p.m.
18. 1,480 per min.
19. 1,335 and 1,913 r.p.m.; 6·72 in. from 400 lb. mass.
20. 3,820 and 9,820 per min.; 15·4 ft. from end along longer portion of shaft.
21. 357 and 761 per min.; 12 in. from first end and 4·64 in. from other end.
23. 1·91 sec.; 0·000296 in.
24. 670 per minute.
25. 1·44 in. to right of B; 83·2 per sec.; 8 in.
26. (*a*) 1·73 lb.; (*b*) 0·295 in.; (*c*) 150 lb.
28. 0·127 sec.; 0·748 in.; 1620 lb. per square inch.
29. 116 lb. 25° to line of stroke.
30. 186 lb. at 18° to the horizontal.
31. 1·17 lb. at gudgeon pin, 1·83 lb. beyond the mass centre, and 0·2 in. beyond crankpin.
33. 1·20 radians per second.
34. 11·3 per second; 15 in. from larger rotor.
35. 374 lb. ft.
36. 12, 30 and 40 in. from one end; 111 per second.
37. $M = 21 \cdot 5$, $a = 0 \cdot 25$.
38. 0·0413 ft.; 29·74°.
39. $x = Ae^{-3t} (\cos 4t + a)$; 4·76.
41. Force on piston is 795 lb.
42. 140 per minute.
43. 1·03 sec.
44. 218 lb.
45. 336 lb.
46. 43·8 per second.
47. 3·23 in., say 3¼ in. 3170 r.p.m.
48. 0·0824 in.
49. 29,200 per min. 0·00175 in.
50. 13,700 and 23,900 per min. 0·00377, 0·00417, 0·00440, and 0·00418 in.
51. 270 lb., 11,900 lb. per ft. No, the mass of the spring is too large.
52. If $A = Ip^2/C$, $\theta_1 = \theta_3 = \dfrac{F_0}{AC(A-3)}$, $\theta_2 = (1 - A)\theta_1$.
53. If $A = Ip^2/C$, $\theta_1 = -\dfrac{F_0\,(A^2 - 3A + 1)}{AC(A-1)(A-3)}$, $\theta_2 = \dfrac{-F_0}{AC(A-3)}$,

$$\theta_3 = -\frac{F_0}{AC\,(A-1)(A-3)}$$

54. 0·00412, 0·00421, 0·00417 in.
58. $ml = 1 \cdot 875$, $\omega = \dfrac{3 \cdot 52}{l^2}\sqrt{\dfrac{12gEI}{w}}$.
60. $ml = 4 \cdot 712$.
61. $ml = 4 \cdot 739$, $ml = 4 \cdot 774$.

EXAMPLES 15. PAGE 500

1. (*a*) 255 lb. at C.G.; 28·1 lb.-ft. (*b*) 255 lb. at 7 in. from crank end, at 39° to the line of stroke.
3. (*a*) 39·6 r.p.m.; (*b*) 24,000 lb.-in.; (*c*) 38·7 degrees; (*d*) 17·7 per sec.; (*e*) 13 in.; 3,130 lb.
4. (*b*) 34° about; (*c*) 47° about.
5. T = torque, x = distance, ft. (*a*) $T = 62{,}500 - 14 \cdot 4x$ lb.-ft. (*b*) $T = 43{,}200 - 14 \cdot 4x$ lb.-ft. (from $x = 61$ to $x = 2{,}770$); (*c*) braking torque = 12,300 lb.-ft.; (*d*) 107 sec.

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$$F_{H(2k)} = \tfrac{1}{2}\left[\frac{\cos\left\{2k\,.\,\theta - \dfrac{N-1}{2N}\,.\,(2k-1)\,.\,2\pi\right\}.\sin(2k-1)\pi}{\sin\dfrac{2k-1}{N}\,.\,\pi} - \frac{\cos\left\{2k\,.\,\theta - \dfrac{N-1}{2N}\,.\,(2k+1)\,.\,2\pi\right\}.\sin(2k+1)\pi}{\sin\dfrac{2k+1}{N}\,.\,\pi}\right]$$ 358

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